Delft University of Technology

Bachelor Thesis

Validation of the Vlasov torsion theory for rectangular solid cross-sections and for rectangular closed tube cross-sections

Author:
Oussama Elarras (4731875)

October 17, 2021

Delft
University of
Technology

Validation of the Vlasov torsion theory for rectangular solid cross-sections and for rectangular closed tube cross-sections

by

Oussama Elarras

Thesis Comittee:

Dr. ir. P.C.J.Hoogenboom and Prof. Dr. M. Veljkovic

A thesis is presented for the degree of Bachelor of Science
in Civil Engineering

Faculty of Civil Engineering and Geosciences
Delft University of Technology
17 October 2021

Preface

In order to obtain the degree of Bachelor of Science, it is essential to utilise all my gained knowledge from my bachelor period at the Faculty of Civil Engineering and Geosciences to write my Bachelor thesis. In this report I will discuss the validity of the Vlasov torsion theory for rectangular solid cross-sections and for rectangular closed tube cross-section. I want to thank my supervisors Dr. ir. P.C.J. Hoogenboom and Prof. Dr. M. Veljkovic for their support and guidance during this process. I also want to thank R. Yan, MSc for helping me to obtain the FEM software Abaqus.

Oussama Elarras

Delft, October 2021

Abstract

To describe the behaviour of beams which are loaded on a torsional force it is possible to make use of the Saint-Venant formula, if the beam is unconstrained on both sides. This theory can be used regardless the cross-sectional dimensions of the beam. If the beam is only constrained at one side and unconstrained at the other side, it is possible to make use of the Vlasov torsion theory to describe the behaviour of the beam, however this theory is proven only for open cross-section. In this report it is attempted to prove the Vlaslov torsion theory for rectangular solid cross-sections and for rectangular closed tube cross-section.

It is important to describe a proper method to validate the Vlasov torsion theory. The beam will be constrained only at one side and a torsional external moment will work at the free end of the beam. During the whole validity process it has been chosen to keep the material properties and the external moment the same in order to compare the results. The length of the beams are kept variable to see what the influence is of the length of the beam on the accuracy of the Vlasov torsion theory. First, the maximum horizontal displacement and the maximum normal stress has been calculated by use of the Vlasov equations. Second, the maximum horizontal displacement and the maximum normal stress have been calculated by use of the FEM software Abaqus. These outcomes have been compared with each other by use of a ratio check, namely if the maximum error is within 15 percent it can be stated that the theory is valid for the tested beam length.

From the obtained results it can be stated that the Vlasov torsion theory seems to be valid regarding the maximum horizontal displacement for the rectangular solid cross-sections and the rectangular closed tube cross-sections. The obtained results for the maximum normal stress exceed the maximum allowable error of 15 percent. Regarding the obtained results for the maximum normal stress, the Vlasov torsion theory can not be declared valid for the mentioned cross-sections.

Contents

1 Introduction

1.1 Problem description

The first theory that was published to describe the behaviour of beams loaded with a torsional moment is the theory of Saint-Venant. The theory of Saint-Venant can be used for beams which can warp and rotate unimpededly. This theory can be used for all different types of cross-sections. In 1933 the Vlasov torsion theory was published, this theory describes the behaviour of thin walled beams which are prevented of rotation and warping on one side of the beam. Since these profiles are often loaded on torsional forces which can cause warping of the thin walled beams, this theory is mostly relevant for thin walled profiles (P.C.J.Hoogenboom, 2008).

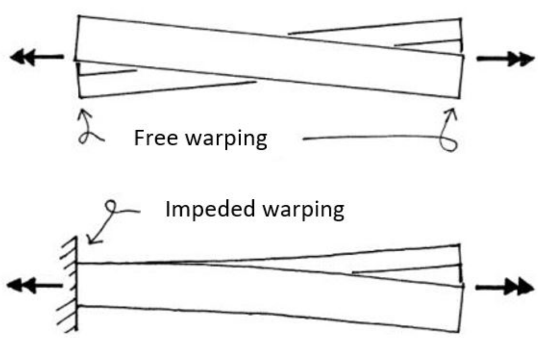

Figure 1: Top views of two I-sections considering respectively the set-up for the Saint-Venant theory and the Vlasov torsion theory (P.C.J.Hoogenboom, 2008)

The Vlasov torsion theory is only proven for thin walled cross-sections, therefore it is not allowed to use the Vlasov torsion theory for other cross-sections. Because of this limitations commercial framework programs make only use of the theory of Saint-Venant. If a constrained thin walled profile is modelled in these framework programs and is tested on a torsional moment the program gives a warning. Because of this limitations the engineer needs to perform hand calculations to describe the behaviour of the mentioned thin walled beam. To unburden the engineer and to improve the framework programs it would be ideal if it can be proven that the Vlasov theory is valid for all types of cross-sections (P.C.J.Hoogenboom, 2008). Therefore, the research question can be formulated as:

Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?

1.2 Approach

The first step is to understand the equations derived from the Vlasov torsion theory. After the theory is understood it is possible to apply the Vlasov torsion theory for a determined system. In this report the FEM software Abaqus will be used to check the results obtained by use of the Vlasov torsion theory. Abaqus will be used on the online workstation of the TU Delft. Which means that the used software is not limited in performing element calculation, due to licence limitations. By use of this software it is attempted to prove the validity of the Vlasov torsion theory. Furthermore the influence of the mesh size on the accuracy of the obtained results will be discussed. The length of the beams that will be checked is variable, this will be done to check the influence of the beam length on the accuracy of the results.

1.3 Previous performed studies

A study about the given research question has been performed by a couple of Bsc students. The first study that has been performed, was done by T.B Raaphorst. From this study it can be concluded that the results are positive for the mentioned cross-sections, however the done calculations have been performed for only a length of 2540 mm (T.B Raaphorst, 2020). The Vlasov torsion theory needs to be checked on a couple of lengths before it can be valid for the mentioned cross-sections. A follow up study has been performed by A.Yildirim, from this study it can be concluded that the stresses calculated with the FEM software were a factor 10 bigger then the calculated values from the Vlasov torsion theory (A.Yildirim, 2021). Due to the inconsistent results a follow up study has been performed by F.Hilmer to prove the validity of the Vlasov torsion theory for rectangular solid cross-sections and for rectangular closed tube cross-sections . The results from this report were promising, however not conclusive enough to prove the validity. The outcomes of the Vlasov torsion theory has been compared with the outcomes obtained from the Finite Element Method software Ansys. Due to the limitation of the element calculations performed by the used student version of Ansys it was not possible to prove the validity of the Vlasov torsion theory (F.Hilmer, 2021).

2 Method

In this chapter the method will be described for the validity of the Vlasov torsion theory. It is essential that a method will be formulated, which makes it possible to perform the check for beams with different types of cross-sections.

2.1 Vlasov Torsion Theory

Before a method can be described to test the validity of the Vlasov torsion theory it is important to understand the theory and the equations which can be derived from this theory. The Vlasov torsion theory can be described by the following formula (P.C.J.Hoogenboom, 2008):

$$EC_w \frac{d^4\phi}{dx^4} - GI_w \frac{d^2\phi}{dx^2} = m_x \quad (1)$$

E = *Young's modulus [MPa]*

$C_w = Warping\ constant\ [mm^6]$

$\phi = Angular\ deviation\ [rad]$

G = *Shear modulus [MPa]*

$I_w = Torsional\ moment\ of\ inertia\ [mm^4]$

$m_x = Distributed\ torsional\ moment\ along\ the\ beam\ [\frac{Nmm}{mm}]$

The warping constant can be described by the following formula:

$$C_w = \int_A \psi^2 dA \quad (2)$$

The Bi-moment can be described by the formula:

$$B = -\int_A \sigma_{xx}\psi dA \quad (3)$$

If warping of a cross-section is constrained, then a Bi-moment will work in the constrained side of the beam. By solving the differential equation it is possible to determine the Bi-moment. Thereafter the wringing moment can be determined:

$$B = -EC_w \frac{d^2\phi}{dx^2} \quad (4)$$

$$M_w = GI_w \frac{d\phi}{dx} + \frac{dB}{dx} \quad (5)$$

By combining formula's 4 and 5 it is possible to derive a differential equation for the torsional moment:

$$GI_w \frac{d\phi}{dx} - EC_w \frac{d\phi^3}{dx^3} = M_w \quad (6)$$

The characteristic length can be determined by use of the formula:

$$l_c = \sqrt{\frac{EC_w}{GI_w}} \quad (7)$$

2.2 Set up of the system

The beam that will be tested has to be constrained at one side and be free of any constraining at the other side. At the constrained side warping and angular displacement will be hindered. At the free side a torsional moment will be introduced.

Figure 2: Set up of the system

It has been chosen to model a beam of the material structural steel (F.Hilmer, 2021). Structural steel has the following material properties:

Table 1: Material properties of the beam

material property	*value*
E	$210000[N/mm^2]$
v	$0.29[-]$
G	$81000[N/mm^2]$

The only external Load that will work on the beam is a torsional moment, which will be equal to :

Table 2: External torsional load on the beam

External Load	*value*
T	$10^6[Nmm]$

2.3 Cross-sectional properties

In order to solve the Vlasov equation it is necessary to determine the cross-sectional properties of the chosen cross-sections that will be analysed. 3 parameters are necessary to solve the equation: the Torsional moment of inertia $[I_w]$, the Warping constant $[C_w]$ and the Warping function $[\psi]$. These parameters are dependent of the dimensions of each cross-section. In this validity process a rectangular solid cross-section and a rectangular closed cross-section will be analysed.

The torsional moment of inertia for a solid rectangular cross-section can be determined by use of the formula (Paul A. Seaburg, 2003):

$$I_w = \frac{hb^3}{3}; \frac{b}{t} \geq 10 \quad (8)$$

The torsional moment of a inertia for a closed rectangular cross-section with a thickness t can be determined by use of the formula (Paul A. Seaburg, 2003):

$$I_w = \frac{2t^2b^2h^2}{bt + ht}; \frac{b}{t} \geq 10 \quad (9)$$

The formula for the warping constant is described in paragraph 2.1. The warping constant is dependent of the dimensions of the cross-section and the warping function. The warping function describes the deformation of a cross-section (P.C.J.Hoogenboom, 2008).

If the dimensions of the cross-section are determined, it is possible to model the cross-section in the software program Shapebuilder. This program returns all the cross-sectional parameters including the three mentioned parameters who are needed to solve the Vlasov equation.

2.4 Aplication of Vlasov torsion theory

In this paragraph the application of the Vlasov torsion theory will be described for the mentioned set up in paragraph 2.2 . The following boundary condition can be derived from the chosen set up (P.C.J.Hoogenboom, 2008):

Constrained at x = 0:

$$1)\ \phi(0) = 0,$$

$$2)\ \frac{d\phi}{dx}(0) = 0$$

Free support and external torsional moment at x = L:

$$3)\ GI_w \frac{d\phi}{dx}(L) - EC_w \frac{d\phi^3}{dx^3}(L) = T,$$

$$4)\ \frac{d\phi^2}{dx^2}(L) = 0$$

Since the differential equation described by Vlasov is a fourth power differential equation, at least four equations are needed to solve this equation. The made set up satisfies this requirement. By use of the software Maple the differential equations can be solved. By solving this equation two relevant properties of the beam will be derived. The rotation of the beam and the maximum longitudinal (normal) warping stress in the beam.

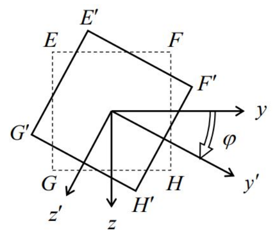

Figure 3: Rotation of a rectangular cross-section (P.C.J.Hoogenboom, 2008)

By use of the rotation of the beam it is possible to determine the maximum displacement of the beam. To make it possible to compare the outcomes it has been chosen to determine only the horizontal displacement. Since the chosen cross-sections are rectangular it is possible to determine the horizontal displacement by use of the formula :

$$u_{max,horizontal} = \phi * 0.5h \quad (10)$$

The maximum longitudinal (normal) warping stress can be formed by combining the formulas 2 and 3 and solving it for σ_{xx} (P.C.J.Hoogenboom, 2008):

$$\sigma_{xx} = -\frac{B}{C_w}\psi \quad (11)$$

$\sigma_{xx} = The\ maximum\ longitudinal\ (normal)\ warping\ stress\ [N/mm^2]$

$B = maximum\ bimoment\ [Nmm^2]$

$C_w = Warping\ constant\ [m^6]$

$\psi = normalized\ unit\ warping\ [mm^2]$

2.5 Finite Element Method calculations

To check the validity of the outcomes from the Vlasov equations a Finite Element Method software will be used to calculate the behaviour of the analysed beams. In the previous done study by F.Himler these calculations were done by a FEM software which was limited in its element calculations. It is expected that due to this limitations it was not possible to formulate a decisive conclusion about the validity of the the Vlasov torsion theory for the mentioned cross-section. In this report a FEM software will be used which can perform element calculation without any restrictions. The software Abaqus will be used to perform the calculations. To make it possible for third parties to perform the same calculations, a small instruction manuel will be given about the steps that need to be taken to get the right outcomes in Abaqus.

Before the Abaqus software can be used it is important to perform the steps mentioned in the paragraphs 2.2, 2.3 and 2.5. By performing these steps the needed parameters and boundary conditions will be obtained.

The first step is to model the dimensions of the beam in Abaqus. Which can be done in the model window using the parts module. First the cross-sectional dimensions have to be modelled followed with the length of the beam. After the dimensions are modelled it is important to assign the whole beam to a section assignment. It is important to assign the section as a homogeneous solid beam.

If the beam is assigned to a section it will be possible to model the material properties of the beam. This can be done in the module property. The mechanical properties need to be determined, in order to be more specific it is important to change the elastic material behavior. In this window it is possible to apply the desired Young's modulus and the Poisson's ratio according to the chosen material.

After the material properties are determined it is possible to model the boundary conditions of the beam. According to the set up in chapter 2.2 the beam needs to be constrained at one side. This can be done in the load module by making use of the Boundary conditions tab. The surface of the to constraining part should be selected first, after the selection is approved it is possible to encastre the chosen section.

The last mechanical step is to apply the load on the beam. Before we can apply a load it is necessary to determine the location of the load. The torsional moment should be located at the unconstrained part of the beam. It is important to apply this moment at the centre of the cross-section. First, the coordinate system needs to be placed at this location. Then, it is possible to make this point visible by making use of the tools tab and creating a reference point. Now it is possible to assign a load to the created reference point. This can be done under the load module. It is important to apply the moment in the right direction so a torsional moment will be created. There are 3 options CM1(moment around the x-axis), CM2(moment around the y-axis) and CM3(moment around the z-axis). The coordinate system has been placed so that the z-axis is the depth of the beam. Thus, to apply a torsional moment CM3 should be chosen.

Before the model can be run, the mesh size should be determined. First, a random value has to be chosen for the element size, thenit is able to run the model. If the running has been completed, it is possible to view the results. If the results do not approach the desired values, a solution could be to refine the mesh size to a smaller value. This step can be iterated until the desired value is approached sufficiently.

After the results are obtained it is important to process the results correctly. For this research the maximum horizontal displacement and the maximum longitudinal (normal) stress should determined. In the visualization module it is possible to display the desired displacement and stresses. The coordinate system has been placed so that the horizontal displacement will happen in line with the x-axis, therefore the maximum horizontal displacement can be found under the U1 tab. The maximum longitudinal (normal) stress will be in the same direction as the z-axis, so to display this stress the tab S33 should be used.

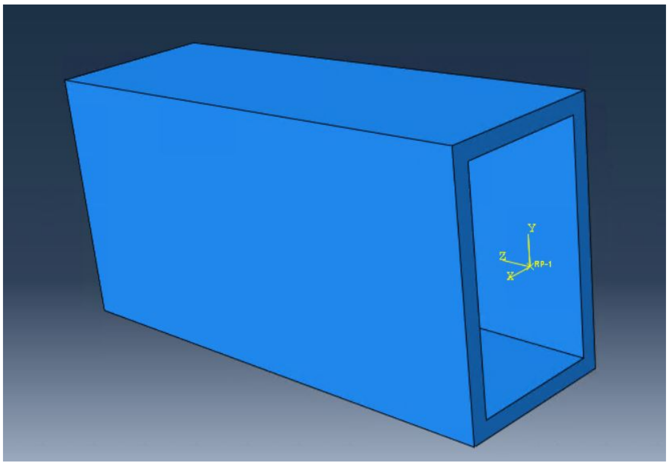

Figure 4: Orientation of the coordinate system

2.6 Validation of the results

If the results from the Vlasov equation and from the FEM software calculations are obtained, it is possible to compare these results. The validation will be done by use of a ratio check of the results. It is given that an error of maximum 15% is allowed for engineering purposes. This ratio check will be performed for both cross-section with variable length. This will be done to analyse the influence of the length of the beam on the validity of the Vlasov torsion theory.

$$Ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} \quad (12)$$

3 Results

In this chapter the results of the calculation will be described. The calculations will be performed for a rectangular solid cross-section and a rectangular closed cross-sections. These calculations will be performed for various lengths of each cross-section. The set up of the system and the material properties can be found in chapter 2.2.

3.1 Rectangular solid cross-section

3.1.1 Cross-section properties

The dimensions of the cross-section are given in figure 5 it has been chosen to have a width height ratio of 1:1.5 . The first beam will have a length of 150 [mm] each following beam will have a two times bigger length as the previous one (F.Hilmer, 2021). It has been chosen to check 5 beams, the lengths of the beams can be found in the table below .

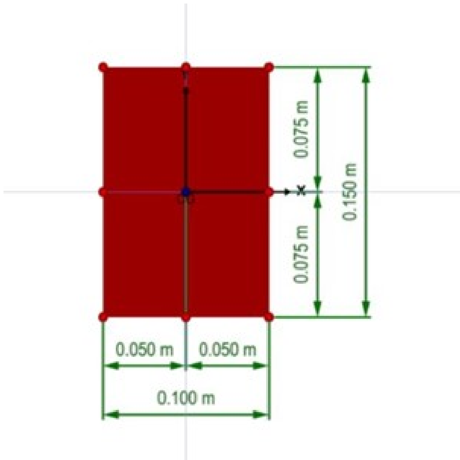

Figure 5: Dimensions of the solid cross-section

Table 3: Lengths of the tested beams

beam	beam length
beam 1	150 [mm]
beam 2	300 [mm]
beam 3	600 [mm]
beam 4	1200 [mm]
beam 5	2400 [mm]

By use of the cross-sectional dimensions, the cross-sectional properties are determined with the software Shapebuilder:

Table 4: Cross-sectional properties of the beams

cross-sectional property	value
I_w	$2.94 * 10^7$ $[mm^4]$
C_w	$3.79 * 10^9 [mm^6]$
ψ	$1.4 * 10^3 [mm^2]$

3.1.2 Beam 1

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set-up has been determined. These calculations have been performed in maple for the determined properties of beam 1. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.005506[rad]$

$u_{max,horizontal} = 0.0413[mm]$

$\sigma_{max} = 67.3666[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results for the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 6 and 7 .

Table 5: Results of the FEM calculations for the different element sizes

element size $[mm]$	maximum displacement $[mm]$	maximum stress$[N/mm^2]$
5	0.04230	38.06
3	0.04230	46.02
2	0.04228	52.73
1.8	0.04228	54.57

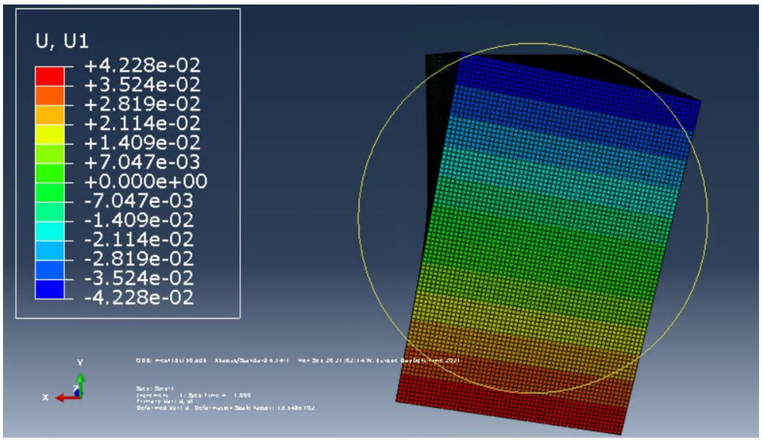

Figure 6: Maximum Displacement of beam 1

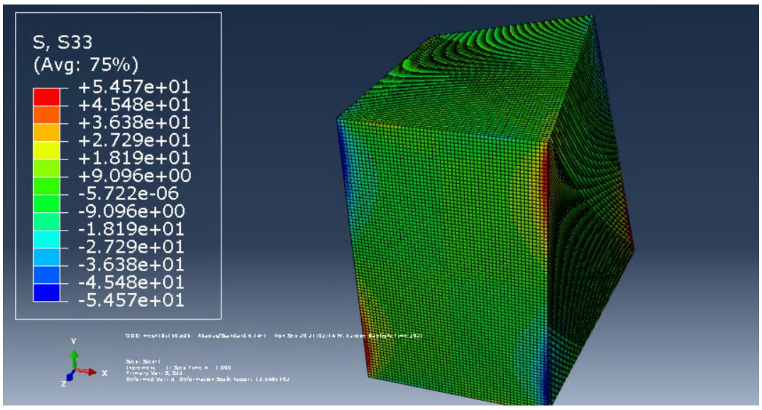

Figure 7: Maximum Stress of beam 1

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 1.8 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.0413\ [mm]}{0.04228\ [mm]} = 0.977$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{67.3666\ [N/mm^2]}{54.57\ [N/mm^2]} = 1.23$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6

to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

The obtained outcomes have also been compared with the results obtained by F.Hilmer (F.Hilmer, 2021). The set-up, material properties, cross-sectional properties, length and mesh size are kept the same to have a proper comparison of the results. From this comparison it can be seen that the outcomes obtained by use of the Vlasov torsion theory are exactly the same. Regarding the maximum horizontal displacement and the maximum normal stress, there were some significant differences for the values obtained by use of the FEM softwares Ansys and Abaqus. The maximum displacement determined by use of the Abaqus software approaches the results obtained by the Vlasov torsion theory better then the Ansys software. But the maximum normal stress is approached better by use of the Ansys software compared to the Abaqus software.

To check the model on inconsistent results it has been chosen to check the model on singularities. This has been done by first changing the boundary conditions of the model. In the used model the beam was constrained on all axes (encastred). To check if singularity has took place in the fully constrained model, it has been chosen to repeat the FEM calculations with only the z-axis being constrained. The results that are obtained are from the smallest possible mesh size of 1.8 mm. From the results in figure 9, it can be concluded that singularity due to fixed x and y can be excluded, because the stresses are almost similar to the obtained results from the fully constrained model. The horizontal displacements gets significantly less accurate if only the z-axes is constrained, as can be seen in figure 8 . Regarding the values obtained for the maximum stresses it can be observed that the difference is approximately 1.4 percent between the obtained stresses.

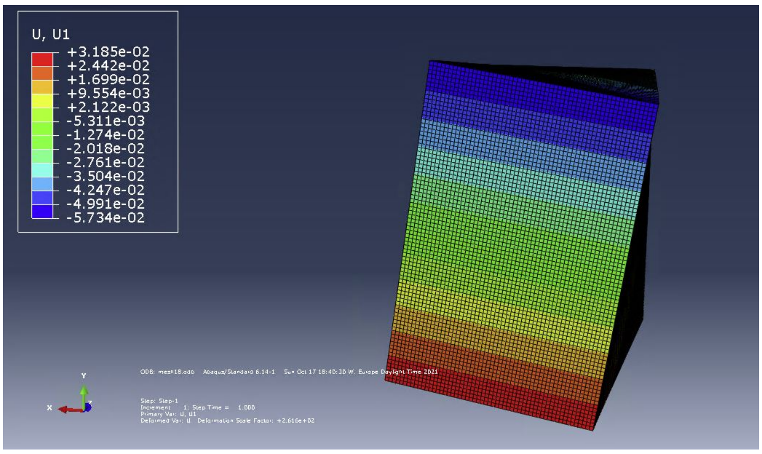

Figure 8: Maximum Displacement of beam 1 only constrained in the z-axis

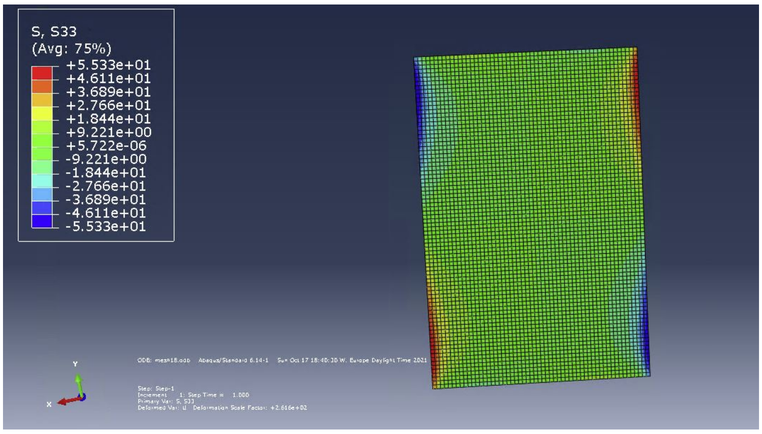

Figure 9: Maximum Stress of beam 1 only constrained in the z-axis

3.1.3 Beam 2

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 2. The performed calculations can be found in Apendix A. From these calculations we can derive the following outcomes:

$\phi = 0.001177[rad]$

$u_{max,horizontal} = 0.08831[mm]$

$\sigma_{max} = 67.3666[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 10 and 11 .

Table 6: Results of the FEM calculations for different element sizes

element size $[mm]$	maximum displacement $[mm]$	maximum stress $[N/mm^2]$
6	0.08935	36.33
3	0.08922	46.74

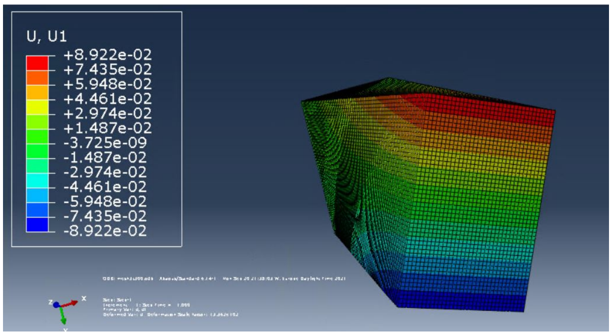

Figure 10: Maximum Displacement of beam 2

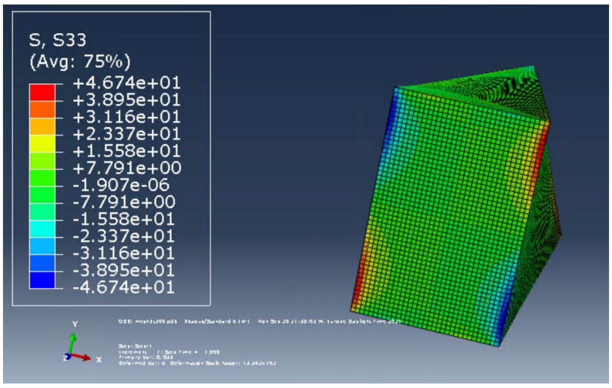

Figure 11: Maximum Stress of beam 2

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 3 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.08831\ [mm]}{0.08935\ [mm]} = 0.988$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{67.3666\ [N/mm^2]}{46.74\ [N/mm^2]} = 1.441$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.1.4 Beam 3

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 3. The performed calculations can be found in Apendix A. From these calculations we can derive the following outcomes:

$\phi = 0.002431[rad]$

$u_{max,horizontal} = 0.1823[mm]$

$\sigma_{max} = 67.3666[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 12 and 13 .

Table 7: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
7	0.1838	33.38
3	0.1834	46.74

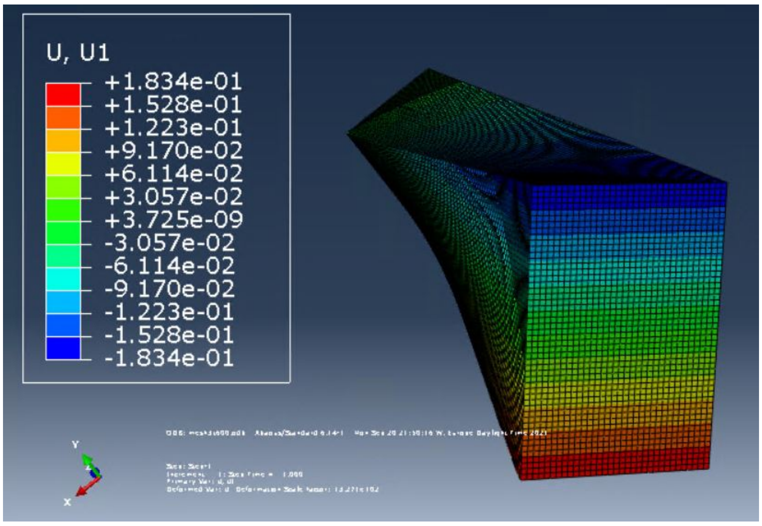

Figure 12: Maximum Displacement of beam 3

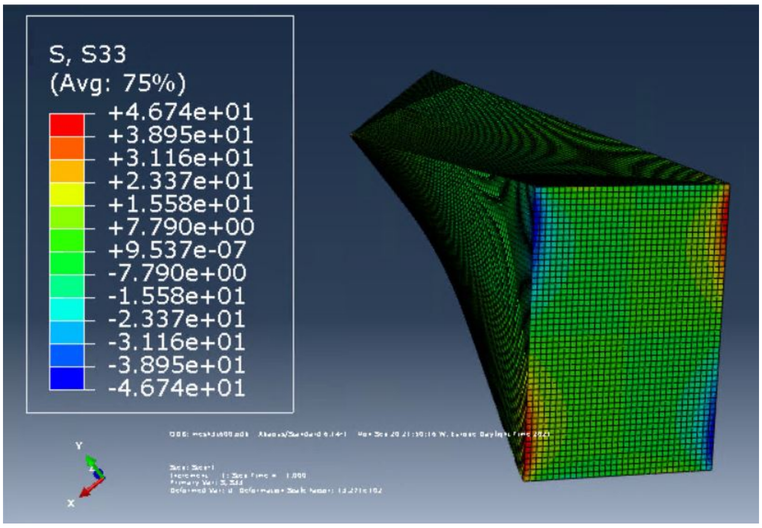

Figure 13: Maximum Stress of beam 3

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 3 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.1823\ [mm]}{0.1834\ [mm]} = 0.994$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{67.3666\ [N/mm^2]}{46.74\ [N/mm^2]} = 1.44$$

From the performed unity checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.1.5 Beam 4

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 4. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.004938[rad]$

$u_{max,horizontal} = 0.3704[mm]$

$\sigma_{max} = 67.3666[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below. The visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 14 and 15 .

Table 8: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
9	0.3733	30.08
3	0.3718	46.74

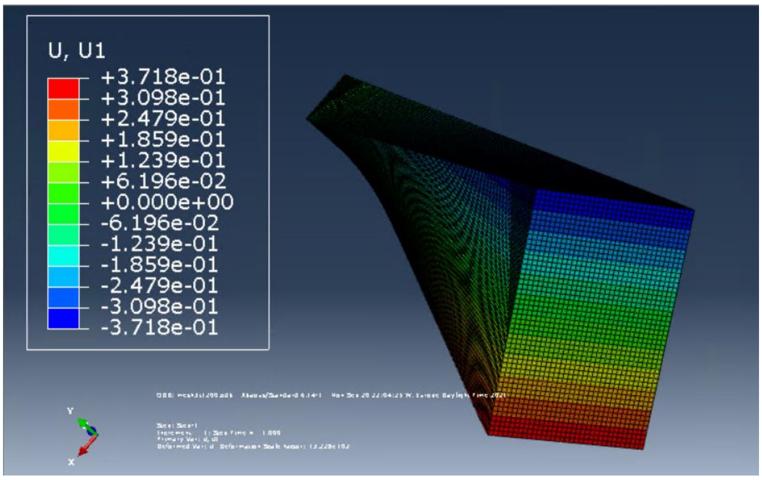

Figure 14: Maximum Displacement of beam 4

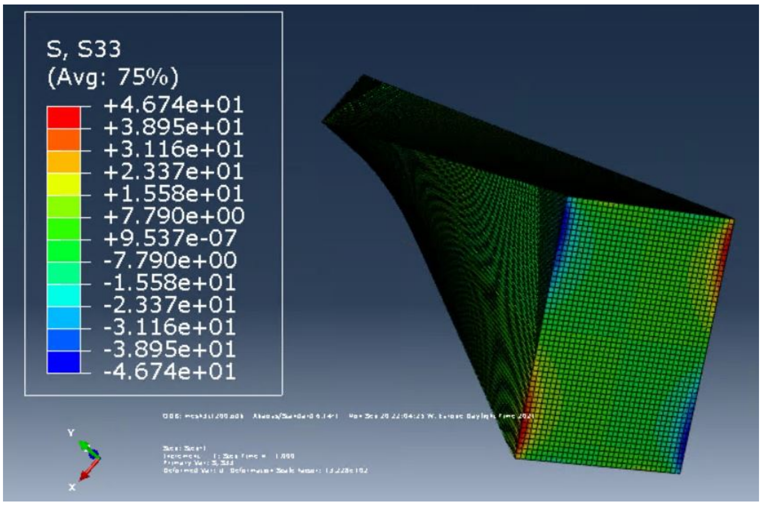

Figure 15: Maximum Stress of beam 3

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 3 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.3704\ [mm]}{0.3717\ [mm]} = 0.997$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{67.3666\ [N/mm^2]}{46.74\ [N/mm^2]} = 1.44$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.1.6 Beam 5

Calculations Vlasov theory

In chapter 2.4 the application of the Vlasov theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 5. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.009953[rad]$

$u_{max,horizontal} = 0.7465[mm]$

$\sigma_{max} = 67.3666[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 16 and 17.

Table 9: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
12	0.7547	25.69
6	0.7496	36.33

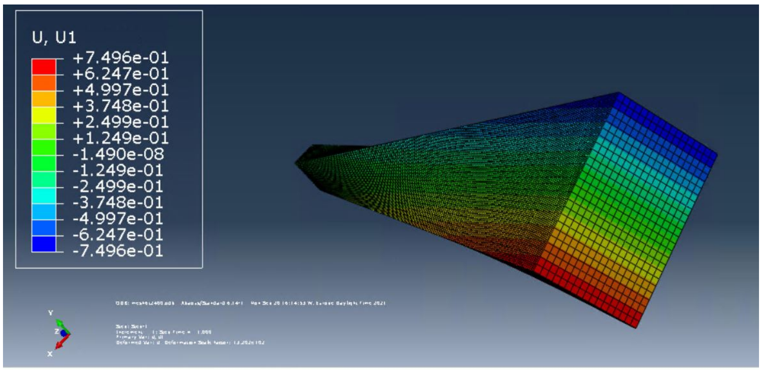

Figure 16: Maximum Displacement of beam 5

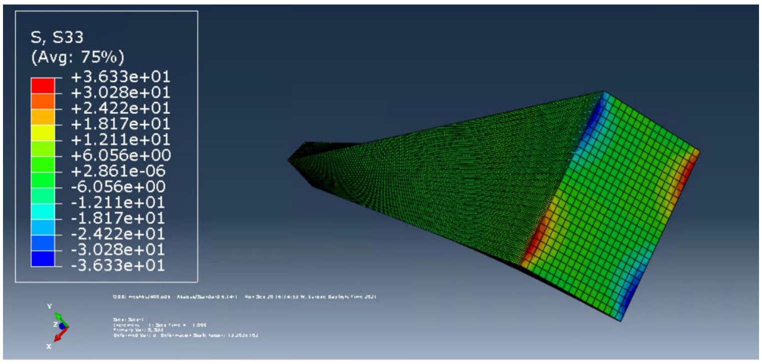

Figure 17: Maximum Stress of beam 5

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 6 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.7465\ [mm]}{0.7496\ [mm]} = 0.996$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{67.3666\ [N/mm^2]}{36.33\ [N/mm^2]} = 1.85$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.2 Rectangular closed cross-section

3.2.1 Cross-section properties

The dimensions of the cross-section are given in figure 18 it has been chosen to have a width height ratio of 1:1.5 and a thickness of 10 mm . The first beam will have a length of 150 [mm] each following beam will have a two times as big length as the previous one (F.Hilmer, 2021). It has been chosen to check 5 beams the lengths of the beams can be found in the table below .

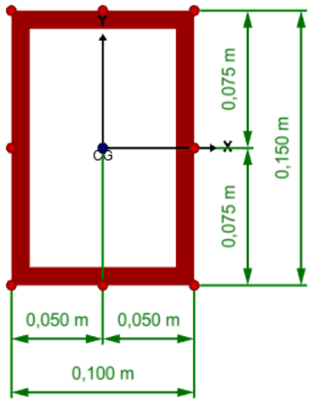

Figure 18: Dimensions of the rectangular closed cross-section

Table 10: Lengths of the tested beams

beam	beam length
beam 6	150 [mm]
beam 7	300 [mm]
beam 8	600 [mm]
beam 9	1200 [mm]
beam 10	2400 [mm]

By use of the cross-sectional dimensions the cross-sectional properties are determined with the software Shapebuilder:

Table 11: Cross-sectional properties of the beams

cross-sectional property	value
I_w	$1.45 * 10^7 [mm^4]$
C_w	$9.187 * 10^8 [mm^6]$
ψ	$1 * 10^3 [mm^2]$

3.2.2 Beam 6

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set-up has been determined. These calculations have been performed in maple for the determined properties of beam 6. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.001171[rad]$

$u_{max,horizontal} = 0.08785[mm]$

$\sigma_{max} = 140.9137[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results for the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 19 and 20 .

Table 12: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
2.5	0.09201	79.34
1	0.09202	111.9

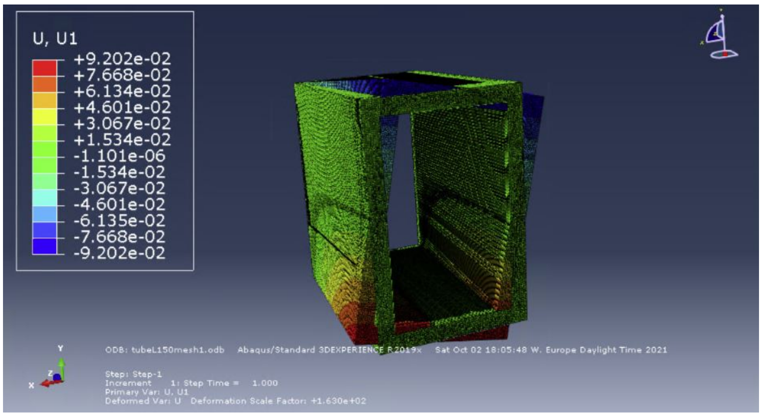

Figure 19: Maximum Displacement of beam 6

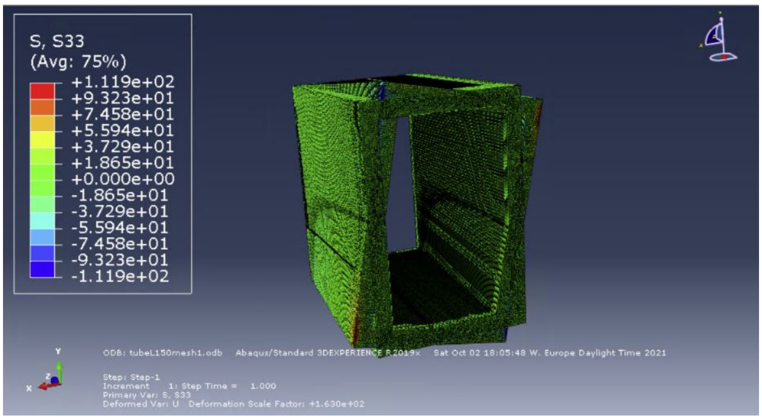

Figure 20: Maximum Stress of beam 6

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 1 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.08785\ [mm]}{0.09202\ [mm]} = 0.951$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{140.9137\ [N/mm^2]}{111.9[N/mm^2]} = 1.259$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.2.3 Beam 7

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 7. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.002451[rad]$

$u_{max,horizontal} = 0.1838[mm]$

$\sigma_{max} = 140.9137[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 21 and 22 .

Table 13: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
3	0.1881	70.24
2	0.1880	82.89

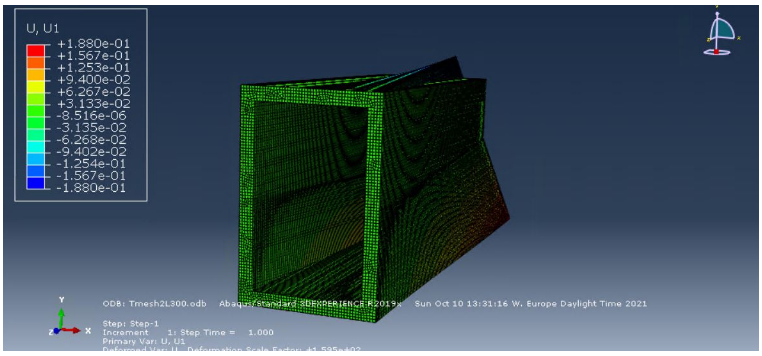

Figure 21: Maximum Displacement of beam 7

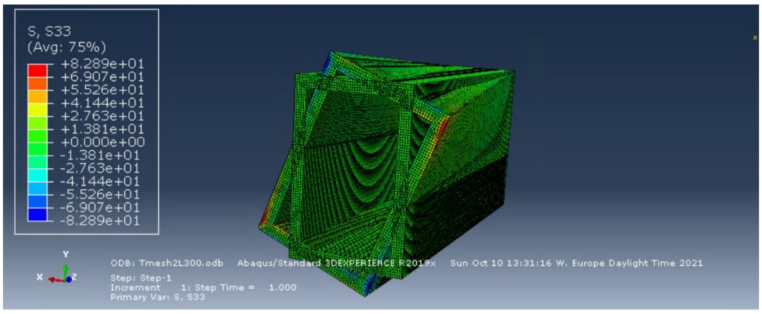

Figure 22: Maximum Stress of beam 7

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 2 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.1838\ [mm]}{0.1880\ [mm]} = 0.978$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{140.9137\ [N/mm^2]}{82.89\ [N/mm^2]} = 1.7$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.2.4 Beam 8

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 8. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.005011[rad]$

$u_{max,horizontal} = 0.3758[mm]$

$\sigma_{max} = 140.9137[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below. and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 23 and 24 .

Table 14: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress[N/mm^2]
4	0.3803	53.73
2	0.3805	73.81

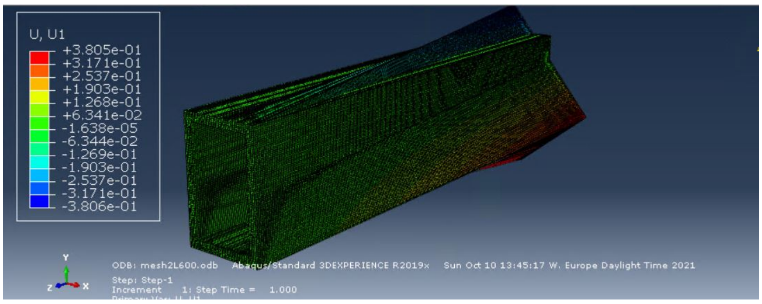

Figure 23: Maximum Displacement of beam 8

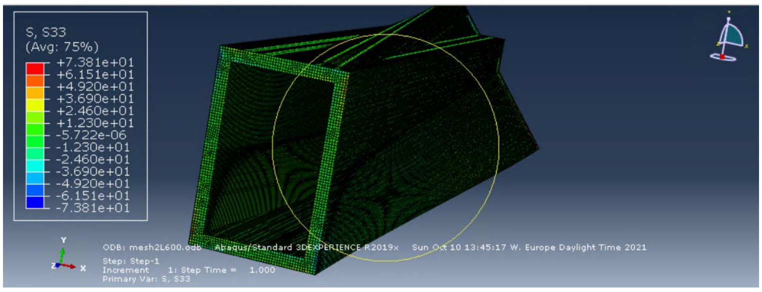

Figure 24: Maximum Stress of beam 8

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 2 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.3758\ [mm]}{0.3805\ [mm]} = 0.987$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{140.9137\ [N/mm^2]}{73.81\ [N/mm^2]} = 1.91$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.2.5 Beam 9

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 9. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.001013[rad]$

$u_{max,horizontal} = 0.7597[mm]$

$\sigma_{max} = 140.9137[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 25 and 26 .

Table 15: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
5	0.7638	43.99
2	0.7641	71.98

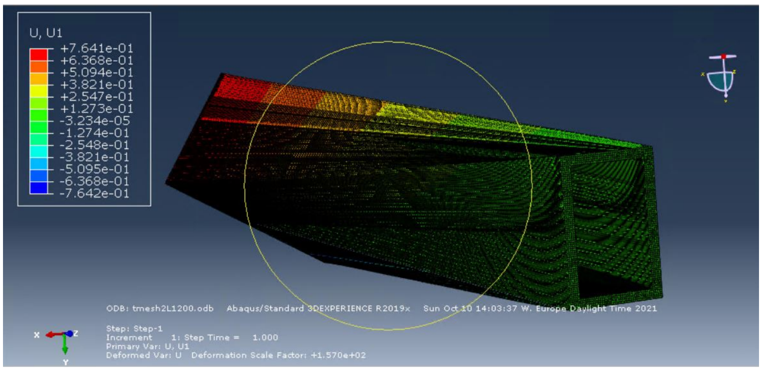

Figure 25: Maximum Displacement of beam 9

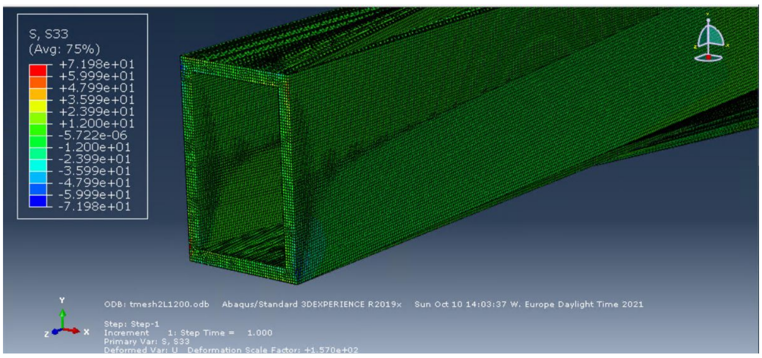

Figure 26: Maximum Stress of beam 9

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 2 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{0.7597\ [mm]}{0.7641\ [mm]} = 0.994$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{140.9137\ [N/mm^2]}{71.98\ [N/mm^2]} = 1.958$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.2.6 Beam 10

Calculations Vlasov torsion theory

In chapter 2.4 the application of the Vlasov torsion theory for the mentioned set up has been determined. These calculations have been performed in maple for the determined properties of beam 10. The performed calculations can be found in Apendix A . From these calculations we can derive the following outcomes:

$\phi = 0.02037[rad]$

$u_{max,horizontal} = 1.5276[mm]$

$\sigma_{max} = 140.9137[N/mm^2]$

Calculations Finite Element Method

The calculations performed with the Finite Element Method have been performed for different element sizes as has been described in chapter 2.6. The results from the different element sizes can be found in the table below and the visual display of the maximum displacement and the maximum stress of the smallest mesh size can be found in respectively figure 27 and 28 .

Table 16: Results of the FEM calculations for different element sizes

element size [mm]	maximum displacement [mm]	maximum stress $[N/mm^2]$
7	1.529	40.01
3	1.531	60.58

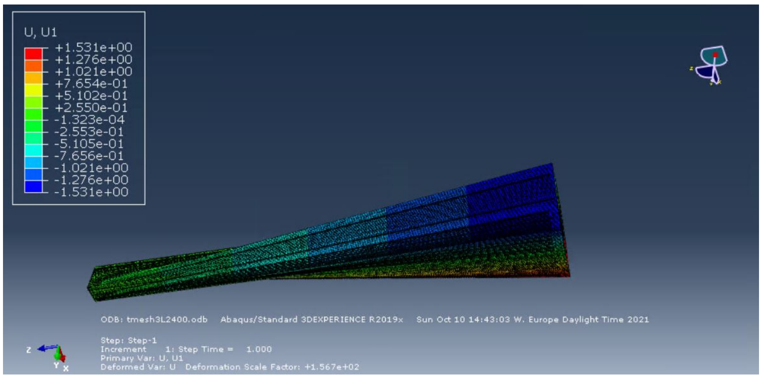

Figure 27: Maximum Displacement of beam 10

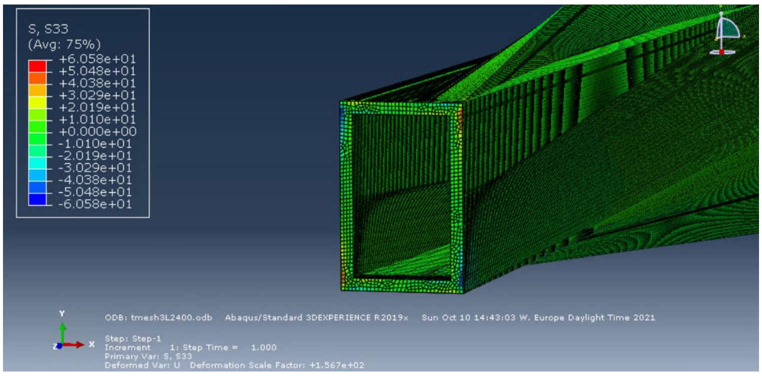

Figure 28: Maximum Stress of beam 10

Comparison of the results

The comparison will be performed for the values obtained from the smallest mesh size. The smallest mesh size that could be performed is 3 mm.

Maximum displacement check:

$$Displacement\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{1.5276\ [mm]}{1.531\ [mm]} = 0.998$$

maximum longitudinal (normal) stress check:

$$Stress\ ratio = \frac{value\ obtained\ from\ Vlasov\ theory}{value\ obtained\ from\ FEM\ software} = \frac{140.9137\ [N/mm^2]}{60.58\ [N/mm^2]} = 2.32$$

From the performed ratio checks it can be concluded that the ratio for the maximum horizontal displacement satisfies the conditions mentioned in chapter 2.6 to be valid. The ratio for the maximum longitudinal(normal) stress exceeds the maximum allowable error. Which means that the found outcomes are invalid.

3.3 Overall comparison of the results

In the table below the overall results are displayed. The first 5 beams have a solid rectangular cross-ssection and the beams 6 until 10 have a rectangular closed tube cross-section.

Table 17: Overall results of the ratio's for the smallest possible mesh size

beam	beam length [mm]	element size [mm]	ratio maximum displacement	ratio maximum stress
beam 1	150	1.8	0.977	1.23
beam 2	300	3	0.988	1.441
beam 3	600	3	0.994	1.44
beam 4	1200	3	0.997	1.44
beam 5	2400	6	0.996	1.85
beam 6	150	1	0.951	1.259
beam 7	300	2	0.978	1.7
beam 8	600	2	0.987	1.91
beam 9	1200	2	0.994	1.958
beam 10	2400	3	0.998	2.32

After performing several refining steps of the mesh size, it could be observed that the mesh size had a decisive influence on the results of the maximum stress. Multiple attempts has been done to refine the mesh size to an even smaller size, however due to an insufficient random access memory of the TU Delft work station, where the Abaqus software was installed, it was not possible to perform these refining steps. To give more understanding of the influence of the mesh size on the maximum stress, it has been chosen to plot the mesh size vs the maximum stress, this has been done for beam 1. Between these points an interpolation function has been used to give a more realistic behaviour of the function, as can be seen in figure 29 . Furthermore, the length of the beam has influence on the accuracy of the ratio's. It can be observed that the longer the beam the more inaccurate the ratio's become. The influence of the length on the accuracy of the maximum stress ratio's has been plot in figure 31 .

Regarding the ratio's for the maximum displacement, it can be observed that the mesh size does not have a significant influence on the results. Furthermore, the length of the beams does not influence the outcome of the ratio's for the maximum displacement, as can be seen in figure 31. The ratio's are consistently close to the desired value of 1 with a maximum error of 4.9 percent.

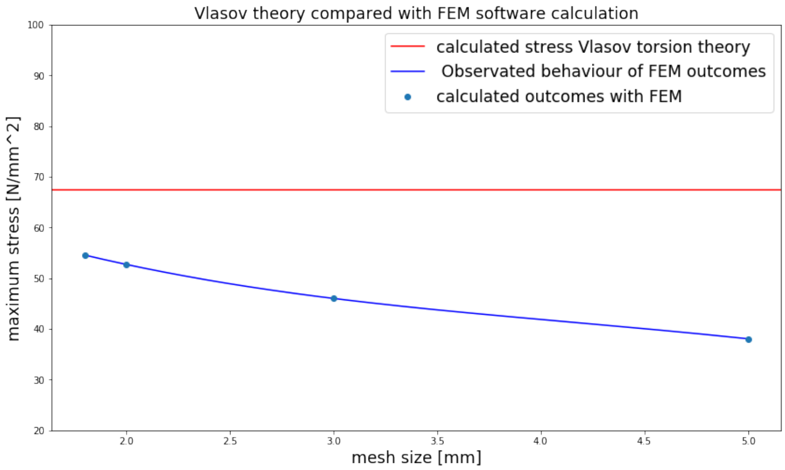

Figure 29: The Observed behaviour of the maximum stress

It can be observed that the smaller the mesh size the more the maximum stress approaches the desired value. In figure 30 the desired behaviour of the maximum stress is displayed, it should be mentioned that singularities have not been taken into account. By following the behaviour of the obtained results it is also possible that the results of the maximum normal stress will go to infinity instead of converging to the desired normal stress. This would cause an undesirable situation and can not be excluded.

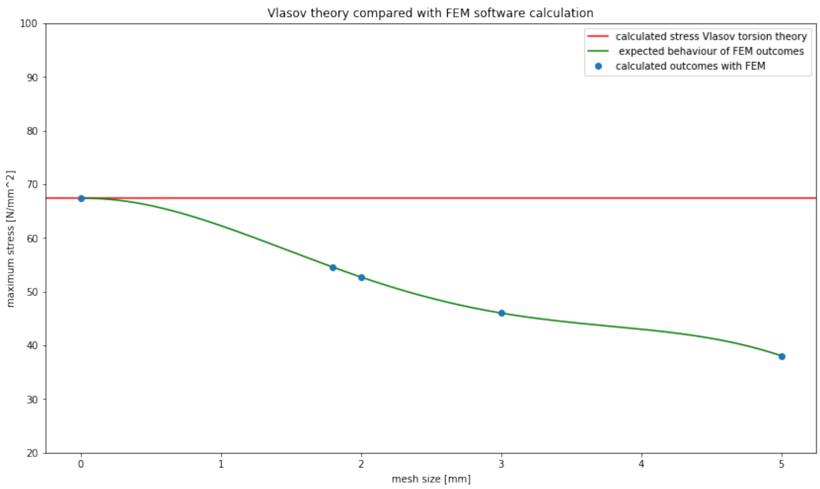

Figure 30: The expected behaviour of the maximum stress

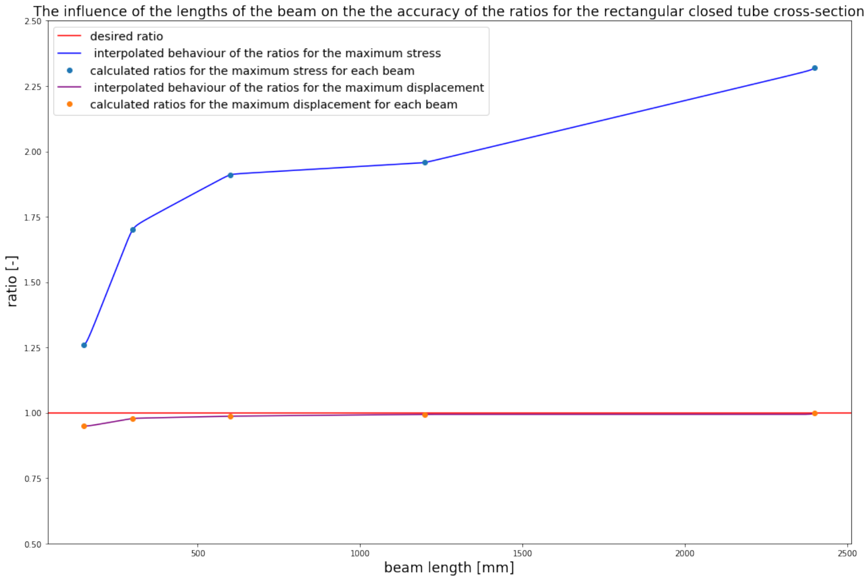

Figure 31: The influence of the lengths of the beam on the the accuracy of the ratios for the rectangular closed open cross-section

It can be observed that all the outcomes obtained for the maximum displacement by use of the Vlasov torsion theory satisfy the mentioned error conditions. To show the possible importance of the Vlasov torsion theory it has been chosen to also calculate the maximum displacements by use of the Saint-Venant displacement equations. By comparing these results it is possible to determine if the Vlasov torsion theory is the only option to calculate the maximum horizontal displacement for a one side constrained beam, loaded with a torsional moment. The performed calculations can be found in Appendix B. In the table below the results from the Vlasov torsion theory and the results from the Saint-Venant theory are displayed.

Table 18: Comparison of the Vlasov torsion theory with the Saint-Venant-theory

beam	beam length [mm]	element size [mm]	ratio maximum displacement Vlasov torsion theory	ratio maximum displacement Saint-Venant theory
beam 1	150	1.8	0.977	1.11
beam 2	300	3	0.988	1.05
beam 3	600	3	0.994	1.03
beam 4	1200	3	0.997	1.01
beam 5	2400	6	0.996	1.0076
beam 6	150	1	0.951	1.035
beam 7	300	2	0.978	1.014
beam 8	600	2	0.987	1.002
beam 9	1200	2	0.994	0.99799
beam 10	2400	3	0.998	0.99616

From the obtained results we can observe that all the ratio's obtained by use of the Saint-Venant theory also satisfy the maximum allowable error, with a minimum error of 0.201 percent and a maximum error of 11 percent. It can be noted that all the outcomes obtained for the Rectangular solid cross-section are unconservative even tough the results satisfy the error conditions. Regarding the values obtained for the rectangular closed tube cross-sections, it can be observed that the results are more accurate and at the lengths of 1200 mm and 2400 mm it can be concluded that the outcomes are conservative. From these results we can state that both theories can be used to calculate the maximum displacement for a rectangular closed tube cross-section and a rectangular solid cross-section, which is constrained at one side.

4 Discussion

In the first study done by F.Hilmer about the validity of the torsion theory for rectangular solid cross-sections and rectangular closed tube cross-sections the student version of Ansys caused the results to be not conclusive enough. This was due to the limitation of the element calculations of the software (F.Hilmer, 2021). In this report the FEM software Abaqus has been used on the workstation of the TU Delft. The software itself can perform unlimited element calculation, which means that the mesh size can be chosen as small as possible. After performing some FEM calculations in Abaqus with relative small mesh sizes, it became clear that the work station of the TU Delft does not have enough random access memory (RAM) to perform the calculations. This observation was strengthened by the fact that the bigger the length of the beams the more inaccurate the outcomes became.This can be explained due to the fact that a longer beam will consist of more elements compared to a relative smaller beam, if the same mesh size has been used. If the maximum allowable elements that can be used is exceeded the Abaqus software will abort the job. Due to this limitation it was not possible to check the validity of the Vlasov theory for smaller mesh sizes.

It is also possible that the inconsistent results for the maximum normal stresses was caused due to singularity, because it has been chosen to constrain the x-axis, y-axis and z-axis. One scenario was that due to the encastring of the beam it was not possible for the beam to expand which caused singularity of the elements and prevented the stresses to converge to the maximum normal stress. To check if this scenario has took place, it has been chosen to simulate the rectangular solid cross-section with a length of 150 mm and the constraining of only the z-axis. This FEM calculation has been done for the smallest possible mesh size. From these results we can exclude singularity of the encastred model due to the fixation of the x-axis and y-axis, because the obtained results from the simulation were almost similar to the results obtained from the fully constrained model.

5 Conclusion and recommendation

From the obtained results of the rectangular solid cross-section and the rectangular closed tube cross-section we can conclude that the maximum horizontal displacements calculated by use of the Vlasov torsion theory are almost the same as the values calculated by use of the FEM software. From the ratio checks it can be observed that all the obtained results are below 1, which means that the Vlasov displacement predictions are all conservative. From the results it is also notable that the influence of the length on the accuracy of the ratio was negligible, because the maximum error was equal to 4.9 percent. Which means that concerning the maximum displacement the Vlasov torsion theory is valid for rectangular solid cross sections and rectangular closed tube cross sections. From the results obtained from the comparison of the Vlasov torsion theory with the Saint-Venant theory it can be observed that both theories satisfy the maximum allowable error for the maximum displacement raio's.

From the obtained results of the rectangular solid cross-section and the rectangular closed tube cross-section we can conclude that the maximum normal stresses calculated by use of the Vlasov torsion theory and the values calculated by use of the FEM software have a significant difference. The minimum error is 23 percent and the maximum error is 132 percent. It is expected that this significant difference is caused due to the limitation of applying a smaller mesh size. Concluding, the errors for the maximum normal stress ratio's exceed the maximum allowable error for engineering purposes of 15 percent. Which means that concerning the maximum normal stress that the Vlasov torsion theory is not valid for rectangular solid cross sections and rectangular closed tube cross sections.

The overall conclusion is that with the obtained results it is not possible to prove the Vlasov torsion theory for rectangular solid cross-sections and for rectangular closed tube cross-sections. It is expected that the maximum normal stresses obtained by the FEM software will approach the calculated outcomes with the Vlasov torsion theory sufficiently to be valid, if a sufficient small mesh size will be used. If this can be proven it means that the Vlasov torsion theory is valid for rectangular solid cross-sections and for rectangular closed tube cross-sections.

For third parties who want to do a follow up study about the mentioned research question, it is recommended to obtain the licence of a FEM software and download it on a computer or work station with enough RAM. The workstation of the TU Delft has a RAM of 32 GB. So to refine the mesh size a computer should be used with at least 64 GB.

Furthermore, it is advised to make use of the Abaqus software, because it can be observed that the outcomes obtained by use of the Ansys software and outcomes of the Abaqus software have a significant difference. This difference in the outcomes could be due to a model error in one of the software or due to singularity of one of the models. In order to find out what caused this difference, it is advised to compare both methods of modelling. By applying the software Abaqus it is possible to do a follow up study more properly, because the instruction manuel for abaqus for the mentioned set-up is described in paragraph 2.5 .

6 References

F. Hilmer, Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?, Bachelor end report, Delft University of Technology, 21 June 2021, Online: http://homepage.tudelft.nl/p3r3s/BSc_projects/eindrapport_hilmer.pdf (September 2021)

P.C.J. Hoogenboom, "Aantekeningen over wringing", Betonvereniging, Gouda, 3 Aug. 2008 (In Dutch), Online: http://homepage.tudelft.nl/p3r3s/dictaatwringing.pdf (September 2021)

T.B. Raaphorst,De wringvervormingstheorie van Vlasov; geldigheid voor massieve en kokervormige doorsneden, Bachelor end report, Delft University of Technology, 23 June 2020, Online:http://homepage.tudelft.nl/p3r3s/BSc_projects/eindrapport_raaphorst.pdf (October 2021)

Paul A. Seaburg, PhD, PE, "Torsional Analysis of Structural Steel Members", Head, Department of Architectural Engineering ,Pennsylvania State University, oct. 2003 , Online: file:///C:/Users/Oussama/Downloads/AISC_DG_9_Torsion.pdf (October 2021)

A. Yildirim ,Checks of the Vlasov torsion theory, Bachelor end report , Delft University of Technology, 7 April 2021, Online: http://homepage.tudelft.nl/p3r3s/BSc_projects/eindrapport_yildirim.pdf (October 2021)

7 Appendix A

Rectangular solid cross-section

beam with L = 150 mm

parameters:

```
h := 150; #mm
b := 100;#mm
l := 150;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1+v));#MPa
Iw := 2.94E7;#mm^4
Cw := 3.79E9;#mm^6
ψ := 1.4E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$l := 150$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffness:

```
ECw := E·Cw;

GIw := G·Iw;
```

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Vlason Equation :

```
with(DEtools) :
```

Vlason Equation :

```
with(DEtools) :
ODE := ECw·diff(phi(x),x,x,x,x) - GIw·diff(phi(x),x,x) = mx ;
```

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{3}$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) - ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 2.393023256\ 10^{12}\ D(\phi)(150) - 7.9590000\ 10^{14}\ D^{(3)}(\phi)(150) = 1.0\ 10^{7}, D^{(2)}(\phi)(150) = 0 \tag{4}$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.00007620944719 + 4.178814383\ 10^{-6}\,x - 5.468545857\ 10^{-12}\ e^{0.05483327774\,x} + 0.00007620945263\ e^{-0.05483327774\,x}$$
$$\phi := -0.00007620944719 + 4.178814383\ 10^{-6}\,x - 5.468545857\ 10^{-12}\ e^{0.05483327774\,x} + 0.00007620945263\ e^{-0.05483327774\,x} \tag{5}$$

Bimoment and torsional moment:

```
B := -ECw·diff(phi,x,x);
Mw1 := GIw·diff(B,x);
Mw2 := diff(B,x);
Mwtot := Mw1 + Mw2;
```

$$B := 13.08635741\ e^{0.05483327774\,x} - 1.823709925\ 10^{8}\ e^{-0.05483327774\,x}$$
$$Mw1 := 1.717156602\ 10^{12}\ e^{0.05483327774\,x} + 2.393023084\ 10^{19}\ e^{-0.05483327774\,x}$$
$$Mw2 := 0.7175678705\ e^{0.05483327774\,x} + 9.999999283\ 10^{6}\ e^{-0.05483327774\,x}$$
$$Mwtot := 1.717156602\ 10^{12}\ e^{0.05483327774\,x} + 2.393023084\ 10^{19}\ e^{-0.05483327774\,x} \tag{6}$$

```
x := 0 : B; x := 'x';
#plot(B, x = 0 ..l);
```

$$-1.823709794\ 10^{8}$$
$$x := x \tag{7}$$

rotations and displacements:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.0005506127102 \tag{8}$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.04129595326 \tag{9}$$

```
x := 0 : sigma_max := -B/Cw·ψ
```

$$sigma\_max := 67.36658870 \tag{10}$$

beam with L = 300 mm

parameters:

$h := 150;$ #mm
$b := 100;$#mm
$l := 300;$#mm
$v := 0.29;$#'-'
$T := 10E6;$#Nmm
$E := 210000;$#MPa
$G := \frac{E}{2\cdot(1+v)};$#$MPa$
$Iw := 2.94E7;$#mm^4
$Cw := 3.79E9;$#mm^6
$\psi := 1.4E3;$#mm^2
$mx := 0;$ #$\frac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 300$$
$$v := 0.29$$
$$T := 1.0\ 10^7$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^7$$
$$Cw := 3.79\ 10^9$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffness:
$ECw := E\cdot Cw;$

$GIw := G\cdot Iw;$

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Vlason Equation :
$with(DEtools):$
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\phi(x) = 0 \tag{3}$$

Boundary conditions:
$bound\_con := \mathrm{phi}(0) = 0, \mathrm{D}(\mathrm{phi})(0) = 0, GIw\cdot\mathrm{D}(\mathrm{phi})(l) - ECw\cdot(\mathrm{D}@@3)(\mathrm{phi})(l) = T, (\mathrm{D}@@2)(\mathrm{phi})(l) = 0;$

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 2.393023256\ 10^{12}\ \mathrm{D}(\phi)(300) - 7.9590000\ 10^{14}\ \mathrm{D}^{(3)}(\phi)(300) = 1.0\ 10^7, \mathrm{D}^{(2)}(\phi)(300) = 0 \tag{4}$$

Solving Vlasov theory:
$Sol := evalf(dsolve(\{ODE, bound\_con\}, \{\mathrm{phi}(x)\}));$
$assign(Sol); \mathrm{phi} := \mathrm{phi}(x)$

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.924053381\ 10^{-19}\ e^{0.05483327774\ x} + 0.00007620945812\ e^{-0.05483327774\ x}$$
$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.924053381\ 10^{-19}\ e^{0.05483327774\ x} + 0.00007620945812\ e^{-0.05483327774\ x} \tag{5}$$

Bimoment and torsional moment:
$B := -ECw\cdot diff(\mathrm{phi}, x, x);$
$Mw1 := GIw\cdot diff(B, x);$
$Mw2 := diff(B, x);$
$Mwtot := Mw1 + Mw2;$

$$B := 9.390351001\ 10^{-7}\ e^{0.05483327774\ x} - 1.823710056\ 10^{8}\ e^{-0.05483327774\ x}$$
$$Mw1 := 123217.6587\ e^{0.05483327774\ x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\ x}$$
$$Mw2 := 5.149037245\ 10^{-8}\ e^{0.05483327774\ x} + 1.000000000\ 10^{7}\ e^{-0.05483327774\ x}$$
$$Mwtot := 123217.6587\ e^{0.05483327774\ x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\ x} \tag{6}$$

$x := 0 : B; x := 'x';$
#$plot(B, x = 0..l);$

$$-1.823710056\ 10^{8}$$
$$x := x \tag{7}$$

rotations and displacements:
$x := l : phi\_max := \mathrm{phi};$

$$phi\_max := 0.001177434857 \tag{8}$$

$u\_max := phi\_max\cdot 0.5\cdot h;$

$$u\_max := 0.08830761430 \tag{9}$$

$x := 0 : sigma\_max := -\frac{B}{Cw}\cdot\psi$

$$sigma\_max := 67.36659838 \tag{10}$$

beam with L = 600 mm

parameters:

```
h := 150; #mm
b := 100;#mm
l := 600;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + v));#MPa
Iw := 2.94E7;#mm^4
Cw := 3.79E9;#mm^6
ψ := 1.4E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$l := 600$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffness:

```
ECw := E·Cw;

GIw := G·Iw;
```

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Vlason Equation :

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) - GIw·diff(phi(x), x, x) = mx ;
```

4 2

$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Vlason Equation :

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) - GIw·diff(phi(x), x, x) = mx ;
```


$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{3}$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) - ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 2.393023256\ 10^{12}\ D(\phi)(600) - 7.9590000\ 10^{14}\ D^{(3)}(\phi)(600) = 1.0\ 10^{7}, D^{(2)}(\phi)(600) = 0 \tag{4}$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 2.020509704\ 10^{-33}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x}$$
$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 2.020509704\ 10^{-33}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x} \tag{5}$$

Bimoment and torsional moment:

```
B := -ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$B := 4.835126712\ 10^{-21}\ e^{0.05483327774\,x} - 1.823710056\ 10^{8}\ e^{-0.05483327774\,x}$$
$$Mw1 := 6.344523150\ 10^{-10}\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x}$$
$$Mw2 := 2.651258459\ 10^{-22}\ e^{0.05483327774\,x} + 1.000000000\ 10^{7}\ e^{-0.05483327774\,x}$$
$$Mwtot := 6.344523150\ 10^{-10}\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x} \tag{6}$$

```
x := 0 : B; x :='x';
#plot(B, x = 0 ..l);
```

$$-1.823710056\ 10^{8}$$
$$x := x \tag{7}$$

rotations and displacements:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.002431079172 \tag{8}$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.1823309379 \tag{9}$$

```
x := 0 : sigma_max := -B/Cw·ψ
```

$$sigma\_max := 67.36659838 \tag{10}$$

beam with L = 1200 mm

parameters:

$h := 150;\#mm$
$b := 100;\#mm$
$l := 1200;\#mm$
$v := 0.29;\#'-'$
$T := 10E6;\#Nmm$
$E := 210000;\#MPa$
$G := \frac{E}{2\cdot(1+v)};\#MPa$
$Iw := 2.94E7;\#mm^4$
$Cw := 3.79E9;\#mm^6$
$\psi := 1.4E3;\#mm^2$
$mx := 0;\#\frac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 1200$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffness:
$ECw := E\cdot Cw;$

$GIw := G\cdot Iw;$

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Vlason Equation :
$with(DEtools):$
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 7.9590000\ 10^{14} \frac{d^4}{dx^4}\phi(x) - 2.393023256\ 10^{12}\frac{d^2}{dx^2}\phi(x) = 0$$

Boundary conditions:
$bound\_con := \mathrm{phi}(0) = 0, \mathrm{D}(\mathrm{phi})(0) = 0, GIw\cdot \mathrm{D}(\mathrm{phi})(l) - ECw\cdot(\mathrm{D}@@3)(\mathrm{phi})(l) = T, (\mathrm{D}@@2)(\mathrm{phi})(l) = 0;$

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 2.393023256\ 10^{12}\,\mathrm{D}(\phi)(1200) - 7.9590000\ 10^{14}\,\mathrm{D}^{(3)}(\phi)(1200) = 1.0\ 10^{7}, \mathrm{D}^{(2)}(\phi)(1200) = 0$$

Solving Vlasov theory:
$Sol := evalf(dsolve(\{ODE, bound\_con\}, \{\mathrm{phi}(x)\}));$
$assign(Sol); \mathrm{phi} := \mathrm{phi}(x)$

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\,x - 5.356893223\ 10^{-62}\,e^{0.05483327774\,x} + 0.00007620945812\,e^{-0.05483327774\,x}$$
$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\,x - 5.356893223\ 10^{-62}\,e^{0.05483327774\,x} + 0.00007620945812\,e^{-0.05483327774\,x}$$

Bimoment and torsional moment:
$B := -ECw\cdot diff(\mathrm{phi}, x, x);$
$Mw1 := GIw\cdot diff(B, x);$
$Mw2 := diff(B, x);$
$Mwtot := Mw1 + Mw2;$

$$B := 1.281917006\ 10^{-49}\,e^{0.05483327774\,x} - 1.823710056\ 10^{8}\,e^{-0.05483327774\,x}$$
$$Mw1 := 1.682096997\ 10^{-38}\,e^{0.05483327774\,x} + 2.393023256\ 10^{19}\,e^{-0.05483327774\,x}$$
$$Mw2 := 7.029171123\ 10^{-51}\,e^{0.05483327774\,x} + 1.000000000\ 10^{7}\,e^{-0.05483327774\,x}$$
$$Mwtot := 1.682096997\ 10^{-38}\,e^{0.05483327774\,x} + 2.393023256\ 10^{19}\,e^{-0.05483327774\,x}$$

$x := 0 : B; x := 'x';$
$\#plot(B, x = 0 .. l);$

$$-1.823710056\ 10^{8}$$
$$x := x$$

rotations and displacements:
$x := l : phi\_max := \mathrm{phi};$

$$phi\_max := 0.004938367802$$

$u\_max := phi\_max\cdot 0.5\cdot h;$

$$u\_max := 0.3703775852$$

$x := 0 : sigma\_max := -\frac{B}{Cw}\cdot\psi$

$$sigma\_max := 67.36659838$$ (

beam with L = 2400 mm

```
h := 150; #mm
b := 100;#mm
l := 2400;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + v));#MPa
Iw := 2.94E7;#mm^4
Cw := 3.79E9;#mm^6
ψ := 1.4E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$l := 2400$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \qquad (1)$$

Stiffness:

```
ECw := E·Cw;

GIw := G·Iw;
```

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \qquad (2)$$

Vlason Equation :

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) - GIw·diff(phi(x), x, x) = mx ;
```

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \qquad (3)$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) - ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 2.393023256\ 10^{12}\ D(\phi)(2400) - 7.9590000\ 10^{14}\ D^{(3)}(\phi)(2400) = 1.0\ 10^{7}, D^{(2)}(\phi)(2400) = 0 \qquad (4)$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.765451918\ 10^{-119}\ e^{0.05483327774\ x} + 0.00007620945808\ e^{-0.05483327774\ x}$$
$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.765451918\ 10^{-119}\ e^{0.05483327774\ x} + 0.00007620945808\ e^{-0.05483327774\ x} \qquad (5)$$

Bimoment and torsional moment:

```
B := -ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$B := 9.010814004\ 10^{-107}\ e^{0.05483327774\ x} - 1.823710055\ 10^{8}\ e^{-0.05483327774\ x}$$
$$Mw1 := 1.182374764\ 10^{-95}\ e^{0.05483327774\ x} + 2.393023255\ 10^{19}\ e^{-0.05483327774\ x}$$
$$Mw2 := 4.940924669\ 10^{-108}\ e^{0.05483327774\ x} + 9.999999996\ 10^{6}\ e^{-0.05483327774\ x}$$
$$Mwtot := 1.182374764\ 10^{-95}\ e^{0.05483327774\ x} + 2.393023255\ 10^{19}\ e^{-0.05483327774\ x} \qquad (6)$$

```
x := 0 : B; x :='x';
#plot(B, x = 0 ..l);
```

$$-1.823710055\ 10^{8}$$
$$x := x \qquad (7)$$

rotations and displacements:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.009952945062 \qquad (8)$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.7464708795 \qquad (9)$$

```
x := 0 : sigma_max := -B/Cw·ψ
```

$$sigma\_max := 67.36659833 \qquad (10)$$

Rectangular closed tube Cross-sections

beam with L = 150 mm

parameters:

```
h := 150; #mm
b := 100;#mm
t := 10;#mm
l := 150;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + v));#MPa
Iw := 1.44E7;#mm^4
Cw := 9.023E8;#mm^6
ψ := 1E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 150$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffness:

```
ECw := E·Cw;
GIw := G·Iw;
```

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:

$$lc := \mathrm{sqrt}\left(\frac{ECw}{GIw}\right);$$

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) − GIw·diff(phi(x), x, x) = mx;
```

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{4}$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) − ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 1.172093023\ 10^{12}\ \mathrm{D}(\phi)(150) - 1.894830000\ 10^{14}\ \mathrm{D}^{(3)}(\phi)(150) = 1.0\ 10^{7}, \mathrm{D}^{(2)}(\phi)(150) = 0 \tag{5}$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 6.140872277\ 10^{-15}\ e^{0.07864949064\ x} + 0.0001084780838\ e^{-0.07864949064\ x}$$
$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 6.140872277\ 10^{-15}\ e^{0.07864949064\ x} + 0.0001084780838\ e^{-0.07864949064\ x} \tag{6}$$

Bimoment and torsional moment:

```
B := −ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$B := 0.007197673551\ e^{0.07864949064\ x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\ x}$$
$$Mw1 := 6.635140760\ 10^{8}\ e^{0.07864949064\ x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\ x}$$
$$Mw2 := 0.0005660933586\ e^{0.07864949064\ x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\ x}$$
$$Mwtot := 6.635140760\ 10^{8}\ e^{0.07864949064\ x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\ x} \tag{7}$$

Rotation, dispacement and stress:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.001171283821 \tag{8}$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.08784628660 \tag{9}$$

$$x := 0 : sigma\_max := -\frac{B}{Cw}\cdot\psi;$$

$$sigma\_max := 140.9136708 \tag{10}$$

beam with L = 300 mm

parameters:

```
h := 150; #mm
b := 100;#mm
t := 10;#mm
l := 300;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + v));#MPa
Iw := 1.44E7;#mm^4
Cw := 9.023E8;#mm^6
ψ := 1E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 300$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \qquad (1)$$

Stiffness:

```
ECw := E·Cw;
GIw := G·Iw;
```

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \qquad (2)$$

Characteristic length:

```
lc := sqrt(ECw/GIw);
```

$$lc := 12.71464051 \qquad (3)$$

Vlasov equation:

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) − GIw·diff(phi(x), x, x) = mx;
```

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \qquad (4)$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) − ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 1.172093023\ 10^{12}\ \mathrm{D}(\phi)(300) - 1.894830000\ 10^{14}\ \mathrm{D}^{(3)}(\phi)(300) = 1.0\ 10^{7}, \mathrm{D}^{(2)}(\phi)(300) = 0 \qquad (5)$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\,x - 3.476307080\ 10^{-25}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x}$$
$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\,x - 3.476307080\ 10^{-25}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x} \qquad (6)$$

Bimoment and torsional moment:

```
B := −ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$B := 4.074555276\ 10^{-13}\ e^{0.07864949064\,x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\,x}$$
$$Mw1 := 0.03756109192\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x}$$
$$Mw2 := 3.204616970\ 10^{-14}\ e^{0.07864949064\,x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\,x}$$
$$Mwtot := 0.03756109192\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x} \qquad (7)$$

Rotation, dispacement and stress:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.002451045726 \qquad (8)$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.1838284294 \qquad (9)$$

```
x := 0 : sigma_max := −(B/Cw)·ψ;
```

$$sigma\_max := 140.9136708 \qquad (10)$$

beam with L = 600 mm

parameters:

```
h := 150; #mm
b := 100;#mm
t := 10;#mm
l := 600;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + v));#MPa
Iw := 1.44E7;#mm^4
Cw := 9.023E8;#mm^6
ψ := 1E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 600$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffness:

```
ECw := E·Cw;
GIw := G·Iw;
```

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:

```
lc := sqrt(ECw/GIw);
```

Vlasov equation:

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) - GIw·diff(phi(x), x, x) = mx;
```

$$ODE := 1.894830000\ 10^{14} \frac{d^4}{dx^4}\phi(x) - 1.172093023\ 10^{12} \frac{d^2}{dx^2}\phi(x) = 0 \tag{4}$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) - ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12}\, D(\phi)(600) - 1.894830000\ 10^{14}\, D^{(3)}(\phi)(600) = 1.0\ 10^{7}, D^{(2)}(\phi)(600) = 0 \tag{5}$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6} x - 1.114023266\ 10^{-45} e^{0.07864949064 x} + 0.0001084780838\, e^{-0.07864949064 x}$$
$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6} x - 1.114023266\ 10^{-45} e^{0.07864949064 x} + 0.0001084780838\, e^{-0.07864949064 x} \tag{6}$$

Bimoment and torsional moment:

```
B := -ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$B := 1.305738897\ 10^{-33} e^{0.07864949064 x} - 1.271464052\ 10^{8} e^{-0.07864949064 x}$$
$$Mw1 := 1.203689125\ 10^{-22} e^{0.07864949064 x} + 1.172093024\ 10^{19} e^{-0.07864949064 x}$$
$$Mw2 := 1.026956992\ 10^{-34} e^{0.07864949064 x} + 1.000000001\ 10^{7} e^{-0.07864949064 x}$$
$$Mwtot := 1.203689125\ 10^{-22} e^{0.07864949064 x} + 1.172093024\ 10^{19} e^{-0.07864949064 x} \tag{7}$$

Rotation, dispacement and stress:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.005010569536 \tag{8}$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.3757927152 \tag{9}$$

```
x := 0 : sigma_max := -B/Cw·ψ;
```

$$sigma\_max := 140.9136708 \tag{10}$$

beam with L = 1200 mm

parameters:

```
h := 150; #mm
b := 100;#mm
t := 10;#mm
l := 1200;#mm
v := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + v));#MPa
Iw := 1.44E7;#mm^4
Cw := 9.023E8;#mm^6
ψ := 1E3;#mm^2
mx := 0; #Nm/m
```

$$
\begin{gathered}
h := 150 \\
b := 100 \\
t := 10 \\
l := 1200 \\
v := 0.29 \\
T := 1.0\ 10^{7} \\
E := 210000 \\
G := 81395.34883 \\
Iw := 1.44\ 10^{7} \\
Cw := 9.023\ 10^{8} \\
\psi := 1000. \\
mx := 0
\end{gathered} \tag{1}
$$

Stiffness:

```
ECw := E·Cw;
GIw := G·Iw;
```

$$
\begin{gathered}
ECw := 1.894830000\ 10^{14} \\
GIw := 1.172093023\ 10^{12}
\end{gathered} \tag{2}
$$

Characteristic length:

```
lc := sqrt(ECw/GIw);
```

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) − GIw·diff(phi(x), x, x) = mx;
```

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{4}$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) − ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 1.172093023\ 10^{12}\ \mathrm{D}(\phi)(1200) - 1.894830000\ 10^{14}\ \mathrm{D}^{(3)}(\phi)(1200) = 1.0\ 10^{7}, \mathrm{D}^{(2)}(\phi)(1200) = 0 \tag{5}$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$
\begin{gathered}
Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.144053959\ 10^{-86}\ e^{0.07864949064\ x} + 0.0001084780838\ e^{-0.07864949064\ x} \\
\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.144053959\ 10^{-86}\ e^{0.07864949064\ x} + 0.0001084780838\ e^{-0.07864949064\ x}
\end{gathered} \tag{6}
$$

Bimoment and torsional moment:

```
B := −ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$
\begin{gathered}
B := 1.340937663\ 10^{-74}\ e^{0.07864949064\ x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\ x} \\
Mw1 := 1.236136938\ 10^{-63}\ e^{0.07864949064\ x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\ x} \\
Mw2 := 1.054640642\ 10^{-75}\ e^{0.07864949064\ x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\ x} \\
Mwtot := 1.236136938\ 10^{-63}\ e^{0.07864949064\ x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\ x}
\end{gathered} \tag{7}
$$

Rotation, dispacement and stress:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.01012961716 \tag{8}$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 0.7597212870 \tag{9}$$

```
x := 0 : sigma_max := −B/Cw·ψ;
```

$$sigma\_max := 140.9136708 \tag{10}$$

beam with L = 2400 mm

parameters:

```
h := 150; #mm
b := 100;#mm
t := 10;#mm
l := 2400;#mm
ν := 0.29;#'-'
T := 10E6;#Nmm
E := 210000;#MPa
G := E/(2·(1 + ν));#MPa
Iw := 1.44E7;#mm^4
Cw := 9.023E8;#mm^6
ψ := 1E3;#mm^2
mx := 0; #Nm/m
```

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 2400$$
$$\nu := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \qquad (1)$$

Stiffness:

```
ECw := E·Cw;
GIw := G·Iw;
```

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \qquad (2)$$

Characteristic length:

```
lc := sqrt(ECw/GIw);
```

$$lc := 12.71464051 \qquad (3)$$

Vlasov equation:

Vlasov equation:

```
with(DEtools) :
ODE := ECw·diff(phi(x), x, x, x, x) − GIw·diff(phi(x), x, x) = mx;
```

$$ODE := 1.894830000\ 10^{14} \frac{d^4}{dx^4} \phi(x) - 1.172093023\ 10^{12} \frac{d^2}{dx^2} \phi(x) = 0 \qquad (4)$$

Boundary conditions:

```
bound_con := phi(0) = 0, D(phi)(0) = 0, GIw·D(phi)(l) − ECw·(D@@3)(phi)(l) = T, (D@@2)(phi)(l) = 0;
```

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12} D(\phi)(2400) - 1.894830000\ 10^{14} D^{(3)}(\phi)(2400) = 1.0\ 10^{7}, D^{(2)}(\phi)(2400) = 0 \qquad (5)$$

Solving Vlasov theory:

```
Sol := evalf(dsolve({ODE, bound_con}, {phi(x)}));
assign(Sol); phi := phi(x)
```

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\,x - 1.206565900\ 10^{-168}\, e^{0.07864949064\,x} + 0.0001084780838\, e^{-0.07864949064\,x}$$
$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\,x - 1.206565900\ 10^{-168}\, e^{0.07864949064\,x} + 0.0001084780838\, e^{-0.07864949064\,x} \qquad (6)$$

Bimoment and torsional moment:

```
B := −ECw·diff(phi, x, x);
Mw1 := GIw·diff(B, x);
Mw2 := diff(B, x);
Mwtot := Mw1 + Mw2;
```

$$B := 1.414207473\ 10^{-156}\, e^{0.07864949064\,x} - 1.271464052\ 10^{8}\, e^{-0.07864949064\,x}$$
$$Mw1 := 1.303680360\ 10^{-145}\, e^{0.07864949064\,x} + 1.172093024\ 10^{19}\, e^{-0.07864949064\,x}$$
$$Mw2 := 1.112266974\ 10^{-157}\, e^{0.07864949064\,x} + 1.000000001\ 10^{7}\, e^{-0.07864949064\,x}$$
$$Mwtot := 1.303680360\ 10^{-145}\, e^{0.07864949064\,x} + 1.172093024\ 10^{19}\, e^{-0.07864949064\,x} \qquad (7)$$

Rotation, dispacement and stress:

```
x := l : phi_max := phi;
```

$$phi\_max := 0.02036771240 \qquad (8)$$

```
u_max := phi_max·0.5·h;
```

$$u\_max := 1.527578430 \qquad (9)$$

```
x := 0 : sigma_max := −(B/Cw)·ψ;
```

$$sigma\_max := 140.9136708 \qquad (10)$$

8 Appendix B

Rectangular solid cross-section

Iw =	29400000	[mm^4]	$M_w = GI_w \frac{\varphi}{l}.$	$\frac{M_w}{GI_w} * l = \varphi$	
G =	81395.34883	[N/mm^2]			
Mw =	10000000	[Nmm]			
h =	150	[mm]			
	l [mm]	phi	umaxhor [mm] (Saint-Venant)	umaxhor [mm] (FEM)	ratio
beam 1	150	0.000627	0.047011662	0.04228	1.111913
beam 2	300	0.001254	0.094023324	0.08935	1.052304
beam 3	600	0.002507	0.188046647	0.1834	1.025336
beam 4	1200	0.005015	0.376093294	0.3717	1.011819
beam 5	2400	0.010029	0.752186589	0.7465	1.007618

Rectangular closed tube cross-section

Iw =	14500000	[mm^4]	$M_w = GI_w \frac{\varphi}{l}.$	$\frac{M_w}{GI_w} * l = \varphi$	
G =	81395.34883	[N/mm^2]			
Mw =	10000000	[Nmm]			
h =	150	[mm]			
	l [mm]	phi	umaxhor [mm] (Saint-Venant)	umaxhor [mm] (FEM)	ratio
beam 6	150	0.001271	0.095320197	0.09202	1.035864
beam 7	300	0.002542	0.190640394	0.188	1.014045
beam 8	600	0.005084	0.381280788	0.3805	1.002052
beam 9	1200	0.010167	0.762561576	0.7641	0.997987
beam 10	2400	0.020335	1.525123153	1.531	0.996161