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Bachelor Thesis

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Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?

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Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?

by

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Preface

Before you lies the Bachelor Thesis "Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?", I carried out for the completion of the Bachelor of Science in Civil Engineering at the Faculty of Civil Engineering and Geosciences of the Delft University of Technology. I was engaged in researching and writing this Bachelor Thesis from April to June 2021.

This Bachelor Thesis is about checking the validity of the theory of Vlasov for different types of cross-sections and setting its boundaries and is meant for students and researchers interested in the application of the torsion theory of Vlasov and the possible limitations of the theory.

I would like to thank my supervisors and in particular Dr. ir. P.C.J. Hoogenboom for the weekly meetings and the helpful feedback during this process, especially during these strange times.

Floris Hilmer
Delft, June 2021

Abstract

There are a few techniques to describe the torsional behaviour of beams. One of these techniques is the torsion theory of Vlasov. This theory describes the torsional behaviour of a beam with a fixed end at one side which constrains warping[1]. The theory of Vlasov is proven to be valid for thin-walled open cross-sections (Hoogenboom, 2019), however, it is uncertain if the theory can be applied for other types of cross-sections. This study looks at the validity of the theory of Vlasov and its limitations for different types of cross-sections and lengths and will formulate an answer to the research question: "Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?".

To answer the research question a specific method and setup is used for the different types of cross-sections and beams. This study looks at a beam fixed at one side and a torsional moment at the free end. For each of the cross-sections five beams with different lengths are tested, i.e. beams 1 - 5 for rectangular solid cross-sections and beams 6 - 10 for rectangular closed tube cross-sections. The first beams of each cross-section, i.e. beams 1 and 6, have a length of 150 mm, which matches the height of the beam, and the following beams have twice the length of the previous beam. First of all, the cross-sectional parameters, i.e. warping constant, torsional moment of inertia and warping function, are calculated with a cross-section design program called ShapeBuilder. These parameters are used as input for the differential equation of Vlasov. The differential equation of Vlasov, in combination with specific boundary conditions, returns a value for the angular deviation which results in the maximum horizontal deformation of the beam. The maximum normal stress in the beam can be calculated with the use of the differential equation of Vlasov as well. The results obtained with the theory of Vlasov are compared to the results of a Finite Element Method program called ANSYS Workbench in terms of a ratio for the maximum horizontal deformation *Ratio* u_{max} and a ratio for the maximum normal stress *Ratio* σ_{max}. A maximum error of 15% is allowed for engineering purposes.

The study found that the *Ratio* u_{max} for rectangular solid cross-sections is the most accurate for the longest beam, i.e. beam 5, and is equal to 0.99. This is an error of 1%. For beams 2 - 4 ratios of respectively 0.90, 0.95 and 0.97 are found, which means that these beams are within the error margin of 15%. For beam 1 a *Ratio* u_{max} of 0.82 is found. The *Ratio* σ_{max} of beam 1 is equal to 0.99 which means an error of 1%. Beams 2 - 5 are less accurate, respectively ratios of 1.07, 1.10, 1.18 and 1.25. A similar result is found for beams 6 - 10, which have rectangular closed tube cross-sections. The ratios for the maximum horizontal displacement are respectively 0.83, 0.94, 0.97, 0.99 and 1.04 and the ratios for the maximum normal stress are respectively 1.04, 1.29, 1.34, 1.67 and 1.87.

The theory of Vlasov probably gives an accurate result for the normal stresses due to constraint warping for both types of cross-sections and the different lengths. Due to the educational license of ANSYS Workbench the number of elements for the mesh of the longer beams, e.g. beams 2 - 5 and beams 7 - 10, is limited. Furthermore, it can be observed that the theory of Vlasov is valid for both types of cross-sections for the horizontal deformation, except for extreme short beams where the length of the beams matches the height of the cross-sections.

[1] Warping is defined as the deformation of an initial flat cross-section and can occur due to a torsional moment or an eccentric force (Hoogenboom, 2019).

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1 | Introduction

1.1 Background information

Besides extension, bending and shearing, beams can experience torsion as well as a result of a torsional moment or an eccentric force, which can lead to warping[1] of a beam. Over the past years several techniques are developed to describe the torsional behaviour of beams. The first technique is a theory found by Aldémar Barré de Saint-Venant, also called *circulatory torsion* or *uniform torsion*. This theory does not take the constraint of warping at the fixed support into account, therefore, it is only valid from a certain distance of the fixed support, see Figure 1.1. This distance is called the characteristic length (l_c in [mm]) and is defined as:

$$l_c = \sqrt{\frac{EC_w}{GI_w}} \tag{1.1}$$

Where:
EC_w [Nmm^4] = warping stiffness
GI_w [Nmm^2] = torsional stiffness

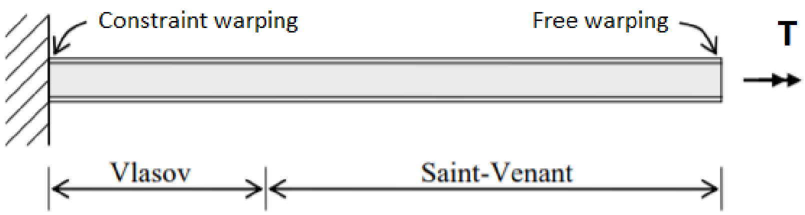


Figure 1.1: Influence of constraint warping, From: Hoogenboom, 2019

1.2 Theory of Vlasov

Because of this difficulty, Vasiliy Vlasov developed a new theory that does take the constraint of warping at the fixed support into account. This theory is referred as *warping torsion* or *non-uniform torsion*. The warping along the longitudinal axis is not constant in this theory and the angular deviation φ of the cross-section follows from the differential equation of Vlasov:

$$EC_w \frac{\mathrm{d}^4\varphi}{\mathrm{d}x^4} - GI_w \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = m_x \tag{1.2}$$

Where:
E [MPa] = Young's modulus
C_w [mm] = warping constant
φ [rad] = angular deviation
G [MPa] = shear modulus
I_w [mm^4] = torsional moment of inertia
m_x [Nmm/mm] = distributed torsional moment along the beam

[1]Warping is defined as the deformation of an initial flat cross-section and can occur due to a torsional moment or an eccentric force (Hoogenboom, 2019).

1.3 Problem definition and research question

The theory of Vlasov is valid for thin-walled open cross-sections, however, it is very uncertain if the theory applies for other types of cross-sections as well. A study performed by T.B. Raaphorst (Raaphorst, 2020) researched the validity of the Vlasov theory for different types of solid and tube cross-sections. One of the conclusions of this study is that the Vlasov theory is accurate for stresses in closed tube cross-sections. An additional study, performed by F. Yildirim (Yildirim, 2021), found inconsistent results. The study states that the Vlasov theory is not accurate for stresses in closed tube cross-sections. The difference with the performed Finite Element Method is a factor 10. Most likely a calculation error is made, but not found by the author. This contradiction between both studies needs additional research to check the validity of the Vlasov theory for different types of cross-sections. This report will look deeper in this subject and will formulate an answer to the research question:

Is the Vlasov torsion theory valid for rectangular solid cross-sections and for rectangular closed tube cross-sections?

2 | Method

The validity of the theory of Vlasov is checked with the use of different computer programs. Different types of cross-sections are analysed with the method described in this chapter. Furthermore, beams with different lengths are analysed to check whether this will give the same results or not.

2.1 Setup of the beam

The different types of cross-sections have the same setup for each of the beams. This study analyses a beam with a fixed end at one side which constrains the warping. Due to a torsional moment which is attached at the other end of the beam, torsion occurs in the beam, see Figure 2.1.

Figure 2.1: Setup of the beam, From: Hoogenboom, 2019

The material that the beam is made out is structural steel. Structural steel has typical properties which are used for all cross-sections that are analysed in this study. Some standard parameters that are used for all cross-sections and the typical properties of structural steel are listed in Table 2.1. The shear modulus G is calculated with Equation (2.1).

$$G = \frac{E}{2(1+\nu)} = \frac{210000}{2(1+0.29)} = 81395\,MPa \tag{2.1}$$

Table 2.1: Standard parameters and properties of structural steel

Symbol	Property	Value	Unit
T	Torsional moment	10E6	Nmm
E	Young's modulus	2.1E5	MPa
ν	Poisson's ratio	0.29	-
G	Shear modulus	8.1E4	MPa

2.2 Determination of the cross-sectional parameters

First of all the cross-sectional parameters of each beam are calculated with the use of a cross-section design program called ShapeBuilder. The dimensions of the cross-sections are chosen arbitrary, as the theory of Vlasov has to be valid for all dimensions. Subsequently the cross-section is modelled in ShapeBuilder with the parameters and properties mentioned in Table 2.1. The mesh refinement is chosen as fine as possible to get the most exact results. The program returns a set of cross-sectional parameters and displays the results in an analysis view. The cross-sectional parameters that are of use for the application of the theory of Vlasov are: the warping constant C_w, the torsional moment of inertia I_w and the warping function ψ. The warping constant shows whether warping could occur in a cross-section or not. If the warping constant equals zero, warping does not occur in the cross-section, for example in circular cross-sections (Hoogenboom, 2019). The warping function shows the distribution of the warping over the cross-section. The torsional moment of inertia describes the torsional stiffness of a beam. The results of the warping function can be displayed in the analysis view and are of use to compare with the results of the Finite Element Method.

2.3 Application of the theory of Vlasov

The cross-sectional parameters obtained from ShapeBuilder are used to check the validity of the theory of Vlasov. The angular deviation, and consequently the horizontal deformation, and the normal stress are calculated with the theory of Vlasov using Maple. To start with the warping stiffness and the torsional stiffness are determined. Now the different cross-sectional parameters and properties of the structural steel are put in the differential equation of Vlasov, described by Equation (1.2). This differential equation is solved with specified boundary conditions. For a one side fixed beam, known as a cantilever, with a torsional moment at the free end specific boundary conditions are used (Hoogenboom, 2019).

$$\varphi(0) = 0$$

$$\frac{d\varphi}{dx}(0) = 0$$

$$GI_w \cdot \frac{d\varphi}{dx}(l) - EC_w \cdot \frac{d^3\varphi}{dx^3}(l) = T$$

$$\frac{d^2\varphi}{dx^2}(l) = 0$$

The differential equation of Vlasov, Equation (1.2), in combination with the boundary conditions mentioned above, results in a value for the angular deflection φ as a function of the longitudinal axis of the beam. Multiplying the angular deflection with 0.5h gives the maximum horizontal deformation of the beam, see Equation (2.2).

$$u_{max} = \varphi \cdot 0.5h \tag{2.2}$$

Subsequently the bimoment B [Nmm^2] as a function of the longitudinal axis is calculated. A bimoment shows the distributions at the cross-section of warping stress in cases of torsional warping (Maisel, Roll, Cement, & Association, 1974) and is calculated with Equation (2.3).

$$B = -EC_w \frac{d^2\varphi}{dx^2} \tag{2.3}$$

The maximum bimoment is used to find the maximum normal stress in the beam, which is located at the fixed end. The maximum normal stress is calculated with Equation (2.4).

$$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi \tag{2.4}$$

Where:
σ_{max} [MPa] = maximum normal stress
B_{max} [Nmm^2] = maximum bimoment
ψ [mm^2] = warping function

2.4 Calculation with the Finite Element Method

The beam is modelled with a Finite Element Method after the calculations with the theory of Vlasov in ShapeBuilder and Maple. The program used for the Finite Element Method is ANSYS Workbench. First of all, as in ShapeBuilder, the parameters and properties as mentioned in Table 2.1 are specified in the program. Subsequently the beam is modelled with the specified dimensions, fixed support and torsional moment. The element size of the mesh is chosen as fine as possible to get the most accurate result and is mentioned for each beam. With this information ANSYS Workbench is able to analyse the model and calculate the horizontal deformation and the normal stress distribution.

2.5 Comparing of the results

In this last stage of the study the results of the theory of Vlasov are compared to the calculations with the Finite Element Method. The horizontal deformations and normal stresses are compared and expressed with a ratio, see Equation (2.5). If the ratio is close to one, an error of 15% or 0.15 is allowed for engineering purposes, it means that the theory of Vlasov is valid for that type of cross-section and beam.

$$Ratio = \frac{Theory\,of\,Vlasov}{Finite\,Element\,Method} \tag{2.5}$$

3 | Rectangular solid cross-sections

The validity of the theory of Vlasov is checked for rectangular solid cross-sections in this chapter. The dimensions of the considered cross-section are indicated in Figure 3.1a. The validity of the theory of Vlasov is researched for different lengths. The length of the first beam is chosen arbitrary, the lengths of the following beams are twice the length of the previous beam. The method described in Chapter 2 is applied in the following subsections.

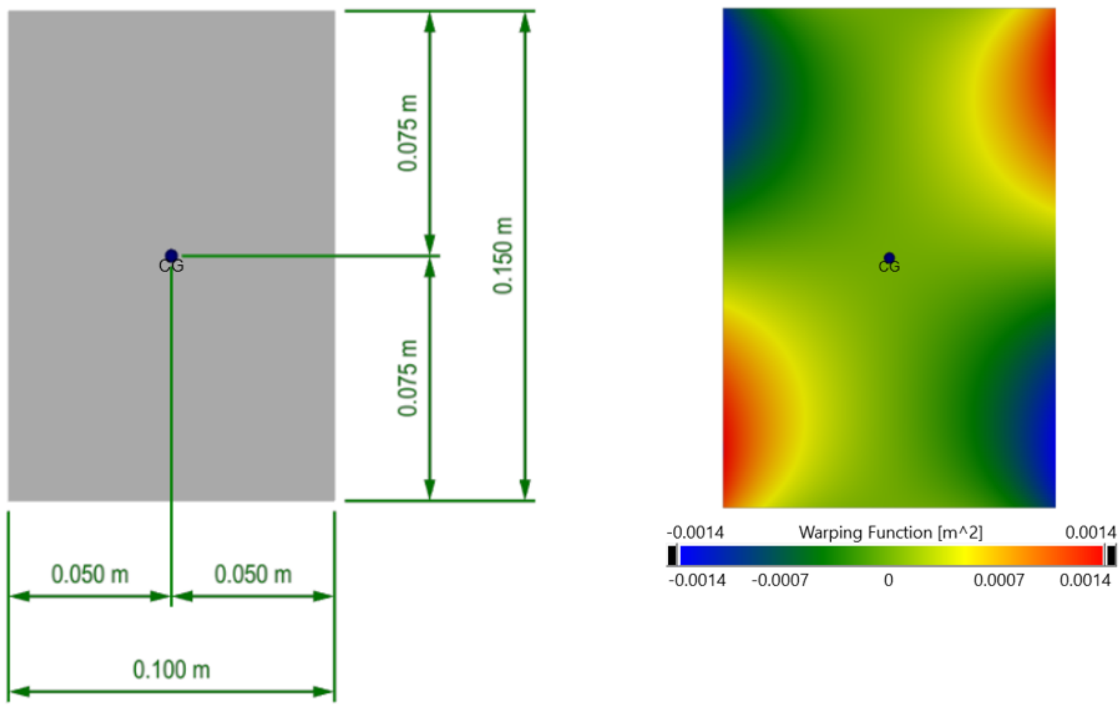


Figure 3.1: (a) Rectangular solid cross-section; (b) Warping function rectangular solid cross-section

3.1 Determination of the cross-sectional parameters

The rectangular solid cross-section with dimensions shown in Figure 3.1a used for beams 1 - 5 is modelled in ShapeBuilder, with a mesh refinement as fine as possible. The input values are defined in Table 2.1 and are inserted in ShapeBuilder. The program returns values for the warping constant, torsional moment of inertia and the warping function. The cross-sectional parameters for the rectangular solid cross-section that are discussed in this part of the study are listed in Table 3.1 and are used as input for the theory of Vlasov. The warping function is visualised in Figure 3.1b.

Table 3.1: Cross-sectional parameters rectangular solid cross-section

Symbol	Property	Value	Unit
C_w	Warping constant	3.79E9	mm^6
I_w	Torsional moment of inertia	2.94E7	mm^4
ψ	Warping function	1.4E3	mm^2

3.2 Beam 1: length = 150 mm

Application of the theory of Vlasov

The cross-sectional parameters obtained from the model in ShapeBuilder, as listed in Table 3.1, are used as input for the theory in Vlasov. Equation (1.2) is the differential equation of Vlasov. Solving this equation, with the boundary conditions mentioned in Section 2.3, gives the angular deflection φ.

The maximum horizontal displacement u_{max} is obtained by multiplying the angular deflection at length x = l with 0.5 the height of the beam, see Equation (2.2). The maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.0005506$ rad

$u_{max} = \varphi \cdot 0.5h = 0.0005506 \cdot 0.5 \cdot 150 = 0.04130$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.82E8}{3.79E9} \cdot 1.4E3 = 67.3666$ MPa

Calculation with the Finite Element Method

The calculations with the Finite Element Method are performed in ANSYS Workbench with an element size of 5.0 mm for the mesh. The parameters and properties as mentioned in Table 2.1 are used as input for the model. The dimensions of the cross-section are given in Figure 3.1a and the length of the beam is equal to 150 mm. Figure 3.2 shows the maximum horizontal deformation of the beam. The maximum horizontal deformation occurs at the free end of the beam. Figure 3.3 shows the maximum normal stress in the beam. The maximum normal stress occurs at the fixed end of the beam.

$u_{max} = 0.05039$ mm

$\sigma_{max} = 67.8250$ MPa

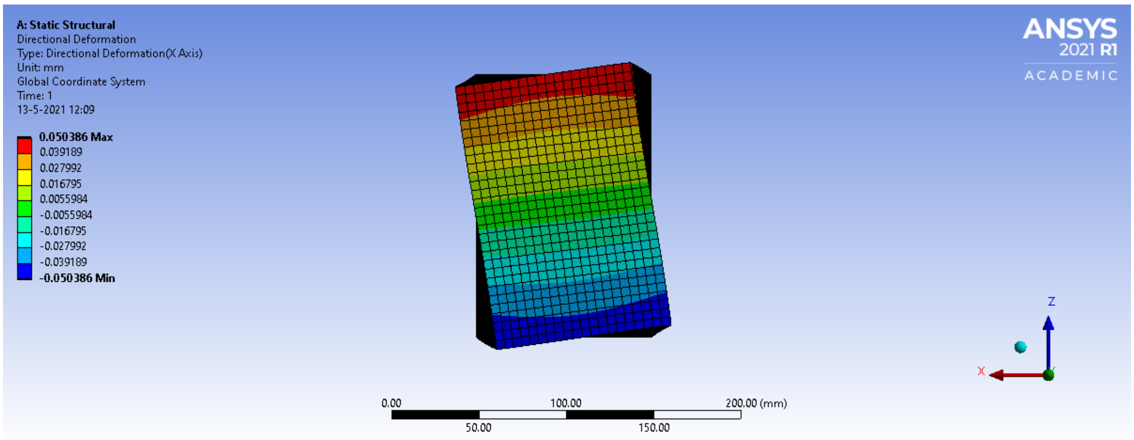


Figure 3.2: Deformation of beam 1 analysed in ANSYS Workbench

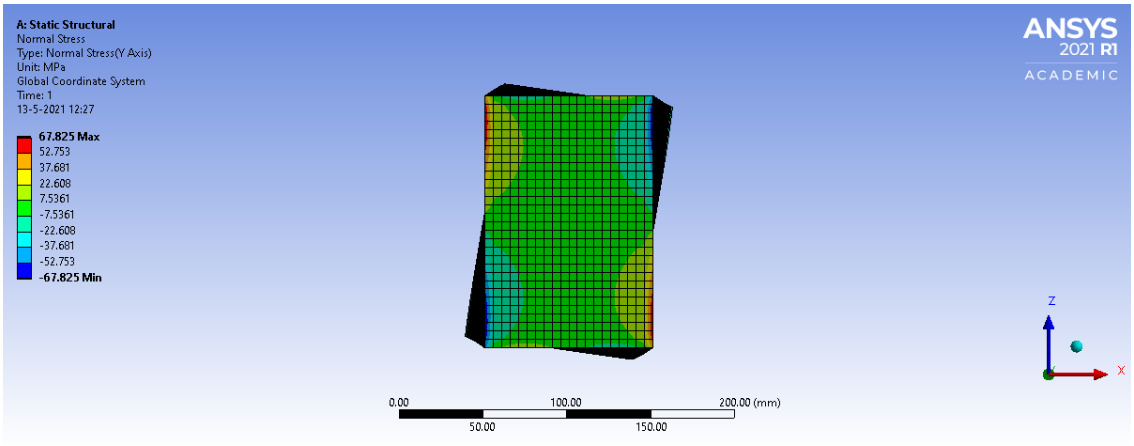


Figure 3.3: Normal stress in beam 1 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio, see Equation (2.5).

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.04130\, mm}{0.05039\, mm} = 0.82$$

The ratio is below one. This means that the theory of Vlasov underestimates the maximum horizontal displacement in beam 1. The error is 18%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{67.3666\, MPa}{67.8250\, MPa} = 0.99$$

The ratio is approximately equal to one. This means that the theory of Vlasov is valid for the maximum normal stress in beam 1. The error is 1%.

3.3 Beam 2: length = 300 mm

Application of the theory of Vlasov

For this beam the same cross-sectional parameters and dimensions are used as in Section 3.2, only the length of the beam differs. These parameters are used as input for the theory in Vlasov. Equation (1.2) is the differential equation of Vlasov. Solving this equation, with the boundary conditions mentioned in Section 2.3, gives the angular deflection φ.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.001177$ rad

$u_{max} = \varphi \cdot 0.5h = 0.001177 \cdot 0.5 \cdot 150 = 0.08831$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.82E8}{3.79E9} \cdot 1.4E3 = 67.3666$ MPa

Calculation with the Finite Element Method

The calculations with the Finite Element Method for this beam are performed in ANSYS Workbench as well but now with an element size of 6.0 mm for the mesh. This is a less accurate element size than used in Section 3.2 due to licence errors. The parameters and properties as mentioned in Table 2.1 are used as input for the model. The dimensions of the cross-section are given in Figure 3.1a and the length of the beam is equal to 300 mm. Figure 3.4 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 3.5 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.09784$ mm

$\sigma_{max} = 63.1910$ MPa

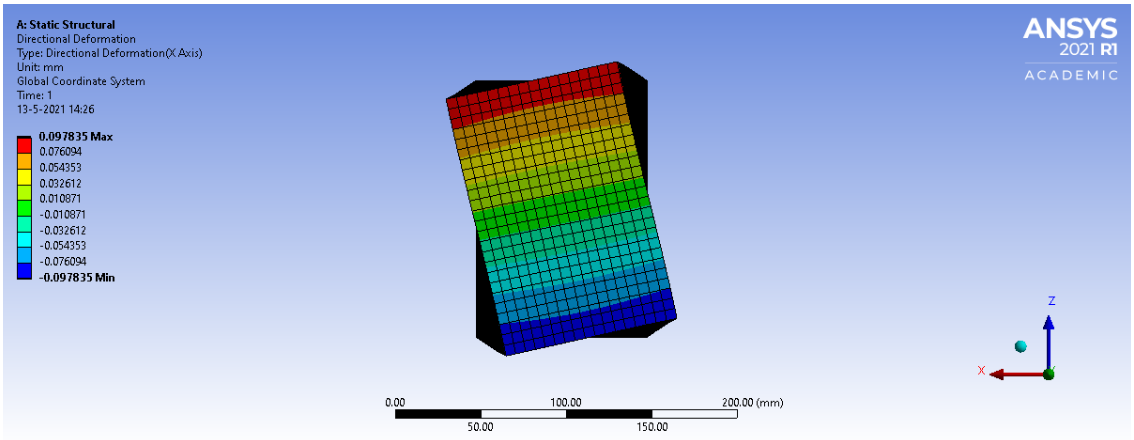


Figure 3.4: Deformation of beam 2 analysed in ANSYS Workbench

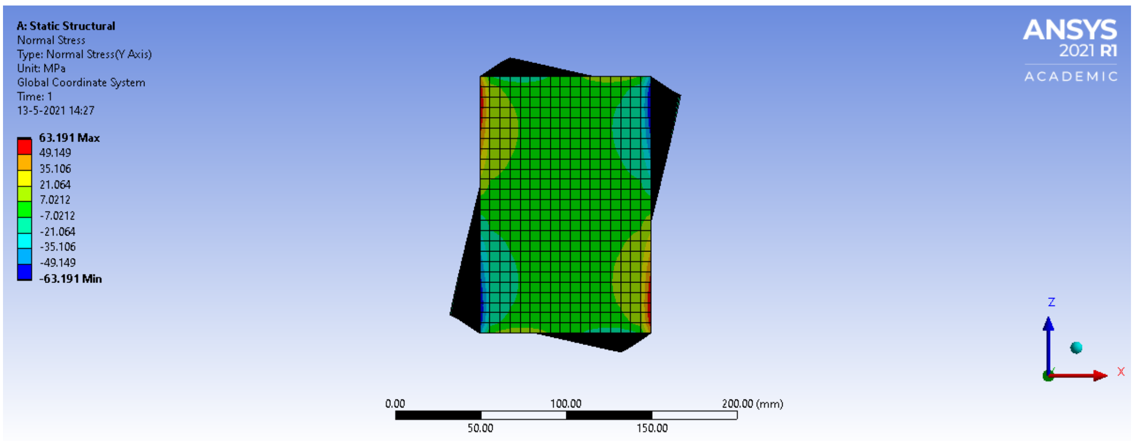


Figure 3.5: Normal stress in beam 2 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio in the same way as in Section 3.2.

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\ Element\ Method} = \frac{0.08831\, mm}{0.09784\, mm} = 0.90$$

The ratio is below one but in the error margin. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 2. The error is 10%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\ Element\ Method} = \frac{67.3666\, MPa}{63.1910\, MPa} = 1.07$$

The ratio is a bit more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 2. This can be due to the larger element size for the mesh. The error is 7%.

3.4 Beam 3: length = 600 mm

Application of the theory of Vlasov

The same cross-sectional parameters, as listed in Table 3.1, are used for this beam as the cross-section remains the same. The length of the beam is 600 mm. The same boundary conditions as mentioned in Section 2.3 are used to solve the differential equation of Vlasov, see Equation (1.2). This gives the results listed below.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.002431$ rad

$u_{max} = \varphi \cdot 0.5h = 0.002431 \cdot 0.5 \cdot 150 = 0.1823$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.82E8}{3.79E9} \cdot 1.4E3 = 67.3666$ MPa

Calculation with the Finite Element Method

The element size for the mesh is 7 mm for this beam. This is less accurate than the previous beams due to licence errors. The input for the model is mentioned in Table 2.1 and the dimensions of the cross-section remain the same as in the previous sections. The length of the beam is 600 mm. Figure 3.6 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 3.7 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.1920$ mm

$\sigma_{max} = 61.2370$ MPa

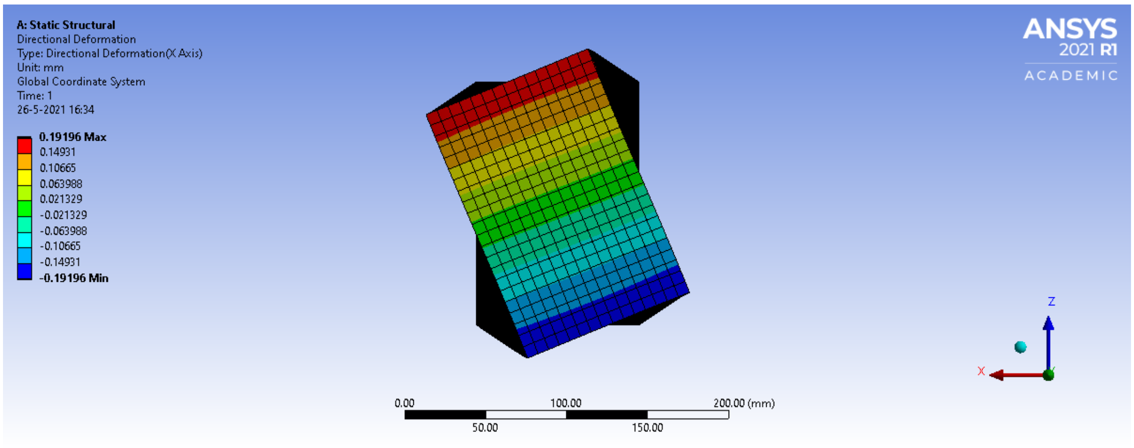


Figure 3.6: Deformation of beam 3 analysed in ANSYS Workbench

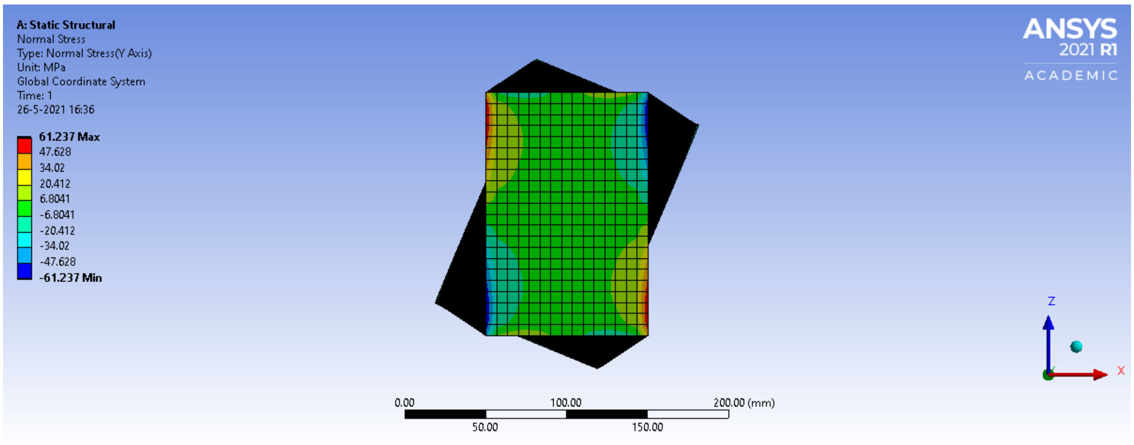


Figure 3.7: Normal stress in beam 3 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio as defined with Equation (2.5).

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.1823\, mm}{0.1920\, mm} = 0.95$$

The ratio is below one but in the error margin. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 3. The error is 5%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{67.3666\, MPa}{61.2370\, MPa} = 1.10$$

The ratio is a bit more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 3. This can be due to the larger element size for the mesh. The error is 10% so within the error margin.

3.5 Beam 4: length = 1200 mm

Application of the theory of Vlasov

To calculate the angular deflection φ of this beam, with the boundary conditions mentioned in Section 2.3, the same cross-sectional parameters are used as in the previous sections as input for the differential equation of Vlasov, see Equation (1.2). The length of the beam is 1200 mm.

The maximum horizontal displacement $u_{\max}$ is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.004938$ rad

$u_{max} = \varphi \cdot 0.5h = 0.004938 \cdot 0.5 \cdot 150 = 0.3704$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.82E8}{3.79E9} \cdot 1.4E3 = 67.3666$ MPa

Calculation with the Finite Element Method

For this beam an element size for the mesh of 9 mm is used. This is less accurate than the shorter beams in the previous sections due to licence errors. The input for the model with the Finite Element Method is mentioned in Table 2.1 because the dimensions of the cross-section are not changed. Figure 3.8 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 3.9 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.3802$ mm

$\sigma_{max} = 57.0340$ MPa

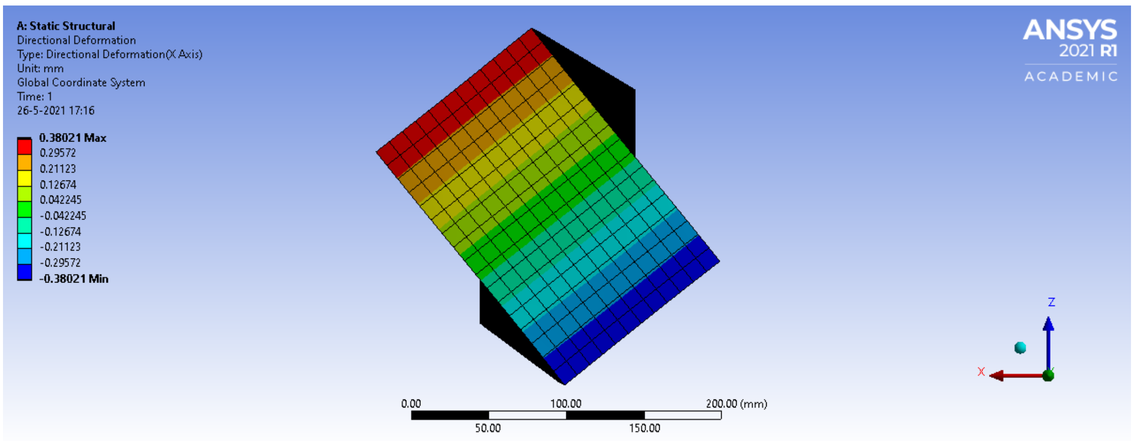


Figure 3.8: Deformation of beam 4 analysed in ANSYS Workbench

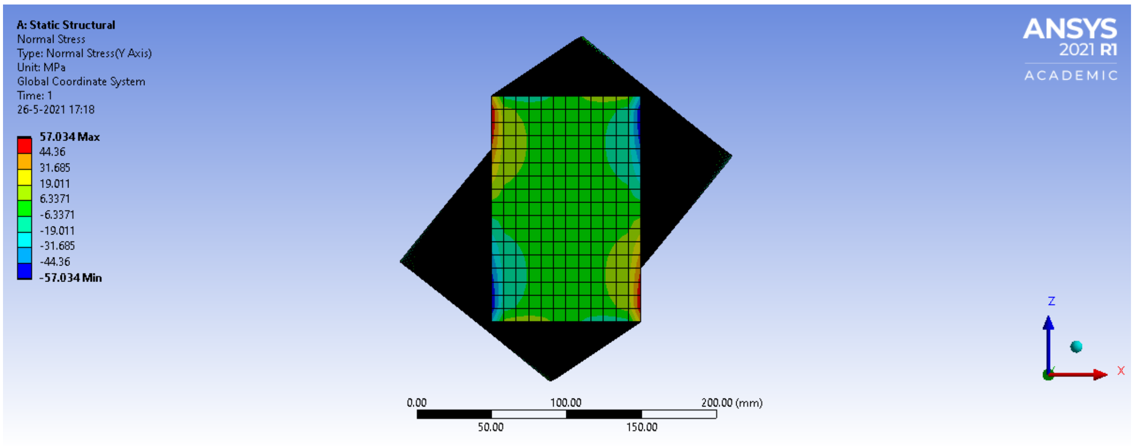


Figure 3.9: Normal stress in beam 4 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio in the same way as in the previous sections.

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.3704\, mm}{0.3802\, mm} = 0.97$$

The ratio is close to one. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 4. The error is 3%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{67.3666\, MPa}{57.0340\, MPa} = 1.18$$

The ratio is more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 4. This can be due to the larger element size for the mesh. The error is 18%.

3.6 Beam 5: length = 2400 mm

Application of the theory of Vlasov

The same boundary conditions, as mentioned in Section 2.3, and cross-sectional parameters are used for this beam as for the previous rectangular solid cross-sections to calculate the angular deflection φ. The differential equation of Vlasov, see Equation (1.2), can be solved with the boundary conditions and input. The length of the beam is 2400 mm.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.009953$ rad

$u_{max} = \varphi \cdot 0.5h = 0.009953 \cdot 0.5 \cdot 150 = 0.7465$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.82E8}{3.79E9} \cdot 1.4E3 = 67.3666$ MPa

Calculation with the Finite Element Method

An element size of 12 mm is used for this beam. This is less accurate than the previous beams with rectangular solid cross-sections due to licence errors. The dimensions of the cross-section remains the same as in the previous sections. Therefore the input for the model is given in Table 2.1. Figure 3.10 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 3.11 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.7567$ mm

$\sigma_{max} = 54.0800$ MPa

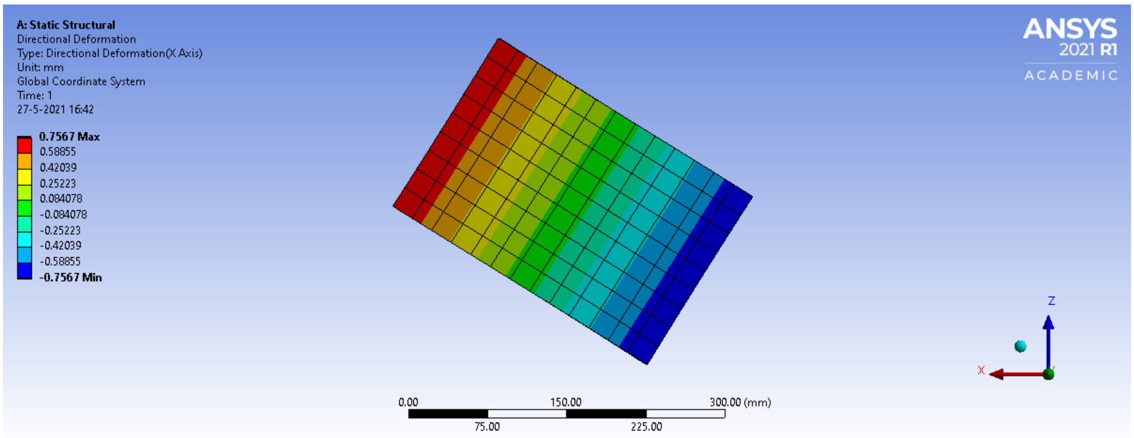


Figure 3.10: Deformation of beam 5 analysed in ANSYS Workbench

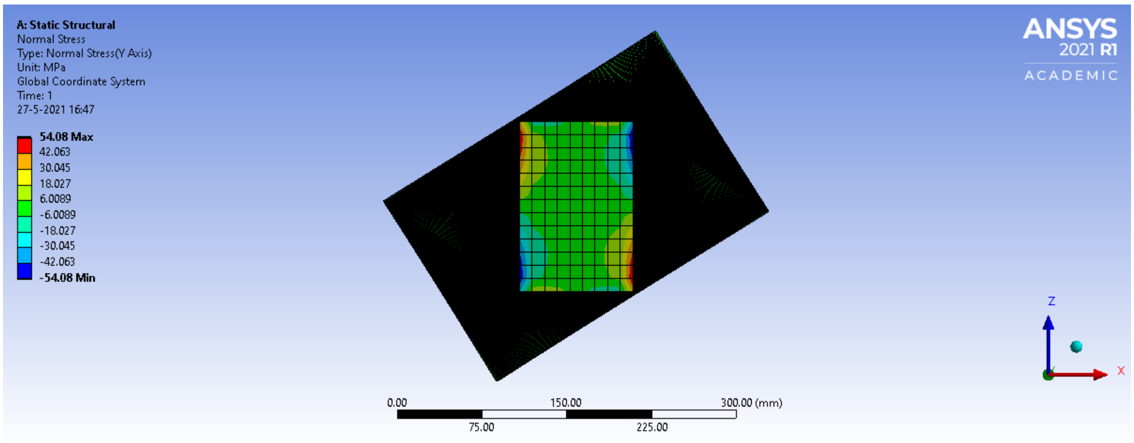


Figure 3.11: Normal stress in beam 5 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio in the same way as in the previous sections.

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.7465\, mm}{0.7567\, mm} = 0.99$$

The ratio is almost one. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 5. The error is 1%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{67.3666\, MPa}{54.0800\, MPa} = 1.25$$

The ratio is more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 5. This can be due to the larger element size for the mesh. The error is 25%.

3.7 Overview rectangular solid cross-sections

An overview of the obtained results for each beam is presented in Table 3.2. As can be seen in the *Mesh refinement* column, the first beam has the finest element size for the mesh. The other beams have a more coarse mesh refinement due to licence errors. This affects the ratio for the maximum normal stress in the beams, as can be seen in the *Ratio* σ_{max} column. Because the cross-section remains the same for each of the five beams, namely 100x150 mm, the ratio for the maximum normal stress has to be the same. The maximum normal stress is only dependent, see Equation (2.4), on the maximum bimoment, warping constant and warping function. These are all cross-sectional properties. Therefore, because of the more coarse mesh refinement, a different result for the maximum normal stress is found. The ratio for the maximum normal stress for the first beam is the most accurate since this beam is modelled with the finest mesh refinement. From the *Ratio* u_{max} column it can be concluded that the ratio is more accurate for the longest beam. This means that the theory of Vlasov is the most accurate for beam five, which has a length of 2400 mm. Actually, since the error margin is 15%, it means that the theory of Vlasov for the *Ratio* u_{max} is accurate for all beams except for beam 1.

Table 3.2: Overview ratios of the five beams with rectangular solid cross-sections

	Mesh refinement [mm]	Ratio u_{max} [-]	Ratio σ_{max} [-]
Beam 1	5	0.82	0.99
Beam 2	6	0.90	1.07
Beam 3	7	0.95	1.10
Beam 4	9	0.97	1.18
Beam 5	12	0.99	1.25

3.8 Influence changing parameters

This section looks at the influence of changing different standard parameters listed in Table 2.1, as well as changing the height of the cross-section and the mesh refinement. One parameter is changed at a time in order to research its influence on u_{max} and σ_{max}. A new ratio for u_{max} and σ_{max} is calculated using ShapeBuilder and Maple for the theory of Vlasov and ANSYS Workbench for the Finite Element Method. This new ratio is compared to the ratio of the *normal* situation; the situation of beam 3 with the standard, unchanged parameters. If the ratio is closer to one, it means that the change has a positive influence on the accuracy of the theory of Vlasov. Table 3.3 gives an overview of the ratios with the changed parameters.

Table 3.3: Influence of changing different parameters

Parameter	Normal	Smaller	Bigger
ν [-]	**0.29**	**0.24**	**0.34**
Ratio u_{max}	0.95	0.95	0.95
Ratio σ_{max}	1.10	1.15	1.05
E [MPa]	**210000**	**190000**	**230000**
Ratio u_{max}	0.95	0.95	0.95
Ratio σ_{max}	1.10	1.10	1.10
Height h [mm]	**150**	**50**	**250**
Ratio u_{max}	0.95	1.00	0.93
Ratio σ_{max}	1.17	1.13	0.80
Mesh refinement [mm]	**7**	**-**	**20**
Ratio u_{max}	0.95	-	0.95
Ratio σ_{max}	1.10	-	1.46

The first parameter that is changed is the Poisson's ratio ν. The ν used for the five beams is equal to 0.29. The smaller value of 0.24 has no influence on the u_{max}, but leads to a slightly less accurate result for σ_{max}. The bigger value of 0.34 has no influence on the u_{max} as well, but leads to a somewhat more accurate result for σ_{max}. The differences are small, i.e. 5%, so it can be concluded that a small change in the Poisson's ratio does not lead to significant differences and therefore has no effect on the theory of Vlasov. If the Poisson's ratio is taken much bigger, this will probably lead to a more accurate result. However, this is very uncertain and should be researched more detailed.

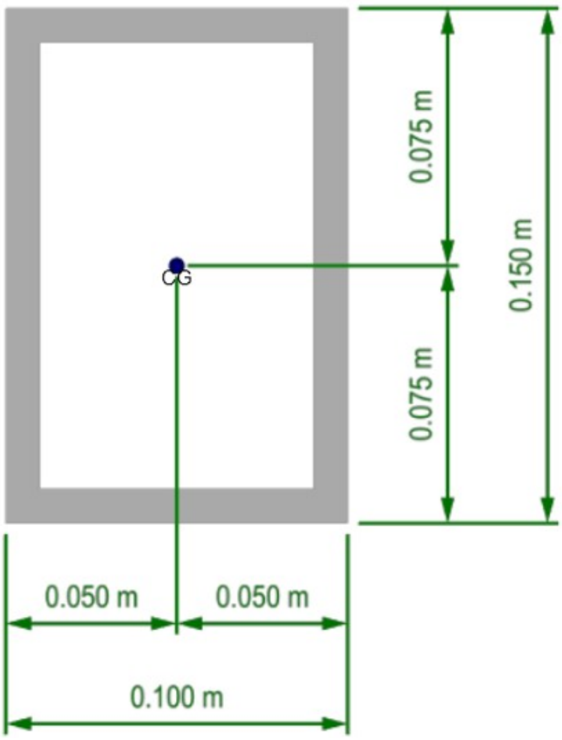
The second parameter that its influence on the theory of Vlasov is checked is the Young's modulus E. The *standard* value that is used in this report is 210000 MPa. Changing this value to 190000 MPa or to 230000 MPa has no influence for both ratios and therefore has no effect on the validity of the theory of Vlasov.

The influence of the height of the cross-section is researched as well. A mesh refinement of 8.5 mm is used for researching the height. The *standard* height of the beam is 150 mm. A smaller height of 50 mm, now the width is bigger than the height, is chosen. This leads to a significantly more accurate ratio for u_{max} and a slightly more accurate ratio for σ_{max}. If the height of the beam is taken 250 mm, it has a negative result on both ratios. A higher beam leads to a less accurate result with the theory of Vlasov for the maximum deformation and the maximum normal stress.

Lastly, the influence of a more coarse mesh refinement is researched. A more fine mesh refinement is not possible due to licence errors and is therefore not included in this report. The mesh refinement used for beam 3 is 7 mm. If the mesh refinement is changed to 20 mm this has no influence on the ratio u_{max}, but it has an influence on the ratio σ_{max}. If the mesh refinement becomes more coarse, the application of the theory of Vlasov will be less accurate.

4 | Rectangular closed tube cross-sections

The second part of this study researches the validity of the theory of Vlasov for rectangular closed tube cross-sections. Five beams are analysed with the same cross-section, see Figure 4.1a, but different lengths. The length of beam 6 is chosen arbitrary and the lengths of the following beams are twice the length of the previous beam, in the same way as in Chapter 3. The method applied in this chapter is described in Chapter 2.



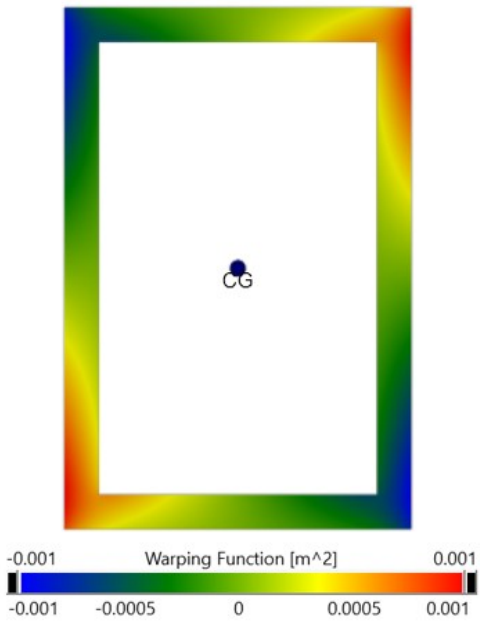


Figure 4.1: (a) Rectangular closed tube cross-section; (b) Warping function rectangular closed tube cross-section

4.1 Determination of the cross-sectional parameters

The dimensions of the rectangular closed tube cross-section are shown in Figure 4.1a. This cross-section is used for beams 6 - 10 and is modelled in ShapeBuilder with a mesh refinement as fine as possible and input values as defined in Table 2.1. The program calculates the model and returns values for the warping constant, torsional moment of inertia and the warping function. These values are listed in Table 4.1. The warping function is visualised in Figure 4.1b.

Table 4.1: Cross-sectional parameters rectangular closed tube cross-section

Symbol	Property	Value	Unit
C_w	Warping constant	9.023E8	mm^6
I_w	Torsional moment of inertia	1.44E7	mm^4
ψ	Warping function	1E3	mm^2

4.2 Beam 6: length = 150 mm

Application of the theory of Vlasov

The cross-sectional parameters listed in Table 4.1, obtained from the model in ShapeBuilder, are used as input for the theory of Vlasov. The differential equation of Vlasov, defined in Equation (1.2), is solved for the angular deflection φ using this cross-sectional parameters and the boundary conditions specified in Section 2.3.

The maximum horizontal displacement u_{max} is obtained by multiplying the angular deflection at length x = l with 0.5 the height of the beam, see Equation (2.2). The maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.001171$ rad

$u_{max} = \varphi \cdot 0.5h = 0.001171 \cdot 0.5 \cdot 150 = 0.08785$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.27E8}{9.023E8} \cdot 1E3 = 140.9137$ MPa

Calculation with the Finite Element Method

ANSYS Workbench is used for the calculations with the Finite Element Method with an element size for the mesh of 2.5 mm. The input parameters for the model are listed in Table 2.1 and the dimensions of the cross-section are given in Figure 4.1a. The length of the beam is 150 mm. Figure 4.2 shows the maximum horizontal deformation of the beam. The maximum horizontal deformation occurs at the free end of the beam. Figure 4.3 shows the maximum normal stress in the beam. The maximum normal stress occurs at the fixed end of the beam.

$u_{max} = 0.1058$ mm

$\sigma_{max} = 135.7900$ MPa

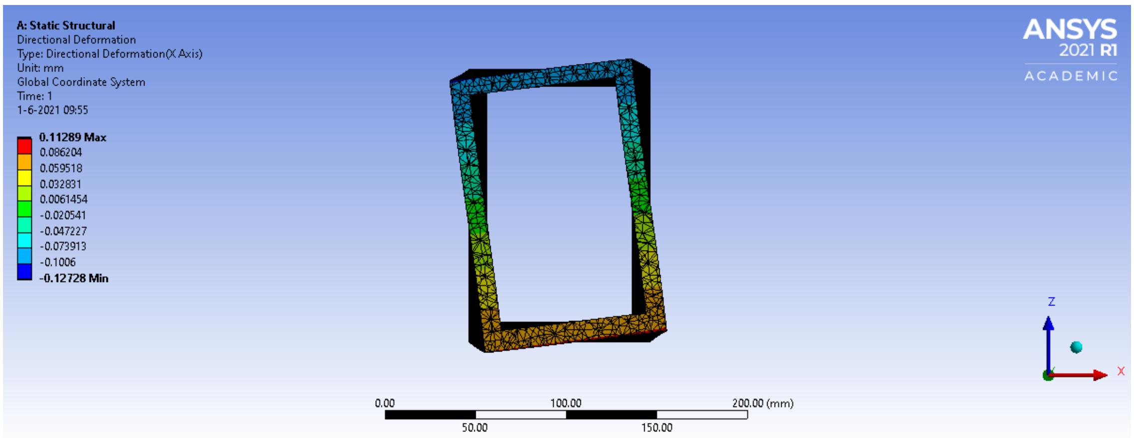


Figure 4.2: Deformation of beam 6 analysed in ANSYS Workbench

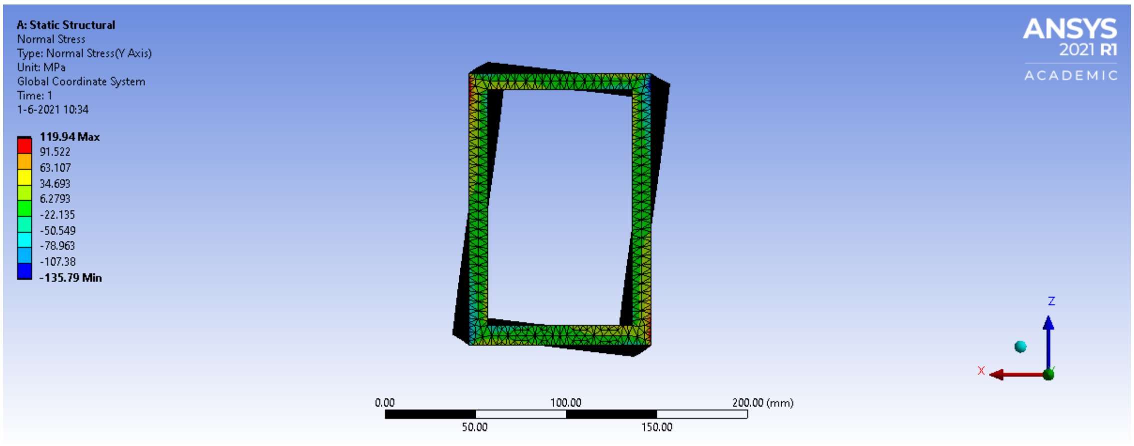


Figure 4.3: Normal stress in beam 6 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio, see Equation (2.5).

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.08785\, mm}{0.1058\, mm} = 0.83$$

The ratio is below one. This means that the theory of Vlasov underestimates the maximum horizontal displacement in beam 6. The error is 17%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{140.9137\, MPa}{135.7900\, MPa} = 1.04$$

The ratio is slightly more than one. This means that the theory of Vlasov is valid for the maximum normal stress in beam 6. The error is 4%.

4.3 Beam 7: length = 300 mm

Application of the theory of Vlasov

The angular deflection φ of this beam can be calculated with the differential equation of Vlasov, see Equation (1.2), in combination with the boundary conditions mentioned earlier. The cross-sectional parameters mentioned in Table 4.1 are used as input for the differential equation. The length of the beam is 300 mm.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.002451$ rad

$u_{max} = \varphi \cdot 0.5h = 0.002451 \cdot 0.5 \cdot 150 = 0.1838$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.27E8}{9.023E8} \cdot 1E3 = 140.9137$ MPa

Calculation with the Finite Element Method

For this beam an element size for the mesh of 3 mm is used. This is slightly more coarse than beam 6 due to licence errors. The parameters mentioned in Table 2.1 are used as input for the Finite Element Method. Figure 4.4 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 4.5 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.1964$ mm

$\sigma_{max} = 109.5600$ MPa

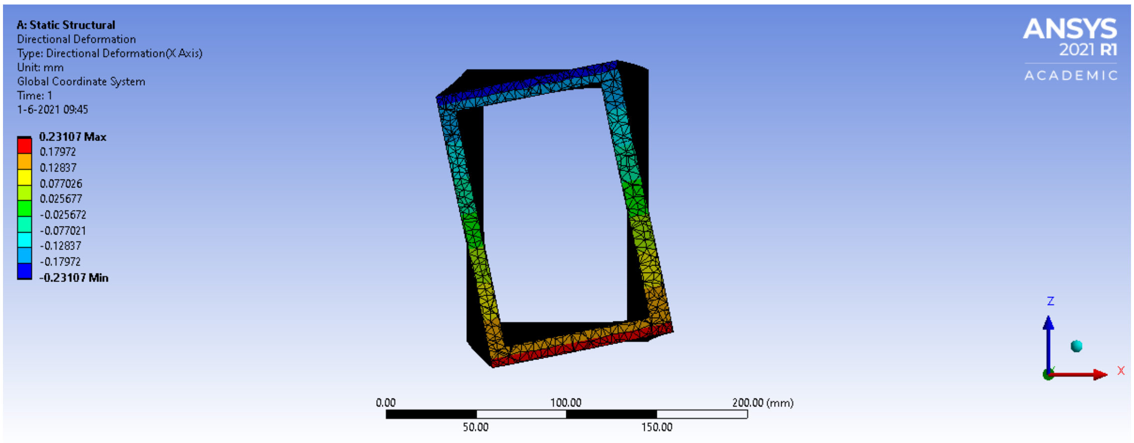


Figure 4.4: Deformation of beam 7 analysed in ANSYS Workbench

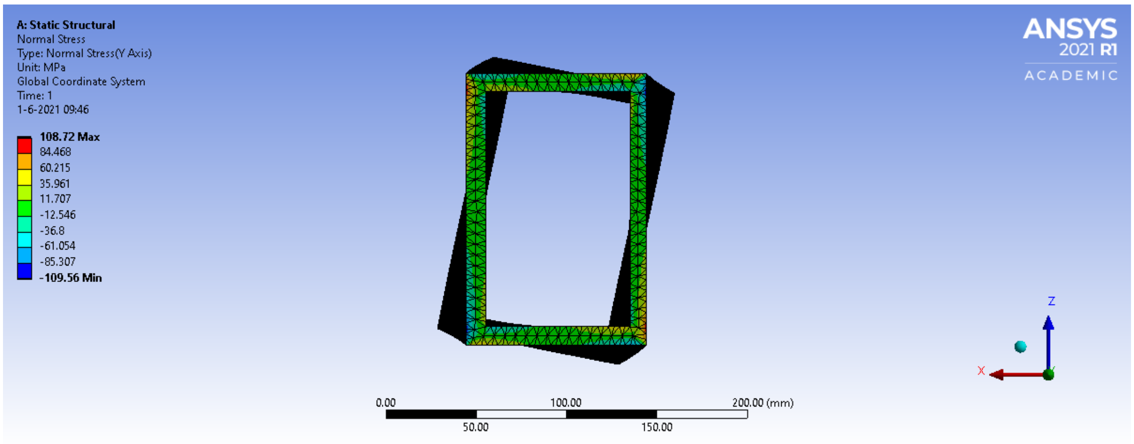


Figure 4.5: Normal stress in beam 7 analysed in ANSYS Workbench

Comparing of the results

The deformations and normal stresses calculated with the theory of Vlasov and with the Finite Element Method are compared using a ratio.

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.1838\, mm}{0.1964\, mm} = 0.94$$

The ratio is below one but within the error margin. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 7. The error is 6%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{140.9137\, MPa}{109.5600\, MPa} = 1.29$$

The ratio is much more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 7. The error is 29%.

4.4 Beam 8: length = 600 mm

Application of the theory of Vlasov

The length of this beam is 600 mm. The same cross-sectional parameters and boundary conditions as the previous sections about rectangular closed tube cross-sections are used to solve the differential equation of Vlasov.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.005011$ rad

$u_{max} = \varphi \cdot 0.5h = 0.005011 \cdot 0.5 \cdot 150 = 0.3758$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.27E8}{9.023E8} \cdot 1E3 = 140.9137$ MPa

Calculation with the Finite Element Method

An element size of 4 mm is chosen for this beam. This is less accurate than the previous beams due to licence errors. ANSYS Workbench uses Table 2.1 as input for the model. Figure 4.6 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 4.7 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.3870$ mm

$\sigma_{max} = 105.3800$ MPa

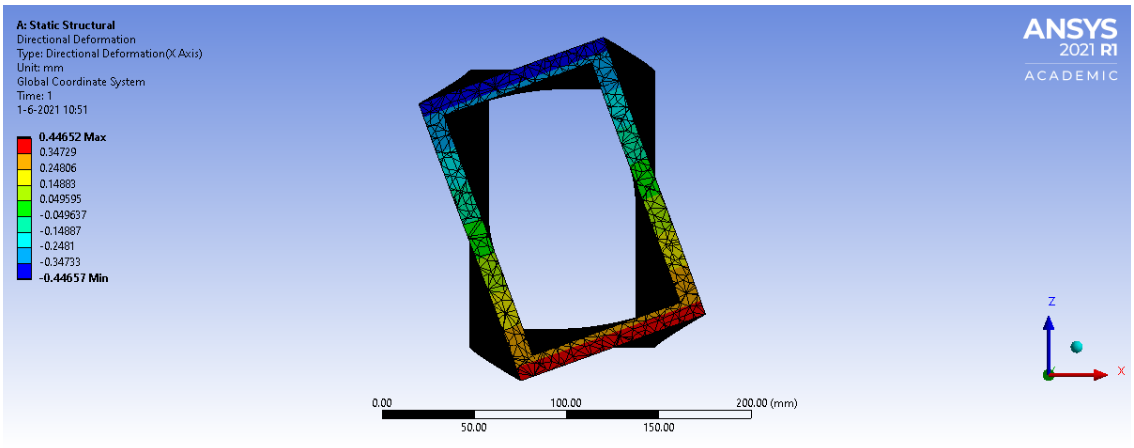


Figure 4.6: Deformation of beam 8 analysed in ANSYS Workbench

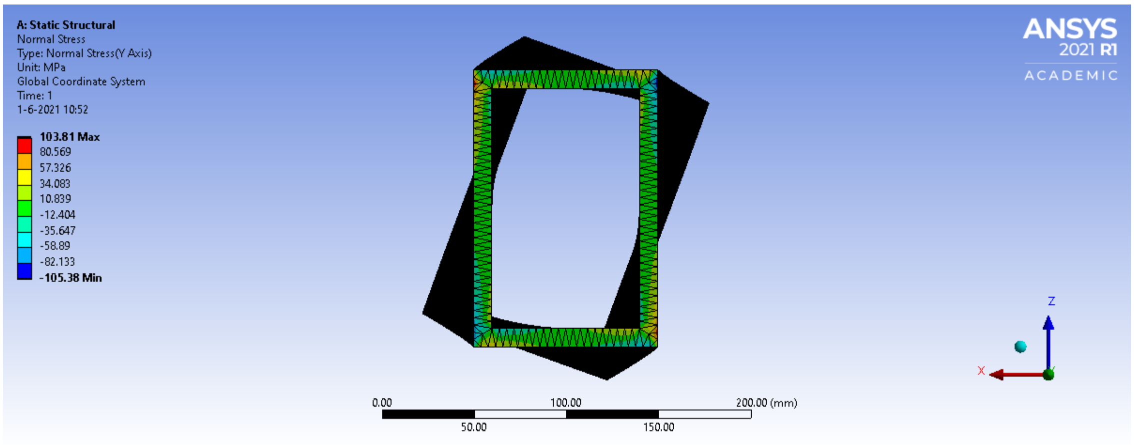


Figure 4.7: Normal stress in beam 8 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio as defined in Equation (2.5).

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.3758\, mm}{0.3870\, mm} = 0.97$$

The ratio is just below one. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 8. The error is 3%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{140.9137\, MPa}{105.3800\, MPa} = 1.34$$

The ratio is more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 8. The error is 34% so the theory of Vlasov is not valid for this beam. This is due to the more coarse element size.

4.5 Beam 9: length = 1200 mm

Application of the theory of Vlasov

The differential equation of Vlasov uses the cross-sectional parameters specified in Table 4.1 and boundary conditions specified in Section 2.3 as input to solve for the angular deviation φ. The length of the beam is 1200 mm.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.01013$ rad

$u_{max} = \varphi \cdot 0.5h = 0.01013 \cdot 0.5 \cdot 150 = 0.7597$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.27E8}{9.023E8} \cdot 1E3 = 140.9137$ MPa

Calculation with the Finite Element Method

The calculations with the Finite Element Method for this beam are performed in ANSYS Workbench as well but now with a more coarse mesh of 5 mm due to licence errors. The parameters and properties mentioned in Table 2.1 are used as input for the model. The dimensions of the cross-section are given in Figure 4.1a and the length of the beam is equal to 1200 mm. Figure 4.8 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 4.9 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 0.7691$ mm

$\sigma_{max} = 84.5960$ MPa

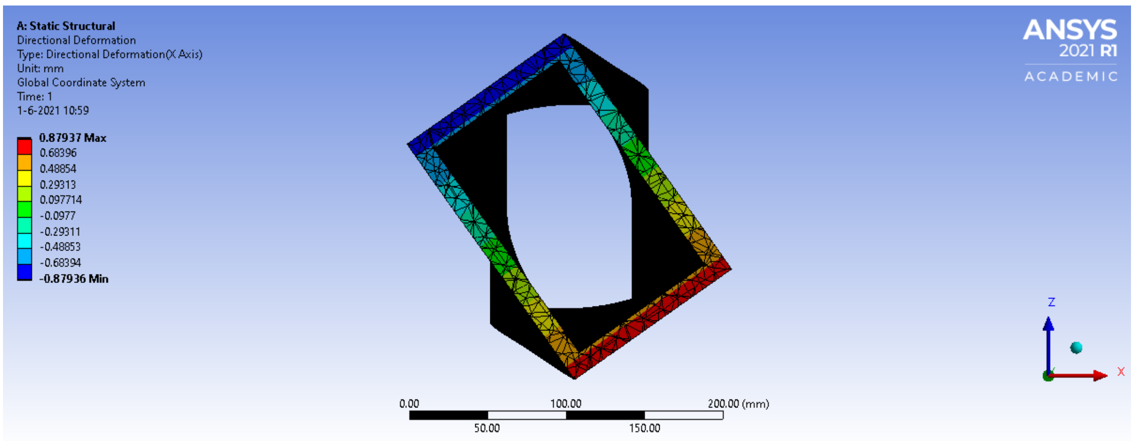


Figure 4.8: Deformation of beam 9 analysed in ANSYS Workbench

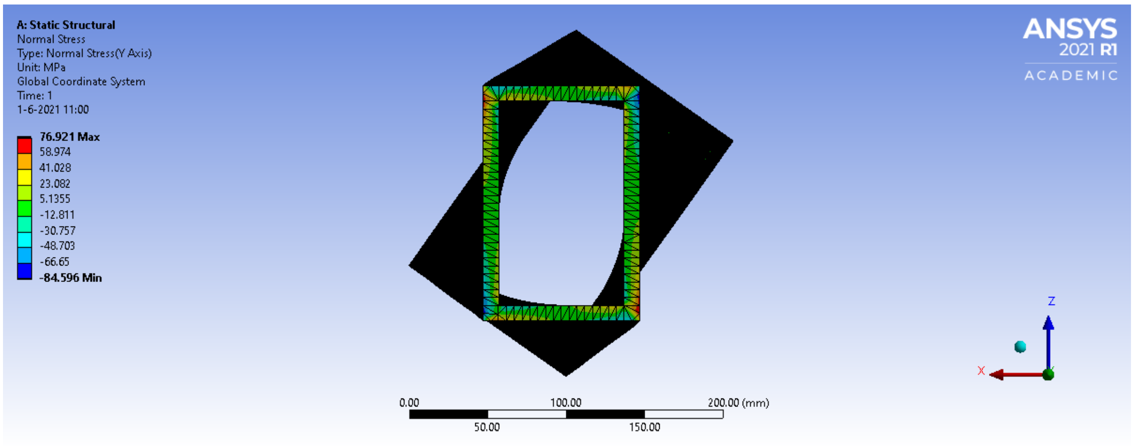


Figure 4.9: Normal stress in beam 9 analysed in ANSYS Workbench

Comparing of the results

The theory of Vlasov and the calculations with the Finite Element Method are compared using a ratio.

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{0.7597\, mm}{0.7691\, mm} = 0.99$$

The ratio is almost equal to one. This means that the theory of Vlasov is valid for the maximum horizontal displacement in beam 9. The error is 1%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{140.9137\, MPa}{84.5960\, MPa} = 1.67$$

The ratio is a much more than one. This means that the theory of Vlasov is not valid for the maximum normal stress in beam 9. This is probably due to the larger element size for the mesh. The error is 67%.

4.6 Beam 10: length = 2400 mm

Application of the theory of Vlasov

The cross-sectional parameters listed in Table 4.1 are used for this beam as the cross-section remains the same as in the previous sections. In order to solve the differential equation of Vlasov, the same boundary conditions are used as in Section 2.3.

The maximum horizontal displacement u_{max} is calculated with Equation (2.2) and the maximum normal stress in the beam is calculated with Equation (2.4). In this equation the bimoment is taken at length x = 0. The calculations in Maple are included in Appendix A.

$\varphi = 0.02037$ rad

$u_{max} = \varphi \cdot 0.5h = 0.02037 \cdot 0.5 \cdot 150 = 1.5276$ mm

$\sigma_{max} = -\frac{B_{max}}{C_w} \cdot \psi = -\frac{-1.27E8}{9.023E8} \cdot 1E3 = 140.9137$ MPa

Calculation with the Finite Element Method

The length of the beam is 2400 mm. The element size for the mess is 7 mm. This is less accurate than the previous beams due to licence errors. As input for the model in ANSYS Workbench the parameters mentioned in Table 2.1 are used. Figure 4.10 shows the maximum horizontal deformation of the beam and occurs at the free end of the beam. Figure 4.11 shows the maximum normal stress in the beam and occurs at the fixed end of the beam.

$u_{max} = 1.4737$ mm

$\sigma_{max} = 75.3800$ MPa

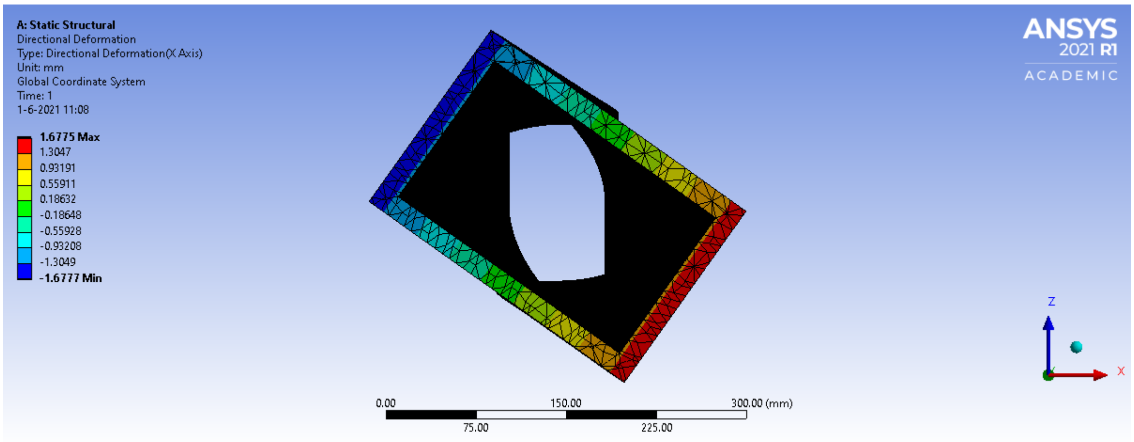


Figure 4.10: Deformation of beam 10 analysed in ANSYS Workbench

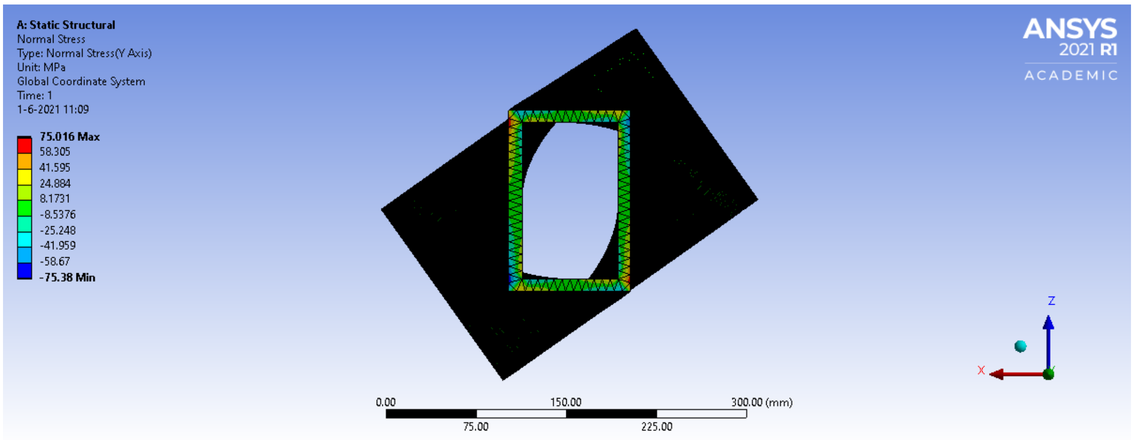


Figure 4.11: Normal stress in beam 10 analysed in ANSYS Workbench

Comparing of the results

The results of the theory of Vlasov and the calculations with the Finite Element Method are compared using the same ratio as in the previous sections.

$$Ratio\, u_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{1.5276\, mm}{1.4737\, mm} = 1.04$$

The ratio is a bit more than one. This means that the theory of Vlasov overestimates the horizontal deformation but is valid for the maximum horizontal displacement in beam 10 because it is within the error margin. The error is 4%.

$$Ratio\, \sigma_{max} = \frac{Theory\, of\, Vlasov}{Finite\, Element\, Method} = \frac{140.9137\, MPa}{75.3800\, MPa} = 1.87$$

The ratio is more than one. This means that the theory of Vlasov overestimates the maximum normal stress in beam 10. This is probably due to a more coarse mesh. The error is 87%.

4.7 Overview rectangular closed tube cross-sections

This section gives an overview of the ratios for the beams with rectangular closed tube cross-sections. The results are presented in Table 4.2. The first beam, i.e. beam 6, has the finest element size for the mesh, as can be seen in the *Mesh refinement* column. This is as fine as possible. The other beams, i.e. beams 7 - 10, have a more coarse mesh refinement due to licence errors. The mesh refinement for these beams are as fine as possible, just as in Chapter 3. Therefore the maximum normal stress, presented in the *Ratio* σ_{max} column, in beams 7 - 10 is less accurate due to the more coarse mesh refinement. The *Ratio* u_{max} column shows the ratios for the maximum horizontal deformations. It can be observed that as the beam is getting longer, the theory of Vlasov will be more accurate. An acceptable result is obtained for beams 7 - 10.

Table 4.2: Overview ratios of the five beams with rectangular closed tube cross-sections

	Mesh refinement [mm]	Ratio u_{max} [-]	Ratio σ_{max} [-]
Beam 6	2.5	0.83	1.04
Beam 7	3	0.94	1.29
Beam 8	4	0.97	1.34
Beam 9	5	0.99	1.67
Beam 10	7	1.04	1.87

4.8 Influence changing parameters

The influence of changing different parameters, cross-sectional dimensions, mesh refinement and thickness of the tube on the maximum horizontal deflection and maximum normal stress are discussed in this section. The procedure of testing the influence is the same as in Section 3.8, but now compared to the situation of beam 8. Table 4.3 gives an overview of the ratios with the changed parameters.

Table 4.3: Influence of changing different parameters

Parameter	Normal	Smaller	Bigger
ν [-]	**0.29**	**0.24**	**0.34**
Ratio u_{max}	0.97	0.97	0.97
Ratio σ_{max}	1.34	1.40	1.26
E [MPa]	**210000**	**190000**	**230000**
Ratio u_{max}	0.97	0.97	0.97
Ratio σ_{max}	1.34	1.34	1.34
Height h [mm]	**150**	**50**	**250**
Ratio u_{max}	0.97	0.99	0.92
Ratio σ_{max}	1.62	1.39	1.25
Mesh refinement [mm]	**4**	**-**	**20**
Ratio u_{max}	0.97	-	1.08
Ratio σ_{max}	1.34	-	2.55
Thickness t [mm]	**10**	**5**	**20**
Ratio u_{max}	0.97	0.98	0.96
Ratio σ_{max}	1.34	1.87	1.21

The first parameter that is looked at is Poisson's ratio ν. This ratio gives the deformation of the steel in directions perpendicular to the direction of loading. The ν used in this study is 0.29. Reducing the ν to 0.24 or increasing it to 0.34 has no effect on the maximum horizontal deformation, the ratio remains 0.97. Changing the ν has a minor effect on the maximum normal stress as can be seen in Table 4.3. These results are comparable to the results obtained for changing the ν of rectangular solid cross-section, discussed in Section 3.8.

Changing the Young's modulus E, both for a smaller and a bigger value, has no effect on the ratios and consequently has no influence on the validity of the theory of Vlasov for rectangular closed tube cross-sections. These results are comparable to the results obtained in Section 3.8.

The next parameter that is changed is the height h of the cross-section. The height used in this study is 150 mm. The influence of both a smaller and bigger height is researched, respectively 50 mm and 250 mm. Choosing a smaller height has a positive effect on both the maximum horizontal deformation and the maximum normal stress and therefore on the validity of the theory of Vlasov on rectangular closed tube cross-sections. Choosing the height bigger has a negative effect on the validity of the theory of Vlasov. An element size for the mesh refinement of 4.5 mm is used for checking the height of the cross-section due to licence errors.

The mesh refinement used for beam 8 is 4 mm. Only the influence of a more coarse mesh refinement is researched as a more fine mesh refinement is not possible for beam 8 due to licence errors. An element size of 20 mm for the mesh refinement has a negative influence on the ratio for the maximum horizontal deformation and for the maximum normal stress. This result is not surprising since the peak u_{max} and σ_{max} is lower.

The last parameter that is researched is the thickness t of the tube. A thickness of 10 mm is used in this study. The influence of both a smaller and bigger thickness, respectively 5 mm and 20 mm, on the ratio u_{max} is negligible. A smaller thickness has a negative influence on the σ_{max}, as a bigger thickness has a positive influence on the σ_{max}.

5 | Observations and conclusions

This study looked at the validity of the theory of Vlasov for rectangular solid cross-sections and rectangular closed tube cross-sections by means of the maximum horizontal displacement and the maximum normal stress. The results for rectangular solid cross-sections are presented in Table 3.2 and for rectangular closed tube cross-sections in Table 4.2.

From Table 3.2 can be concluded that the theory of Vlasov is valid for the maximum horizontal displacement for beams 2 - 5. The *Ratio* u_{max} for beam 1, where the length matches the height, is 0.82 which means that the theory of Vlasov underestimates the maximum horizontal displacement as a maximum error of 15% is allowed. The *Ratio* u_{max} is most accurate for the longest beam, i.e. beam 5 with a *Ratio* u_{max} of 0.99.

The theory of Vlasov gives an accurate result for the maximum normal stress in beams 1 - 5 as the *Ratio* σ_{max} for beam 1 is 0.99. For this beam the most fine mesh refinement is used. Unfortunately it was not possible to use smaller elements for the mesh of the longer beams due to the educational licence of ANSYS Workbench. Therefore inaccurate results are found for the normal stresses and can be ignored for these beams. Equation (2.4) shows that the maximum normal stress is only dependent on cross-sectional parameters, i.e. the bimoment, warping constant and warping function, so consequently the *Ratio* σ_{max} has to be equal for beams 1 - 5 regardless the length of the beams.

According to Table 4.2 the theory of Vlasov is valid for the maximum horizontal displacement for beams 7 - 10 as the error is smaller than 15%. The *Ratio* u_{max} for beam 5, where the length matches the height, is 0.83 which means that the theory of Vlasov underestimates the maximum horizontal displacement in this beam. This result is similar to the result found for the *Ratio* u_{max} for beam 1.

For the *Ratio* σ_{max} similar results are found for the rectangular closed tube cross-sections according to Table 4.2. The theory of Vlasov gives an accurate result for the maximum normal stress in beams 6 - 10 as the *Ratio* σ_{max} is 1.04 for beam 6, which has the finest mesh refinement. Because of the same reason as for the rectangular solid cross-sections, it means that the theory of Vlasov is valid for the longer beams with a larger error as well.

Section 3.8 and Section 4.8 looked at the influence of changing different parameters on the validity of the theory of Vlasov. From these sections can be concluded that changing the Young's modulus E has no influence and changing the Poisson's ratio ν has a minor influence on the validity of the theory of Vlasov for the considered cross-sections. Choosing the height smaller than the width of the cross-section has a positive influence on the validity of the theory of Vlasov. When looking at the mesh refinement it can be observed that a more coarse element size leads to a less accurate result. Lastly, the influence of the thickness t of the tube is researched. It can be concluded that the thickness has a minor influence on the *Ratio* u_{max} and a bigger thickness leads to a more accurate result for *Ratio* σ_{max}.

It could be interesting to do research on the exact length from the fixed end where the theory of Vlasov is valid for the maximum horizontal displacement for both types of cross-sections. This distance will be approximately between one and two times the height of the cross-section, according to the findings of this study.

This study concludes that the theory of Vlasov is valid for two types of cross-sections: rectangular solid cross-sections and rectangular closed tube cross-sections. It could be interesting to look at validity of the theory of Vlasov for more unusual types of cross-sections. In addition, more extreme values for the Poisson's ratio ν could be looked at.

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Appendix A: Calculations in Maple

restart :

Beam 1: rectangular solid cross-section 150 mm

Parameters:
$h := 150; \#mm$
$b := 100; \#mm$
$l := 150; \#mm$
$v := 0.29; \#`-`$
$T := 10E6; \#Nmm$
$E := 210000; \#MPa$
$G := \dfrac{E}{2\cdot(1+v)}; \#MPa$
$Iw := 2.94E7; \#mm4$
$Cw := 3.79E9; \#mm6$
$\mathrm{psi} := 1.4E3; \#mm2$
$mx := 0; \#\dfrac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 150$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffnesses:
$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Characteristic length:
$lc := \mathrm{sqrt}\left(\dfrac{ECw}{GIw}\right);$

$$lc := 18.23710056 \tag{3}$$

Vlasov equation:
with(*DEtools*) :
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{4}$$

Boundary conditions:

$bound\_con := \mathrm{phi}(0) = 0, \mathrm{D}(\mathrm{phi})(0) = 0, GIw\cdot\mathrm{D}(\mathrm{phi})(l) - ECw\cdot(\mathrm{D}@@3)(\mathrm{phi})(l) = T, (\mathrm{D}@@2)(\mathrm{phi})(l) = 0;$

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 2.393023256\ 10^{12}\ \mathrm{D}(\phi)(150) - 7.9590000\ 10^{14}\ \mathrm{D}^{(3)}(\phi)(150) = 1.0\ 10^{7}, \mathrm{D}^{(2)}(\phi)(150) = 0 \quad \textbf{(5)}$$

Solving Vlasov equation:

$Sol := evalf(dsolve(\{ODE, bound\_con\}, \{\mathrm{phi}(x)\}));$
$assign(Sol); \mathrm{phi} := \mathrm{phi}(x)$

$$Sol := \phi(x) = -0.00007620944719 + 4.178814383\ 10^{-6}\ x - 5.468545857\ 10^{-12}\ \mathrm{e}^{0.05483327774\,x} + 0.00007620945263\ \mathrm{e}^{-0.05483327774\,x}$$

$$\phi := -0.00007620944719 + 4.178814383\ 10^{-6}\ x - 5.468545857\ 10^{-12}\ \mathrm{e}^{0.05483327774\,x} + 0.00007620945263\ \mathrm{e}^{-0.05483327774\,x} \quad \textbf{(6)}$$

Bi-moment and torsional moment:

$B := -ECw\cdot diff(\mathrm{phi}, x, x);$
$Mw1 := GIw\cdot diff(B, x);$
$Mw2 := diff(B, x);$
$Mwtot := Mw1 + Mw2;$

$$B := 13.08635741\ \mathrm{e}^{0.05483327774\,x} - 1.823709925\ 10^{8}\ \mathrm{e}^{-0.05483327774\,x}$$

$$Mw1 := 1.717156602\ 10^{12}\ \mathrm{e}^{0.05483327774\,x} + 2.393023084\ 10^{19}\ \mathrm{e}^{-0.05483327774\,x}$$

$$Mw2 := 0.7175678705\ \mathrm{e}^{0.05483327774\,x} + 9.999999283\ 10^{6}\ \mathrm{e}^{-0.05483327774\,x}$$

$$Mwtot := 1.717156602\ 10^{12}\ \mathrm{e}^{0.05483327774\,x} + 2.393023084\ 10^{19}\ \mathrm{e}^{-0.05483327774\,x} \quad \textbf{(7)}$$

$x := 0 : B; x := 'x';$
$\#plot(B, x = 0..l);$

$$-1.823709794\ 10^{8}$$

$$x := x \quad \textbf{(8)}$$

Rotations and displacements:

$x := l : phi\_max := \mathrm{phi};$

$$phi\_max := 0.0005506127102 \quad \textbf{(9)}$$

$u\_max := phi\_max\cdot 0.5\cdot h;$

$$u\_max := 0.04129595326 \quad \textbf{(10)}$$

$x := 0 : sigma\_max := -\frac{B}{Cw}\cdot \mathrm{psi};$

$$sigma\_max := 67.36658870 \quad \textbf{(11)}$$

$restart:$

Beam 2: Rectangular solid cross-section 300 mm

Parameters:

$h := 150; \#mm$
$b := 100; \#mm$
$l := 300; \#mm$
$v := 0.29; \#`-`$
$T := 10E6; \#Nmm$
$E := 210000; \#MPa$
$G := \frac{E}{2\cdot(1+v)}; \#MPa$
$Iw := 2.94E7; \#mm4$
$Cw := 3.79E9; \#mm6$
$\mathrm{psi} := 1.4E3; \#mm2$
$mx := 0; \#\frac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 300$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffnesses:

$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Characteristic length:

$lc := \mathrm{sqrt}\left(\frac{ECw}{GIw}\right);$

$$lc := 18.23710056 \tag{3}$$

Vlasov equation:

$with(DEtools):$
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \quad \textbf{(4)}$$

Boundary conditions:

$bound\_con := \mathrm{phi}(0) = 0, \mathrm{D}(\mathrm{phi})(0) = 0, GIw \cdot \mathrm{D}(\mathrm{phi})(l) - ECw \cdot (\mathrm{D}@@3)(\mathrm{phi})(l) = T, (\mathrm{D}@@2)(\mathrm{phi})(l) = 0;$

$$bound\_con := \phi(0) = 0, \mathrm{D}(\phi)(0) = 0, 2.393023256\ 10^{12}\ \mathrm{D}(\phi)(300) - 7.9590000\ 10^{14}\ \mathrm{D}^{(3)}(\phi)(300) = 1.0\ 10^{7}, \mathrm{D}^{(2)}(\phi)(300) = 0 \quad \textbf{(5)}$$

Solving Vlasov equation:

$Sol := evalf(dsolve(\{ODE, bound\_con\}, \{\mathrm{phi}(x)\}));$
$assign(Sol); \mathrm{phi} := \mathrm{phi}(x)$

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.924053381\ 10^{-19}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x}$$

$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.924053381\ 10^{-19}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x} \quad \textbf{(6)}$$

Bi-moment and torsional moment:

$B := -ECw \cdot diff(\mathrm{phi}, x, x);$
$Mw1 := GIw \cdot diff(B, x);$
$Mw2 := diff(B, x);$
$Mwtot := Mw1 + Mw2;$

$$B := 9.390351001\ 10^{-7}\ e^{0.05483327774\,x} - 1.823710056\ 10^{8}\ e^{-0.05483327774\,x}$$

$$Mw1 := 123217.6587\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x}$$

$$Mw2 := 5.149037245\ 10^{-8}\ e^{0.05483327774\,x} + 1.000000000\ 10^{7}\ e^{-0.05483327774\,x}$$

$$Mwtot := 123217.6587\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x} \quad \textbf{(7)}$$

$x := 0 : B; x :='x';$
$\#plot(B, x = 0..l);$

$$-1.823710056\ 10^{8}$$

$$x := x \quad \textbf{(8)}$$

Rotations and displacements:

$x := l : phi\_max := \mathrm{phi};$

$$phi\_max := 0.001177434857 \quad \textbf{(9)}$$

$u\_max := phi\_max \cdot 0.5 \cdot h;$

$$u\_max := 0.08830761430 \quad \textbf{(10)}$$

$x := 0 : sigma\_max := -\frac{B}{Cw} \cdot \mathrm{psi};$

$$sigma\_max := 67.36659838 \quad \textbf{(11)}$$

>

$restart:$

Beam 3: Rectangular solid cross-section 600 mm

Parameters:
$h := 150; \#mm$
$b := 100; \#mm$
$l := 600; \#mm$
$v := 0.29; \#`-`$
$T := 10E6; \#Nmm$
$E := 210000; \#MPa$
$G := \dfrac{E}{2\cdot(1+v)}; \#MPa$
$Iw := 2.94E7; \#mm4$
$Cw := 3.79E9; \#mm6$
$\text{psi} := 1.4E3; \#mm2$
$mx := 0; \#\dfrac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 600$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffnesses:
$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Characteristic length:
$lc := \text{sqrt}\left(\dfrac{ECw}{GIw}\right);$

$$lc := 18.23710056 \tag{3}$$

Vlasov equation:
$with(DEtools):$
$ODE := ECw\cdot diff(\text{phi}(x), x, x, x, x) - GIw\cdot diff(\text{phi}(x), x, x) = mx;$

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{4}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 2.393023256\ 10^{12}\ D(\phi)(600) - 7.9590000\ 10^{14}\ D^{(3)}(\phi)(600) = 1.0\ 10^{7}, D^{(2)}(\phi)(600) = 0 \tag{5}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));

assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 2.020509704\ 10^{-33}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x}$$

$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 2.020509704\ 10^{-33}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x} \tag{6}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);

Mw1 := *GIw*·*diff*(*B*, *x*);

Mw2 := *diff*(*B*, *x*);

Mwtot := *Mw1* + *Mw2*;

$$B := 4.835126712\ 10^{-21}\ e^{0.05483327774\,x} - 1.823710056\ 10^{8}\ e^{-0.05483327774\,x}$$

$$Mw1 := 6.344523150\ 10^{-10}\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x}$$

$$Mw2 := 2.651258459\ 10^{-22}\ e^{0.05483327774\,x} + 1.000000000\ 10^{7}\ e^{-0.05483327774\,x}$$

$$Mwtot := 6.344523150\ 10^{-10}\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x} \tag{7}$$

x := 0 : *B*; *x* :='*x*';

#*plot*(*B*, *x* = 0 ..*l*);

$$-1.823710056\ 10^{8}$$

$$x := x \tag{8}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.002431079172 \tag{9}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.1823309379 \tag{10}$$

$x := 0 : sigma\_max := -\frac{B}{Cw}\cdot \mathrm{psi};$

$$sigma\_max := 67.36659838 \tag{11}$$

[>

restart :

Beam 4: Rectangular solid cross-section 1200 mm

Parameters:

$h := 150;$ #*mm*
$b := 100;$ #*mm*
$l := 1200;$ #*mm*
$v := 0.29;$ #\`-\`
$T := 10E6;$ #*Nmm*
$E := 210000;$ #*MPa*
$G := \dfrac{E}{2\cdot(1+v)};$ #*MPa*
$Iw := 2.94E7;$ #*mm4*
$Cw := 3.79E9;$ #*mm6*
psi := 1.4E3; #*mm2*
$mx := 0;$ $\#\dfrac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 1200$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffnesses:

$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Characteristic length:

$lc := \mathrm{sqrt}\left(\dfrac{ECw}{GIw}\right);$

$$lc := 18.23710056 \tag{3}$$

Vlasov equation:

with(*DEtools*) :
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \quad \textbf{(4)}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 2.393023256\ 10^{12}\ D(\phi)(1200) - 7.9590000\ 10^{14}\ D^{(3)}(\phi)(1200) = 1.0\ 10^{7}, D^{(2)}(\phi)(1200) = 0 \quad \textbf{(5)}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 5.356893223\ 10^{-62}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x}$$

$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 5.356893223\ 10^{-62}\ e^{0.05483327774\,x} + 0.00007620945812\ e^{-0.05483327774\,x} \quad \textbf{(6)}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 1.281917006\ 10^{-49}\ e^{0.05483327774\,x} - 1.823710056\ 10^{8}\ e^{-0.05483327774\,x}$$

$$Mw1 := 1.682096997\ 10^{-38}\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x}$$

$$Mw2 := 7.029171123\ 10^{-51}\ e^{0.05483327774\,x} + 1.000000000\ 10^{7}\ e^{-0.05483327774\,x}$$

$$Mwtot := 1.682096997\ 10^{-38}\ e^{0.05483327774\,x} + 2.393023256\ 10^{19}\ e^{-0.05483327774\,x} \quad \textbf{(7)}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.823710056\ 10^{8}$$

$$x := x \quad \textbf{(8)}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.004938367802 \quad \textbf{(9)}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.3703775852 \quad \textbf{(10)}$$

x := 0 : *sigma_max* := $-\frac{B}{Cw}$·psi;

$$sigma\_max := 67.36659838 \quad \textbf{(11)}$$

[>

restart :

Beam 5: Rectangular solid cross-section 2400 mm

Parameters:

$h := 150$; #*mm*
$b := 100$; #*mm*
$l := 2400$; #*mm*
$v := 0.29$; #\`-\`
$T := 10E6$; #*Nmm*
$E := 210000$; #*MPa*
$G := \dfrac{E}{2\cdot(1+v)}$; #*MPa*
$Iw := 2.94E7$; #*mm4*
$Cw := 3.79E9$; #*mm6*
psi := 1.4E3; #*mm2*
$mx := 0$; # $\dfrac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$l := 2400$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 2.94\ 10^{7}$$
$$Cw := 3.79\ 10^{9}$$
$$\psi := 1400.$$
$$mx := 0 \tag{1}$$

Stiffnesses:

$ECw := E\cdot Cw$;
$GIw := G\cdot Iw$;

$$ECw := 7.9590000\ 10^{14}$$
$$GIw := 2.393023256\ 10^{12} \tag{2}$$

Characteristic length:

$lc := \mathrm{sqrt}\left(\dfrac{ECw}{GIw}\right)$;

$$lc := 18.23710056 \tag{3}$$

Vlasov equation:

with(*DEtools*) :
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx$;

$$ODE := 7.9590000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 2.393023256\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \quad \textbf{(4)}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 2.393023256\ 10^{12}\ D(\phi)(2400) - 7.9590000\ 10^{14}\ D^{(3)}(\phi)(2400) = 1.0\ 10^{7}, D^{(2)}(\phi)(2400) = 0 \quad \textbf{(5)}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.765451918\ 10^{-119}\ e^{0.05483327774\,x} + 0.00007620945808\ e^{-0.05483327774\,x}$$

$$\phi := -0.00007620945812 + 4.178814383\ 10^{-6}\ x - 3.765451918\ 10^{-119}\ e^{0.05483327774\,x} + 0.00007620945808\ e^{-0.05483327774\,x} \quad \textbf{(6)}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 9.010814004\ 10^{-107}\ e^{0.05483327774\,x} - 1.823710055\ 10^{8}\ e^{-0.05483327774\,x}$$

$$Mw1 := 1.182374764\ 10^{-95}\ e^{0.05483327774\,x} + 2.393023255\ 10^{19}\ e^{-0.05483327774\,x}$$

$$Mw2 := 4.940924669\ 10^{-108}\ e^{0.05483327774\,x} + 9.999999996\ 10^{6}\ e^{-0.05483327774\,x}$$

$$Mwtot := 1.182374764\ 10^{-95}\ e^{0.05483327774\,x} + 2.393023255\ 10^{19}\ e^{-0.05483327774\,x} \quad \textbf{(7)}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.823710055\ 10^{8}$$

$$x := x \quad \textbf{(8)}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.009952945062 \quad \textbf{(9)}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.7464708795 \quad \textbf{(10)}$$

x := 0 : *sigma_max* := $-\frac{B}{Cw}$·psi;

$$sigma\_max := 67.36659833 \quad \textbf{(11)}$$

[>

restart :

Beam 6: rectangular closed tube cross-section 150 mm

Parameters:

$h := 150;$ *#mm*
$b := 100;$ *#mm*
$t := 10;$ *#mm*
$l := 150;$ *#mm*
$v := 0.29;$ #\`-\`
$T := 10E6;$ *#Nmm*
$E := 210000;$ *#MPa*
$G := \dfrac{E}{2\cdot(1+v)};$ *#MPa*
$Iw := 1.44E7;$ *#mm4*
$Cw := 9.023E8;$ *#mm6*
psi $:= 1E3;$ *#mm2*
$mx := 0;$ $\#\dfrac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 150$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffnesses:

$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:

$lc := \text{sqrt}\left(\dfrac{ECw}{GIw}\right);$

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:

with(*DEtools*) :
$ODE := ECw\cdot diff(\text{phi}(x), x, x, x, x) - GIw\cdot diff(\text{phi}(x), x, x) = mx;$

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \quad \textbf{(4)}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12}\ D(\phi)(150) - 1.894830000\ 10^{14}\ D^{(3)}(\phi)(150) = 1.0\ 10^{7}, D^{(2)}(\phi)(150) = 0 \quad \textbf{(5)}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 6.140872277\ 10^{-15}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x}$$

$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 6.140872277\ 10^{-15}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x} \quad \textbf{(6)}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 0.007197673551\ e^{0.07864949064\,x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\,x}$$

$$Mw1 := 6.635140760\ 10^{8}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x}$$

$$Mw2 := 0.0005660933586\ e^{0.07864949064\,x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\,x}$$

$$Mwtot := 6.635140760\ 10^{8}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x} \quad \textbf{(7)}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.271464052\ 10^{8}$$

$$x := x \quad \textbf{(8)}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.001171283821 \quad \textbf{(9)}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.08784628660 \quad \textbf{(10)}$$

$x := 0 : sigma\_max := -\frac{B}{Cw} \cdot \mathrm{psi};$

$$sigma\_max := 140.9136708 \quad \textbf{(11)}$$

[>

$restart:$

Beam 7: rectangular closed tube cross-section 300 mm

Parameters:

$h := 150; \#mm$
$b := 100; \#mm$
$t := 10; \#mm$
$l := 300; \#mm$
$v := 0.29; \#`-`$
$T := 10E6; \#Nmm$
$E := 210000; \#MPa$
$G := \dfrac{E}{2\cdot(1+v)}; \#MPa$
$Iw := 1.44E7; \#mm4$
$Cw := 9.023E8; \#mm6$
$\mathrm{psi} := 1E3; \#mm2$
$mx := 0; \#\dfrac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 300$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffnesses:

$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:

$lc := \mathrm{sqrt}\left(\dfrac{ECw}{GIw}\right);$

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:

$with(DEtools):$
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \qquad \textbf{(4)}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12}\ D(\phi)(300) - 1.894830000\ 10^{14}\ D^{(3)}(\phi)(300) = 1.0\ 10^{7}, D^{(2)}(\phi)(300) = 0 \qquad \textbf{(5)}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 3.476307080\ 10^{-25}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x}$$

$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 3.476307080\ 10^{-25}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x} \qquad \textbf{(6)}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 4.074555276\ 10^{-13}\ e^{0.07864949064\,x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\,x}$$

$$Mw1 := 0.03756109192\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x}$$

$$Mw2 := 3.204616970\ 10^{-14}\ e^{0.07864949064\,x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\,x}$$

$$Mwtot := 0.03756109192\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x} \qquad \textbf{(7)}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.271464052\ 10^{8}$$

$$x := x \qquad \textbf{(8)}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.002451045726 \qquad \textbf{(9)}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.1838284294 \qquad \textbf{(10)}$$

x := 0 : *sigma_max* := $-\frac{B}{Cw}$·psi;

$$sigma\_max := 140.9136708 \qquad \textbf{(11)}$$

[>

restart :

Beam 8: rectangular closed tube cross-section 600 mm

Parameters:
$h := 150;$ *#mm*
$b := 100;$ *#mm*
$t := 10;$ *#mm*
$l := 600;$ *#mm*
$v := 0.29;$ #`-`
$T := 10E6;$ *#Nmm*
$E := 210000;$ *#MPa*
$G := \frac{E}{2\cdot(1+v)};$ *#MPa*
$Iw := 1.44E7;$ *#mm4*
$Cw := 9.023E8;$ *#mm6*
psi $:= 1E3;$ *#mm2*
$mx := 0;\ \#\frac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 600$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffnesses:
$ECw := E\cdot Cw;$
$GIw := G\cdot Iw;$

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:
$lc := \mathrm{sqrt}\left(\frac{ECw}{GIw}\right);$

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:
with(*DEtools*) :
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \quad \textbf{(4)}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12}\ D(\phi)(600) - 1.894830000\ 10^{14}\ D^{(3)}(\phi)(600) = 1.0\ 10^{7}, D^{(2)}(\phi)(600) = 0 \quad \textbf{(5)}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.114023266\ 10^{-45}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x}$$

$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.114023266\ 10^{-45}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x} \quad \textbf{(6)}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 1.305738897\ 10^{-33}\ e^{0.07864949064\,x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\,x}$$

$$Mw1 := 1.203689125\ 10^{-22}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x}$$

$$Mw2 := 1.026956992\ 10^{-34}\ e^{0.07864949064\,x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\,x}$$

$$Mwtot := 1.203689125\ 10^{-22}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x} \quad \textbf{(7)}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.271464052\ 10^{8}$$

$$x := x \quad \textbf{(8)}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.005010569536 \quad \textbf{(9)}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.3757927152 \quad \textbf{(10)}$$

$x := 0 : sigma\_max := -\frac{B}{Cw} \cdot \mathrm{psi};$

$$sigma\_max := 140.9136708 \quad \textbf{(11)}$$

[>

restart :

Beam 9: rectangular closed tube cross-section 1200 mm

Parameters:
$h := 150$; *#mm*
$b := 100$; *#mm*
$t := 10$; *#mm*
$l := 1200$; *#mm*
$v := 0.29$; #\`-\`
$T := 10E6$; *#Nmm*
$E := 210000$; *#MPa*
$G := \frac{E}{2\cdot(1+v)}$; *#MPa*
$Iw := 1.44E7$; *#mm4*
$Cw := 9.023E8$; *#mm6*
psi := 1E3; *#mm2*
$mx := 0$; $\#\frac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 1200$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffnesses:
$ECw := E\cdot Cw$;
$GIw := G\cdot Iw$;

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:
$lc := \mathrm{sqrt}\left(\frac{ECw}{GIw}\right)$;

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:
with(*DEtools*) :
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx;$

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{4}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12}\ D(\phi)(1200) - 1.894830000\ 10^{14}\ D^{(3)}(\phi)(1200) = 1.0\ 10^{7}, D^{(2)}(\phi)(1200) = 0 \tag{5}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.144053959\ 10^{-86}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x}$$

$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.144053959\ 10^{-86}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x} \tag{6}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 1.340937663\ 10^{-74}\ e^{0.07864949064\,x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\,x}$$

$$Mw1 := 1.236136938\ 10^{-63}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x}$$

$$Mw2 := 1.054640642\ 10^{-75}\ e^{0.07864949064\,x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\,x}$$

$$Mwtot := 1.236136938\ 10^{-63}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x} \tag{7}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.271464052\ 10^{8}$$

$$x := x \tag{8}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.01012961716 \tag{9}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 0.7597212870 \tag{10}$$

$x := 0 : sigma\_max := -\frac{B}{Cw} \cdot psi;$

$$sigma\_max := 140.9136708 \tag{11}$$

[>

restart :

Beam 10: rectangular closed tube cross-section 2400 mm

Parameters:

$h := 150$; #*mm*
$b := 100$; #*mm*
$t := 10$; #*mm*
$l := 2400$; #*mm*
$v := 0.29$; #`-`
$T := 10E6$; #*Nmm*
$E := 210000$; #*MPa*
$G := \frac{E}{2\cdot(1+v)}$; #*MPa*
$Iw := 1.44E7$; #*mm4*
$Cw := 9.023E8$; #*mm6*
psi := 1E3; #*mm2*
$mx := 0$; #$\frac{Nm}{m}$

$$h := 150$$
$$b := 100$$
$$t := 10$$
$$l := 2400$$
$$v := 0.29$$
$$T := 1.0\ 10^{7}$$
$$E := 210000$$
$$G := 81395.34883$$
$$Iw := 1.44\ 10^{7}$$
$$Cw := 9.023\ 10^{8}$$
$$\psi := 1000.$$
$$mx := 0 \tag{1}$$

Stiffnesses:

$ECw := E\cdot Cw$;
$GIw := G\cdot Iw$;

$$ECw := 1.894830000\ 10^{14}$$
$$GIw := 1.172093023\ 10^{12} \tag{2}$$

Characteristic length:

$lc := \mathrm{sqrt}\left(\frac{ECw}{GIw}\right)$;

$$lc := 12.71464051 \tag{3}$$

Vlasov equation:

with(*DEtools*) :
$ODE := ECw\cdot diff(\mathrm{phi}(x), x, x, x, x) - GIw\cdot diff(\mathrm{phi}(x), x, x) = mx$;

$$ODE := 1.894830000\ 10^{14}\ \frac{d^4}{dx^4}\ \phi(x) - 1.172093023\ 10^{12}\ \frac{d^2}{dx^2}\ \phi(x) = 0 \tag{4}$$

Boundary conditions:

bound_con := phi(0) = 0, D(phi)(0) = 0, *GIw*·D(phi)(*l*) − *ECw*·(D@@3)(phi)(*l*) = *T*, (D@@2)(phi)(*l*) = 0;

$$bound\_con := \phi(0) = 0, D(\phi)(0) = 0, 1.172093023\ 10^{12}\ D(\phi)(2400) - 1.894830000\ 10^{14}\ D^{(3)}(\phi)(2400) = 1.0\ 10^{7}, D^{(2)}(\phi)(2400) = 0 \tag{5}$$

Solving Vlasov equation:

Sol := *evalf*(*dsolve*({*ODE*, *bound_con*}, {phi(*x*)}));
assign(*Sol*); phi := phi(*x*)

$$Sol := \phi(x) = -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.206565900\ 10^{-168}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x}$$

$$\phi := -0.0001084780838 + 8.531746034\ 10^{-6}\ x - 1.206565900\ 10^{-168}\ e^{0.07864949064\,x} + 0.0001084780838\ e^{-0.07864949064\,x} \tag{6}$$

Bi-moment and torsional moment:

B := −*ECw*·*diff*(phi, *x*, *x*);
Mw1 := *GIw*·*diff*(*B*, *x*);
Mw2 := *diff*(*B*, *x*);
Mwtot := *Mw1* + *Mw2*;

$$B := 1.414207473\ 10^{-156}\ e^{0.07864949064\,x} - 1.271464052\ 10^{8}\ e^{-0.07864949064\,x}$$

$$Mw1 := 1.303680360\ 10^{-145}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x}$$

$$Mw2 := 1.112266974\ 10^{-157}\ e^{0.07864949064\,x} + 1.000000001\ 10^{7}\ e^{-0.07864949064\,x}$$

$$Mwtot := 1.303680360\ 10^{-145}\ e^{0.07864949064\,x} + 1.172093024\ 10^{19}\ e^{-0.07864949064\,x} \tag{7}$$

x := 0 : *B*; *x* :='*x*';
#*plot*(*B*, *x* = 0 ..*l*);

$$-1.271464052\ 10^{8}$$

$$x := x \tag{8}$$

Rotations and displacements:

x := *l* : *phi_max* := phi;

$$phi\_max := 0.02036771240 \tag{9}$$

u_max := *phi_max*·0.5·*h*;

$$u\_max := 1.527578430 \tag{10}$$

$x := 0 : sigma\_max := -\frac{B}{Cw}\cdot \mathrm{psi};$

$$sigma\_max := 140.9136708 \tag{11}$$

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