CT5060 Master Thesis

A Parametric Structural Design Tool for Shell Structures

Delft University of Technology Faculty of Civil Engineering and Geosciences Master track: Building Engineering Specialization: Structural Design

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Preface

This thesis concludes my Master program Building Engineering at the faculty of Civil Engineering and Geosciences at Delft University of Technology, with a specialisation in Structural Design. This research has been performed during the period June 2014 till April 2015. This report is intended to provide an overview of the results and background of this research for both the members of the graduation committee and others who are interested.

During my studies I always had an interest in structures and the (computational) analysis thereof. After following the course *special structures* however, I became especially fascinated by so called *shell* structures. The combination of my interests in computation and shell structures provided the inspiration for this research topic: *A Parametric Structural Design Tool for Shell Structures*.

I would like to express my gratitude towards my supervisors: ir. A. Borgart, ir. S. Pasterkamp, dr. ir.P.C.J. Hoogenboom and prof.dr.ir. J.G. Rots for their guidance, advice and suggestions during the progress of this thesis. I would especially like to thank my first supervisor Andrew Borgart for sharing much of his knowledge and being available almost every week to meet up with me and discuss my progress. Last but not least, I would like to express my appreciation to my family for their love and support and giving me the ability to study, and also my roommates and friends for their support and inspiration during my entire education.

Delft, April 2015 K. W. Riemens

Abstract

This thesis presents a parametric design tool for structural analysis of shallow shells. The tool was developed with Rhino and Grasshopper. Also VB (Visual Basic) scripting was used to create new components in the Grasshopper interface. The techniques and methods involved for the development of the tool consists mainly of classic analytical and numerical methods like the finite difference method.

In contrast to FEM (Finite Element Method) based analysis programs, the current tool offers more qualitative insight into the behaviour of shell structures which is often more important during a conceptual design stage. This is mainly because of the parametric and real time capabilities of the tool and the way in which it presents the results. The tool was tested on several test cases whereby the results were compared to analytical and FEM solutions and showed good correspondence. The results of the developed tool are sufficiently accurate for a conceptual design stage and give fast quantitative and qualitative insight into the behavior of shell structures. Still, several limitations are encountered, especially with respect to the boundary conditions, these provide opportunities for future research.

List of symbols

Symbol	Meaning		
m_{xx}, m_{yy}, m_{xy}	Bending and twisting moments (per unit length) respectively		
n_{xx}, n_{yy}, n_{xy}	Normal and shear forces (per unit length) respectively		
φ	Airy stress function		
W	Displacement perpendicular to the surface		
k_G	Original Gaussian curvature of unrestrained surface		
g	Change of Gaussian curvature		
k_1, k_2	Principal curvatures		
k_{xx}, k_{yy}, k_{xy}	Curvatures and twist of a surface		
$\kappa_{xx}, \kappa_{yy}, \kappa_{xy}$	Curvatures and twist caused by bending moments		
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$	Strains caused by stretching		
D	Flexural rigidity		
Ε	Young's modulus		
t	Thickness of the shell		
R_1, R_2	Radius of principal curvatures		
v_x, v_y	Shear forces		
v _n	Principal shear force		
p	Total load		
p_S, p_B	Load carried by stretching, load carried by bending respectively		
$arphi_x$, $arphi_y$	Angle		
ν	Lateral contraction (or Poission's ratio)		
f	Kirchhoff shear force		

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1. Introduction

1.1. Background and problem statements

Within the field of structural and architectural engineering it can be observed that the use of advanced geometry and computation is increasing (Coenders, 2006). Computation is used to generate complex geometries and to perform structural analysis. An iterative process is often necessary for complex (structural) design, which consists of design-, calculation- and production phases where data exchange can occur between CAD (Computer Aided Design) programs and FEM (Finite Element Method) programs (Borgart, Hoogenboom & de Leeuw, 2005). As a result of these processes and new technological advances, the possibilities for free form architecture have significantly increased. An example of free form architecture are shell structures. Shell structures are lightweight structures, typically (double) curved and can be used to create large covered spaces. An example of a shell structure is given below (Fig. 1.1).



Fig. 1.1 - Meiso no Mori Municipal Funeral Hall (Kakamigahara, Japan)

Much knowledge is currently available regarding the mechanical behavior of geometrically regular curved surfaces like most shells structures are (Flügge 1960). This is mainly caused by the fact that these surfaces are relatively easy to describe by analytical mathematical functions. For describing irregular curved surfaces, like those in free form architecture, few analytical mathematical functions exist and therefore it is hard to derive formulas to describe their mechanical behavior. One way of dealing with this problem is to calculate the stresses and strains of these irregular curved structures with computer programs based on the FEM (finite element method).

However the problem is that one only obtains quantitative information about the results (like the magnitude of the forces) but no qualitative information. This does not always give clear insight into the structural behavior. Qualitative insight in this respect refers to insight in the relation between parameters such as structural geometry, boundary conditions and materials properties and the resulting deformations and stress resultants (quantitative information). For example, what is the relation between the shape of the curved surface and the flow of forces? Because of the lack of insight it can be difficult to design irregular curved surfaces which have shell-like behavior (that is, carrying the load mainly by normal forces and little by bending moments).

During the conceptual stage of the design process, many important design decisions are made with regard to structural considerations, laying the basis for the rest of the project. The first structural setup is usually conceived at this stage. Qualitative- and global quantitative insight in the mechanical behaviour of the structure is therefore very important. When such insight is obtained in the conceptual stage of the design process it could be employed respect to esthetical appearance or constructability at an early stage, could lead to a reduction of risk and cost, and thus reduce problems during later stages.

FEM based structural analysis programs might thus not be the most appropriate computational structural analysis tools in the conceptual design stage. Additionally, these analysis programs require a rather detailed structural model and the results produced are unnecessarily precise for the conceptual design stage. Most of the existing software tools for structural analysis are oriented towards advanced users and require a detailed understanding of the program and its underlying principles. Moreover, the calculation procedures within FEM programs in combination with the necessary interfacing between different CAD programs decreases the speed and flexibility of the design and analysis process.

In contrast to FEM programs, (classical) analytical methods offer, apart from quantitative insight, qualitative insight into the mechanical behaviour for a wide range of structural topologies. Graphic statics is a good example of such a method in which analytical relations between the structural geometry and the corresponding mechanical behaviour are used to generate a graphical representation of the flow and magnitude of forces.

A recent development in the field of computational design are parametric associative design tools which capture design information by defining logical relations between (geometrical) components, controlled by parameters. These techniques offer a very flexible approach to exploring complex geometries and are, currently, mainly used within the field of architectural design. Despite the wide range of possibilities for linking geometry to structural analysis they still find little application within the field of structural engineering at the moment.



Fig. 1.2 - Example of parametric associative design tool (Rhino and Grasshopper)

Successful attempts have been made however. In 2010 M. Oosterhuis developed a tool to analyse plates loaded out-of-plane. In 2012, D. Liang expanded this tool and developed a second tool to analyse plates loaded in-plane. With these models (Fig. 1.3) the internal forces, moments, displacements etc. of a surface can be calculated. This thesis can be seen as an extension of their previous work and will thus combine these in-plane and out-off-plane models and introduce curvature to design and calculate shell structures.



Fig. 1.3 - Parametric structural models by M. Oosterhuis (2010) and D. Liang (2012)

Following M. Oosterhuis and D. Liang the two problem statements, forming the incentive for writing this thesis, are deduced from this background:

1) There is a need for simple structural analysis tools, based on analytical relations, which give both the architect and engineer quantitative insight as well as qualitative insight in the (flow of) forces of (shell)-structures during a conceptual design stage.

2) Parametric design applications are not used to their full potential within the field of structural engineering.

1.2. Objectives and approach

The purpose of this research project is to be able to analyse the structural behavior of shell structures by studying the way (applied) loads flow through the shell's surface to the supports and how this relates to the shell's geometry. To unlock this secret will give fundamental understanding of the behavior of shell structures and thus the means to design shells with efficiency of performance and elegance of form. The main objective for this thesis can be defined as follows:

"Develop a parametric structural design tool for shell structures that can be used by architects and engineers, which is based on simple analytical methods, which gives both quantitative and qualitative (real time) insight in the flow and magnitude of forces during a conceptual design stage."

The development of the new envisioned tool for shallow shell structures which combines bending and stretching is considered the next logical step after the development of the tools by Oosterhuis and Liang. The basic approach which will be used here is thus similar to their approach:

- Provide the theoretical framework covering all the theory and methods that need to be used
- Define which demands have to be fulfilled with respect to functionality and usability by the parametric tool
- Provide a general outline and structure for the parametric tool in accordance with the demands
- Implement the theoretical framework into the structural design tool
- Choose several test cases and use the tool to analyse them
- Validate the results in a qualitative and quantitative manner by comparing the results to analytical solutions or FEM results

1.3. Scope

Although there is a broad range of structural topologies, the scope of this thesis is confined to isotropic shallow shell structures. A shallow shell is defined as a (thin) shell having a relatively small raise as compared to its span. Such shells have a wide application in engineering, for example in roof structures. Moreover many practically important problems lie within the scope of shallow-shell theory.

Furthermore, the following limitations or restrictions are applied in this thesis:

- The thickness of the shell is constant
- Only physically linear behavior is considered
- Confined to shell (or plate) structures with a rectangular projected floorplan
- Only loads perpendicular to the shell will be considered, these do not have to be uniformly distributed however

Finally, the following assumptions are made:

- The effect of the transverse shear forces v_x and v_y in the in-plane-equilibrium equation is negligible (the static assumption, see 3.2.2)
- The influence of the transverse deflections, w, will predominate over the influence of the in-plane displacements u_x and u_y in the bending response of the shell (the geometric assumption)
- The lateral contraction (or Poisson's ratio) is zero ($\nu = 0$)

2. General definitions and fundamentals of surfaces and shells

2.1. Shells in general

Thins shells as structural elements are considered as occupying a special position in engineering, in particular in civil, architectural, aeronautical and marine engineering. Examples of shell structures in civil and architectural engineering are: water tanks, large-span roofs, concrete domes etc. The wide range of application of using shell structures can be explained by their having the following advantages (Ventsel, 2001):

- Efficiency of load-carrying behavior
- High degree of reserved strength integrity
- High strength: weight ratio
- High stiffness
- Containment of space

Moreover, apart from these mechanical advantages, shell structures have the unique position of having high aesthetic value in various architectural designs.



Fig. 2.1 - Shell characteristics (Ventsel, 2001)

The term shell is used for bodies which are bounded by two curved surfaces and whereby the distance between these surfaces is small in comparison with other body dimensions. The surface of points that lie at equal distances from these two curved surfaces is called the middle surface of the shell. The length of the segment, which is perpendicular to the curved surfaces, is called the thickness of the shell and is denoted by t (Fig. 2.1). The geometry of a shell is defined by specifying the geometry of the middle surface and the thickness of the shell at each point. In this thesis only shells of a constant thickness are considered however.

Shells have all the characteristics of plates, along with an additional one: namely curvature. Curvature can be considered as the main classifier of a shell due to the fact that a shell's mechanical behavior is primarily governed by curvature. Due to the curvature of the surface, a shell's behavior is in general more complicated than that of flat plates because their bending cannot, in general, be separated from their stretching. On the other hand, a plate may be considered as a special limit case of a shell that has zero curvature (see also 5.2); consequently, shells are sometimes referred to as curved plates. There are two different classes of shells: thick shells and thin shells. For engineering purposes, a shell may be regarded as thin if the following condition is satisfied:

$$\max\left(\frac{t}{R}\right) \le \frac{1}{20} \tag{2.1}$$

Hence, shells for which this inequality does not hold are referred to as *thick shells*.

The complexity of the governing equations of the general linear theory of thin shells lead to the development of a wide range of approximate theories associated with simplifications of these equations. Donnel (1933),Mushtari (1938) and Vlasov (1964) independently developed a simplified approximate theory of thin shells of a general form. Due to their simplicity, the governing equations (see 3.2.7) of this theory were found to be extremely convenient for solving many engineering shell problems. Apart from the Kirchhoff–Love hypotheses, some additional assumptions that simplify the strain–displacement relations, equilibrium, and compatibility equations were used in deriving these equations. It turned out that the Donnel-Vlasov-Mushtari theory could be applied with sufficient accuracy to shallow shells as well.

As been stated in section 1.3 a shallow shell is defined as a (thin) shell having a relatively small raise as compared to its span. According to Ventsel (2001) a shell is said to be shallow if at any point of its middle surface the following inequalities hold¹:

$$\left(\frac{\partial z}{\partial x}\right)^2 \ll 1, \qquad \left(\frac{\partial z}{\partial y}\right)^2 \ll 1$$
 (2.2)

Or more specifically:

$$\left(\frac{\partial z}{\partial x}\right)^2 < \frac{1}{20}, \qquad \left(\frac{\partial z}{\partial y}\right)^2 < \frac{1}{20}$$
 (2.3)

¹ Vlasov (1964) defined a shallow shell as a shell whose rise does not exceed 1/5 of the smallest dimension of the shell in its plane (projection on the coordinate plane Oxy). It can be shown however that this practical limitation of the applicability of the shallow shell theory corresponds to an error noticeably exceeding 5% (Novozhilov, 1964).

2.2. Coordinate system

Lines and surfaces can be described in a global Cartesian coordinate system $(\overline{x}, \overline{y}, \overline{z})$. Local properties of surfaces can however also be described by a local coordinate system (x, y, z), see (Fig. 2.2). In the local coordinate system the *z*-direction is perpendicular to the surface and the *x*-and *y*-direction are tangent to the surface.



Fig. 2.2 - Global and local coordinate system (Lecture notes, Hoogenboom)

The right-hand-rule is used to determine which axis is x and which is y (Fig. 2.3).



Fig. 2.3 - Right-hand-rule to remember the Cartesian coordinate system (Lecture notes, Hoogenboom)

2.3. Surfaces and curvature

2.3.1. Curvatures of a surface

The curvature is the reciprocal of the radius of curvature:



Fig. 2.4 - Curvatures on surface (Lecture notes, Hoogenboom)

In *local* coordinate system (z-axis perpendicular to the surface):

Curvature in x- and y-direction:

$$k_{xx} = \frac{\partial^2 z}{\partial x^2}, \qquad k_{yy} = \frac{\partial^2 z}{\partial y^2}$$
 (2.5)

Twist of the surface:

$$k_{xy} = \frac{\partial^2 z}{\partial x \partial y} \tag{2.6}$$

In global coordinate system the curvatures become:

$$k_{\overline{x}\overline{x}} = \frac{\frac{\partial^2 \overline{z}}{\partial \overline{x}^2}}{\left(1 + \left(\frac{\partial \overline{z}}{\partial \overline{x}}\right)^2\right)^{\frac{3}{2}}}, \qquad k_{\overline{y}\overline{y}} = \frac{\frac{\partial^2 \overline{z}}{\partial \overline{y}^2}}{\left(1 + \left(\frac{\partial \overline{z}}{\partial \overline{y}}\right)^2\right)^{\frac{3}{2}}}$$
(2.7)

Note that strictly speaking these are not the same as the curvatures of a flat plate in bending, which are defined by:

$$\kappa_x = \frac{\partial \varphi_x}{\partial x} = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = \frac{\partial \varphi_y}{\partial y} = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$
(2.8)

Principal curvatures

At a point of a surface there will be a minimum value k_2 and maximum value k_1 . These are called the principal curvatures and are given by the following formulas:

$$k_{1} = \frac{1}{2} \left(k_{xx} + k_{yy} \right) + \sqrt{\frac{1}{4} \left(k_{xx} - k_{yy} \right)^{2} + k_{xy}^{2}}$$
(2.9)

$$k_{2} = \frac{1}{2} \left(k_{xx} + k_{yy} \right) - \sqrt{\frac{1}{4} \left(k_{xx} - k_{yy} \right)^{2} + k_{xy}^{2}}$$
(2.10)

2.3.2. Gaussian curvature

Carl Friedrich Gauss (1777-1855) was a German scientist who was famous for his work in mathematics. In his paper '*General investigation of curved surfaces*' (Gauss, 1827) he described the product of the principal curvatures as the measure of curvature. Thus, the Gaussian curvature of a surface at a point is defined as the product of the principal curvatures at that point:

$$k_G = k_1 \cdot k_2 \tag{2.11}$$

It can also be shown that:

$$k_G = k_{xx} \cdot k_{yy} - k_{xy}^2 \tag{2.12}$$

$$k_G = k_1 \cdot k_2 = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$
(2.13)

A positive value means the surface is bowl-like. A negative value means the surface is saddle-like. A zero value means the surface is flat in at least one direction (plates, cylinders, and cones have zero Gaussian curvature), see (Fig. 2.5).



Fig. 2.5 - Gaussian curvature (Lecture notes, Hoogenboom)

Due to deformation of a surface (in a shell this is caused by the applied loading) there will be a changes of Gaussian curvature. Following Calladine (1983) this change of Gaussian curvature is denoted by: g

2.3.3. NURBS surfaces

NURBS stands for Non Uniform Rational B-Spline. It is a way to define surfaces in a mathematical way and is commonly used in computer graphics. It was developed in the sixties to model smooth surfaces. A NURBS surface is determined by an order, weighted control points, and knots. A NUBRS surface can be seen as a generalization of B-splines which are lines that have a mathematical definition consisting of a number of curves that are added. The B-spline has a beginning point and an end point which the line goes through but it does not go through the intermediate points, these latter points are the control points.



The software uses this data to generate the surface. The shape can be changed by moving the control points.

3. Theoretical Framework

3.1. Introduction

As been stated in section 1.2, the first step to be undertaken will be the following:

• Provide the theoretical framework covering all the theories and methods that need to be used

The theoretical framework can be considered as the basis for the development of the tool. In accordance with the objective and approach defined in chapter 1, the theoretical framework consists mainly of (classic) analytical analysis methods for shell structures. These analytical methods provide the exact relation between the structural geometry, material properties and boundary conditions as parameters and the resulting internal forces and deformations in an unequivocal way by exact algebraic equations. They therefore give the engineer quantitative as well as qualitative insight on the mechanical behavior of the shell structure.

As with the work of M. Oosterhuis (2010) and D. Liang (2012) the emphasis within this thesis also lies on the computational application of these (classic) analytical theories. It is envisioned that the implementation of these methods results in a faster and more flexible structural analysis process, which is more appropriate for the conceptual design process.

3.2. Shell differential equations

3.2.1. Calladine's two-surfaces shell theory

The interaction between stretching and bending behavior in shell structures can be studied effectively by a so called two-surface theory which was proposed by Calladine (1983). According to this theory the surface of the shell can be conceptually split into two distinct surfaces which are designated the B-(or bending) surface and the S-(or stretching) surface. The S-surface possesses only in-plane stiffness and can carry only membrane forces but it cannot transmit bending (and twisting) moments and shear forces. The B-surface on the other hand only possesses flexural stiffness and can sustain bending (and twisting) moments and also transverse shear forces, but it cannot carry membrane forces.



Fig. 3.1 - Calladine's two surface shell theory: (a) Shell element showing positive sense of pressure loading, all stress resultants and displacement w. (b) The S-Surface (c) The B-surface

The idea of separating the behavior of a shell into two distinct parts affords the possibility of thinking separately about two different aspects of shell behavior while still allowing for the actual interaction between them. Since the "split" is only conceptual the B-surface and the S-surface still must coincide with each other, not only in the original configuration but also in the subsequent distortion of the shell caused by loading. This is achieved by simply stating that the values of g_s and g_B (the changes of Gaussian curvature) are equal to each other, thus the compatibility condition becomes:

$$g_S = g_B \tag{3.1}$$

In this two-surface model for a shell the force interaction is expressed in terms of an interface stress or pressure. Thus the applied load p_S and p_B carried by the S- and B-surface respectively are related to the applied loading p by the equilibrium equation:

$$p = p_S + p_B \tag{3.2}$$

Thus, the load-sharing between the two surfaces provides insight into the regime of behaviour into which a particular problem falls.

3.2.2. Equilibrium equations

The equilibrium equations that follow from considering an element are:

$$\frac{n_{xx}}{R_1} + \frac{n_{yy}}{R_2} - \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} = p \qquad (3.3)$$

$$\frac{\partial n_{xx}}{\partial x} + \frac{\partial n_{xy}}{\partial y} + \left[\frac{v_x}{R_1}\right] = 0 \qquad (3.4)$$

$$\frac{\partial n_y}{\partial y} + \frac{\partial n_{xy}}{\partial x} + \left[\frac{v_y}{R_2}\right] = 0 \qquad (3.5)$$

$$\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y} = v_x \qquad (3.6)$$

$$\frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x} = v_y \qquad (3.7)$$
Fig. 3.2 - Equilibrium of shell element with all stress resultants

These equations are somewhat untidy and can be made more useful by rearranging them into two separate sets of equations (one for bending and the other for stretching) in accordance with the two surface theory. Moreover, since we are dealing with shallow shells we may disregard the terms enclosed in squared brackets in (3.xx):

$$\Rightarrow \frac{v_x}{R_1} \approx 0, \qquad \frac{v_y}{R_2} \approx 0 \tag{3.8}$$

The general justification for this is simply that the denominators R_1 and R_2 in terms $\frac{v_x}{R_1}$ and $\frac{v_y}{R_2}$ are large for shallow shells, and consequently give only a very weak coupling between the bending and stretching effects; a coupling which disappears entirely, of course, in the case of a flat plate.

3.2.3. Equilibrium equations after separation of B-surface and S-surface

In accordance with the two surface concept the equilibrium equations after separation for the stretching surface become:



Fig. 3.3 - Equilibrium of stretching surface

The principal normal forces can be calculated with the following formulas:

$$n_1 = \frac{n_{xx} + n_{yy}}{2} + \sqrt{\left(\frac{n_{xx} - n_{yy}}{2}\right)^2 + n_{xy}^2},$$
(3.12)

$$n_2 = \frac{n_{xx} + n_{yy}}{2} - \sqrt{\left(\frac{n_{xx} - n_{yy}}{2}\right)^2 + n_{xy}^2}$$
(3.13)

The equilibrium equations for the bending surface become:



Fig. 3.4 - Equilibrium of bending surface

These last three equations can be combined and lead to the following equation:

$$\Rightarrow \frac{\partial^2 m_{xx}}{\partial x^2} + 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} = -p_B \tag{3.17}$$

3.2.4. Constitutive equations

The constitutive equations describe the relations between the internal forces and deformations. For a lateral contraction (or Poisson's ratio) equal to zero ($\nu = 0$) the constitutive relations for stretching behavior are:

$$\varepsilon_{xx} = \frac{n_{xx}}{Et}, \qquad \varepsilon_{yy} = \frac{n_{yy}}{Et}, \qquad \gamma_{xy} = \frac{2n_{xy}}{Et}$$
(3.18)



Fig. 3.5 - Constitutive relations stretching behavior for zero-poisson's ratio

For bending behavior the constitutive relations with poisson's ratio is zero are as follows:

$$m_{xx} = D\kappa_{xx}, \qquad m_{yy} = D\kappa_{yy}, \qquad m_{xy} = D\kappa_{xy}$$
(3.19)



Fig. 3.6 - Constitutive relations bending behavior for zero poisson's ratio

Where *D* is the flexural rigidity of the shell:

$$D = \frac{Et^3}{12} \tag{3.20}$$

The principal moments can be calculated with the following formulas:

$$m_{1} = \frac{m_{xx} + m_{yy}}{2} + \sqrt{\left(\frac{m_{xx} - m_{yy}}{2}\right)^{2} + m_{xy}^{2}}$$
(3.21)
$$m_{2} = \frac{m_{xx} + m_{yy}}{2} - \sqrt{\left(\frac{m_{xx} - m_{yy}}{2}\right)^{2} + m_{xy}^{2}}$$
(3.22)

3.2.5. Kinematic equations

The kinematic equations relate the deformations to the displacements. For stretching behavior the kinematic equations are:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2, \qquad (3.23)$$
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$

The strain compatibility equation (see appendix A for the derivation) is as follows:

$$-\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = g_s$$
(3.24)

For bending the kinematic equations are as follows:

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$
(3.25)

$$\Rightarrow \frac{\kappa_{xx}}{R_1} + \frac{\kappa_{yy}}{R_2} = -\Gamma^2 w = g_B \tag{3.26}$$

$$\Rightarrow D\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = D\nabla^4 w = p_B$$
(3.27)

Where Γ^2 is the "shell-operator" defined by:

$$\Gamma^{2}(\dots) = \frac{1}{R_{1}} \frac{\partial^{2}}{\partial y^{2}}(\dots) + \frac{1}{R_{2}} \frac{\partial^{2}}{\partial x^{2}}(\dots)$$
(3.28)

3.2.6. Airy Stress Function

To deal with stretching behavior the Airy stress function is introduced, which is defined in the following way:

$$n_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \qquad n_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \qquad n_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$
 (3.29)

This reduces the amount of variables in the compatibility equation from three to one variable:

$$\Rightarrow -\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = -\frac{1}{Et} \left(\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) = -\frac{1}{Et} \nabla^4 \phi = g_s \tag{3.30}$$

Also the equilibrium equation for stretching becomes:

$$\Rightarrow \frac{n_{xx}}{R_1} + \frac{n_{yy}}{R_2} = \Gamma^2 \phi = p_s \tag{3.31}$$

Where Γ^2 is the "shell-operator" defined by:

$$\Gamma^{2}(\dots) = \frac{1}{R_{1}} \frac{\partial^{2}}{\partial y^{2}}(\dots) + \frac{1}{R_{2}} \frac{\partial^{2}}{\partial x^{2}}(\dots)$$
(3.32)

Physical interpretation of the Airy stress function

According to Pál Csonka (1987) the Airy stress function can be interpreted in the following way:

- The derivative of the stress function with respect to x or y at point P equals disregarding its sign - the component in direction x or y of the specific forces acting along the arc P_0P
- The value of the stress function at point P equals disregarding its sign the moment of the specific forces acting along the arc P_0P about a straight line which passes through P and which is parallel to z-axis.

3.2.7. Static geometric analogy

There is a curious formal analogy between the static equilibrium equations and the geometric compatibility equations in the classical theory of thin shell. It was pointed out by Gol'denweiser (1940) and Lur'e (1940) and examined by Novozhilov (1959) and Calladine(1980). According to the static geometric analogy the same expressions or equations occur for corresponding quantities from bending behavior and stretching behavior. This can be seen in the table below:

Bending behavior	Stretching behavior
- <i>w</i>	φ
$\varphi_x = \frac{\partial w}{\partial x}, \qquad \varphi_y = \frac{\partial w}{\partial y}$	$-X = \left[\frac{\partial \phi}{\partial x}\right]_P, \qquad +Y = \left[\frac{\partial \phi}{\partial y}\right]_P$
$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}, -\kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}$	$n_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \ n_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \ n_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$
$\frac{\kappa_x}{R_1} + \frac{\kappa_y}{R_2} = -\frac{1}{R_1} \frac{\partial^2 w}{\partial x^2} - \frac{1}{R_1} \frac{\partial^2 w}{\partial y^2} = g_B$	$\frac{N_y}{R_1} + \frac{N_x}{R_2} = \frac{1}{R_1} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{R_1} \frac{\partial^2 \phi}{\partial y^2} = p_S$
$m_x = D\kappa_{xx}, \qquad m_y = D\kappa_{yy}, \qquad -m_{xy} = -D\kappa_{xy}$	$\varepsilon_y = \frac{1}{Et} n_{yy}, \qquad \varepsilon_x = \frac{1}{Et} n_{xx}, \qquad \frac{1}{2} \gamma_{xy} = \frac{1}{Et} n_{xy}$
$\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} = -p_B$	$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = -g_s$
$D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = p_B$	$-\frac{1}{Et}\left(\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}\right) = g_s$

Table 1 - Static geometric analogy: corresponding equations

The correspondence exists between the following variables:

 $n_{xx} \leftrightarrow \kappa_{yy} \qquad \qquad \varepsilon_{xx} \leftrightarrow m_{yy} \qquad \qquad \phi \leftrightarrow -w$

 $n_{yy} \leftrightarrow \kappa_{yy} \qquad \qquad \varepsilon_{yy} \leftrightarrow m_{xx} \qquad \qquad g_s \leftrightarrow p_B$

$$n_{xy} \leftrightarrow -\kappa_{xy} \qquad \frac{1}{2} \gamma_{xy} \leftrightarrow -m_{xy} \qquad P_s \leftrightarrow g_B$$

3.2.8. Coupled equations for (shallow) shells

In the general case of a shallow shell we can combine equations [...] which lead to the well-known coupled equations of shallow-shell theory:

$$g_S = g_B \Rightarrow -\frac{1}{Et} \nabla^4 \phi = -\Gamma^2 w = g$$
 (3.33)

$$p_B + p_S = p \Rightarrow D\nabla^4 w + \Gamma^2 \phi = p$$
 (3.34)

Where ∇^4 is the biharmonic operator:

$$\nabla^{4} = \frac{\partial^{4}}{\partial x^{4}}(...) + 2\frac{\partial^{4}}{\partial x^{2}\partial y^{2}}(...) + \frac{\partial^{4}}{\partial y^{4}}(...)$$
(3.35)

And Γ^2 is the "shell-operator" defined by:

$$\Gamma^{2}(\dots) = \frac{1}{R_{1}} \frac{\partial^{2}}{\partial y^{2}}(\dots) + \frac{1}{R_{2}} \frac{\partial^{2}}{\partial x^{2}}(\dots)$$
(3.36)

Now instead of working with the radius of principal curvatures R_1 and R_2 we can also rewrite this Γ^2 operator with curvatures k_{xx} , k_{yy} and k_{xy} :

$$\Gamma^{2}(...) = k_{xx} \frac{\partial^{2}}{\partial y^{2}}(...) - 2k_{xy} \frac{\partial^{2}}{\partial x \partial y}(...) + k_{yy} \frac{\partial^{2}}{\partial x^{2}}(...)$$
^(3.37)

From here on this expression shall be used.

3.2.9. Boundary conditions

One of the most challenging parts in this thesis is to deal with boundary conditions appropriately. In accordance with the two-surface theory discussed in 3.2.1 where the B-surface and the S-surface were endowed with specific mechanical properties, it follows that some boundary conditions have to be applied to the S-surface and others to the B-surface. For example, edge displacement conditions in the tangent plane apply to the S-surface but normal constraints are transferred to the B-surface. It turns out that on each edge of the shell two stretching boundary conditions and two bending boundary conditions have to be specified.



Fig. 3.7 - Common boundary conditions on the edges of a shell

Some common boundary conditions which occur in practice are the following:

- Fixed edge
 - Stretching conditions: $u_t = 0$, $u_n = 0$
 - Bending conditions: w = 0, $\frac{\partial w}{\partial n} = 0$
- Semi-rigid edge (diaphragm wall):
 - Stretching conditions: $u_t = 0$, $n_n = 0$
 - Bending conditions: w = 0, $\frac{\partial^2 w}{\partial n^2} = 0$
- Free edge
 - Stretching conditions: $n_{nt} = 0$, $n_n = 0$
 - Bending conditions: f = 0, $\frac{\partial^2 w}{\partial n^2} = 0$

Here the subscript letter t denotes the direction parallel to the edge and n the direction normal to the edge.

In this thesis it was attempted to express the bending and stretching boundary conditions in terms of the transverse displacements w and the Airy stress function ϕ respectively. With respect to expressing the bending boundary conditions in terms the displacements w this seems to go quite well. With respect to the stretching boundary conditions however, there exists some disagreement in the literature whether these stretching boundary conditions can all be expressed in terms of the Airy stress function, specifically the boundary conditions where displacements (u_x and u_y) are specified. In the theory of plane elasticity this is a well-known problem.

Thus, some authors claim that it is not possible because the Airy stress function can only handle boundary conditions in terms of stresses/loading and not displacements. For example S. R. Ahmed et al. (2004) have written:

"Successful application of the stress function formulation in conjunction with the finite difference technique has been reported for the solution of plane elastic problems where the conditions on the boundary are prescribed in terms of stresses only. (...) Boundary restraints specified in terms of u_x and u_y cannot be satisfactorily imposed on the stress function. As most of the practical problems of elasticity are of mixed boundary conditions, the approach fails to provide any explicit understanding of the stress distribution in the region of restrained boundaries, which are, in general, the most critical zones in terms of stresses."

On the other hand, other authors have claimed that it is possible, for example Zienkiewicz and Gerstner (1959) have written:

"(...) problems in which displacements are specified on part of the boundary, are considerably more difficult. (...) it is proposed to derive here the boundary conditions which would have to be satisfied by the Airy stress function on the portion of a boundary for which the displacements are specified (...)."

They further point out that a striking similarity occurs between the expressions of the Airy stress function which they derive and the expressions for certain boundary conditions for a plate in bending². In this thesis the intent is to use the expressions of Zienkiewicz and Gerstner and it will be seen whether these can successfully be implemented.

² This can actually be expected because of the static geometric analogy for shell structures (3.2.7) which states that there is a correspondence between quantities from the stretching surface and quantities from the bending surface. When the shell has zero curvature everywhere and thereby reduces to a flat plate or disk, this static geometric analogy reduces to the so called *'plate analogy'* (Prager, 1956).

The expressions in terms of the Airy stress function for the stretching boundary conditions where the displacements are specified (and equal zero, i.e. a fixed edge) proposed by Zienkiewicz and Gerstner (1959) are given below. Also the corresponding bending boundary conditions which use expressions in the same form (but then in terms of the displacements w) are given. These boundary conditions are given for a straight edge parallel to the y-axis, but can be used as well for a straight edge parallel to the x-axis by simply changing the subscript letters from x to y in all terms.

Stretching combination 1:		Bending combination 1:	
у	$u_y = 0 \Rightarrow \frac{\partial u_y}{\partial y} = \varepsilon_{yy} = 0$	$m_{xx} = 0$	
	$\Rightarrow n_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$	$\Rightarrow \frac{\partial^2 w}{\partial x^2} = 0$	a da la companya da l
	$u_x = 0 \Rightarrow -\frac{\partial \theta}{\partial y} = \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y} = 0$	$f = \frac{\partial m_{xx}}{\partial x} + 2\frac{\partial m_{xy}}{\partial y} = 0$	$y = \frac{1}{2} \int \frac{dy}{dy} \int \frac{dy}{dy} \int \frac{dy}{dy} = 0$
	$\Rightarrow \left(\frac{\partial^3 \phi}{\partial x^3} + 2\frac{\partial^3 \phi}{\partial x \partial y^2}\right) = 0$	$\Rightarrow \left(\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial^3 w}{\partial x \partial y^2}\right) = 0$	

In this case, it can be seen that a fixed edge on the stretching surface, corresponds to a free edge for the bending surface. Another common stretching boundary condition is where the displacements parallel to the edge is zero while the edge can move freely perpendicular to the edge. The expressions used for these stretching boundary conditions also occur for the bending boundary conditions when the edge is hinged and have a prescribed curvature (though the latter is usually zero).

Stretching combinat	ion 2:	Bending combination 2:	
y T	$u_y = 0 \Rightarrow \frac{\partial u_y}{\partial y} = \varepsilon_y = 0$	$M_x = 0$	
$G_x \xrightarrow{G_x} x$	$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 0$	$\Rightarrow \frac{\partial^2 w}{\partial x^2} = 0$	y-
⇒L	$N_x = G_x$	$\kappa_y = \kappa_{y;0}$ (prescribed curvature)	a vi
	$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = G_x$	$\Rightarrow \frac{\partial^2 w}{\partial y^2} = -\kappa_{y;0}$	

The final stretching boundary conditions which are considered is where the edge can move freely parallel and perpendicular to the edge. The expressions used for these stretching boundary conditions also occur for the bending boundary conditions when the edge have a prescribed twist and curvature parallel to the edge. However, when these are zero the edge becomes a fully clamped edge.

Stretching combination 3:		Bending combination 3:	
y 🔥	$n_{xy} = G_y$	$\kappa_{xy} = \kappa_{xy;0}$ (prescribed twist)	
$G_x \rightarrow \uparrow$	$\Rightarrow \frac{\partial^2 \varphi}{\partial x \partial y} = -G_y$	$\Rightarrow \frac{\partial^2 w}{\partial x \partial y} = -\frac{1}{2} \kappa_{xy;0}$	and a second
	$n_{xx} = G_x$	$\kappa_{yy} = \kappa_{yy;0}$ (prescribed curvature)	
→u	$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = G_x$	$\Rightarrow \frac{\partial^2 w}{\partial y^2} = -\kappa_{yy;0}$	

Note that although there is a correspondence between the expressions used for the stretching boundary conditions and those used for the bending boundary conditions, these corresponding boundary conditions do not have to occur at the same edge. For example, stretching combination 1 can occur with bending combination 3.

3.3. Finite difference method

In mathematics the Finite Difference Method (FDM) is a numerical method to approximate derivatives of a certain function. It can also be used for surfaces. The essence of the FDM lies in the following:

- The middle plane of the surface under consideration is covered by a rectangular, triangular, or other reference network depending on the geometry of the surface. This network is called a finite difference mesh and points of intersection of this mesh are referred to as mesh or nodal points.
- 2. The governing differential equation inside the shell domain is replaced by the corresponding finite difference equations at the mesh points using the finite difference operators.
- 3. Boundary conditions are also formulated with the use of the finite difference operators at nodal points located on the boundary.

As a result, a closed set of linear algebraic equations is obtained for every nodal point within the plate or shell. Solving this system of equations, one obtains a numerical field of the nodal displacements (and Airy stress function in the case of shells). The key point of the FDM is the finite difference approximation of derivatives. Consider the approximations for the derivatives of a one- dimensional, continuous function f(x). It is known that the derivative at point x_i is defined, as follows:

$$\left(\frac{df}{dx}\right)_{i} = \lim_{\Delta \to 0} \frac{f_{i+1} - f_{i}}{\Delta} \quad \text{or} \quad \left(\frac{df}{dx}\right)_{i} = \lim_{\Delta \to 0} \frac{f_{i} - f_{i-1}}{\Delta}$$
(3.38)



Figure 2.4.1 – function f(x) discretised
Where $f_i = f(x_i)$ and Δ is a finite increment of the variable x. If the symbol for limit is left out one obtains:

$$\left(\frac{df}{dx}\right)_{i} \approx \frac{f_{i+1} - f_{i}}{\Delta} \quad \text{or} \quad \left(\frac{df}{dx}\right)_{i} \approx \frac{f_{i} - f_{i-1}}{\Delta}$$
(3.39)

These expressions in 2.xx are called the first forward and first backward approximations of the derivative of f(x) at point x_i , respectively. However, in practice the expression for a central approximation is often used:

$$\left(\frac{df}{dx}\right)_i \approx \frac{f_{i+1} - f_{i-1}}{2\Delta} \tag{3.40}$$

As the increment Δ gets smaller the approximation of the derivative will get more accurate. Applying expressions (2.xx) and (2.xx) as operators we can derive the corresponding differential approximations of the second, third, and fourth derivatives of the function f(x).

$$\left(\frac{d^2 f}{dx^2}\right)_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta^2} \tag{3.41}$$

$$\left(\frac{d^3 f}{dx^3}\right)_i \approx \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2\Delta^3}$$
(3.42)

$$\left(\frac{d^4f}{dx^4}\right)_i \approx \frac{f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}{\Delta^4}$$
(3.43)

The finite difference method can also be used for a continuous function z(x, y) of two variables. One can use a rectangular mesh with the following reference points (Fig. 3.8):



Fig. 3.8 - Rectangular mesh for finite difference method

Here $\Delta x = \lambda_x$ and $\Delta y = \lambda_y$ and in a square mesh we have $\lambda_x = \lambda_y = \lambda$. Now if for example one wants to calculate $\nabla^2 z = \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$ at a specific point in the grid one gets:

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)_0 = \frac{z_N + z_S + z_E + z_W - 4z_0}{\lambda^2}$$
(3.44)

A better way to visualize this finite difference approximation is to use a coefficient pattern. Then equation 2.xx becomes:

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)_0 = \frac{1}{\lambda^2} \cdot \underbrace{1}_{1} \underbrace{4}_{1} \underbrace{1}_{1}$$
(3.45)

In similar ways the following derivatives can be derived and represented:

3.4. Rain flow analogy

3.4.1. Application for principal shear force trajectories

The concept of the so called "*rain flow analogy*" was inspired by Beranek (1976). It can be used to determine the principal shear force trajectories. To do this one visualises the sum of bending moments ($m = m_{xx} + m_{yy}$) as a 'hill'. The rain flow analogy then states that like rain which falls on a surface and flows down in the steepest direction generating streamlines, the principal shear force trajectories too follow in the steepest direction of the *m*-hill (Fig. 3.9). The rain flow analogy can thus be used to determine the load path for out-of-plane structural mechanic behavior.



Fig. 3.9 - Rain flow analogy (image from Blaauwendraad 2010)

3.4.2. Application for magnitude of principal shear forces

To obtain the magnitude of the principal shear forces one can integrate the associated load flows between the stream lines. However, the calculation can be much simpler which will be shown by considering the following equations. Starting with the sum of bending moments:

$$m = m_{xx} + m_{yy} = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
(3.50)

Shear forces:

$$v_{x} = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y} = -D\left(\frac{\partial^{3}w}{\partial x^{3}} + \frac{\partial^{3}w}{\partial x\partial y^{2}}\right) = -D\frac{\partial}{\partial x}\left(\frac{\partial^{2}w}{\partial x^{2}} + \frac{\partial^{2}w}{\partial y^{2}}\right)$$
(3.51)
$$\Rightarrow v_{x} = \frac{\partial}{\partial x}(m)$$

$$v_{y} = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x} = -D\left(\frac{\partial^{3}w}{\partial y^{3}} + \frac{\partial^{3}w}{\partial x^{2}\partial y}\right) = -D\frac{\partial}{\partial y}\left(\frac{\partial^{2}w}{\partial y^{2}} + \frac{\partial^{2}w}{\partial x^{2}}\right)$$

$$\Rightarrow v_{y} = \frac{\partial}{\partial y}(m)$$
(3.52)

This shows that the shear force at a certain point and direction equals the gradient of the m-hill on that specific point in that direction. Next consider a triangular plate part with shear forces acting on the edges.



Fig. 3.10 - Shear forces of an elementary plate (image from Blaauwendraad 2010)

From these equations it follows that:

$$v_n = v_x \cos\beta + v_y \sin\beta$$
(3.53)
$$v_t = -v_x \sin\beta + v_y \cos\beta$$

If v_n is the maximum shear force it follows that β has to be chosen such that the following condition is satisfied:

$$\frac{\partial v_n}{\partial \beta} = -v_x \sin \beta + v_y \cos \beta = 0 \Rightarrow \tan \beta = \frac{v_y}{v_x}$$
(3.54)

The maximum value of v_n can then be calculated with the following formula:

$$v_{n;max} = \sqrt{(v_x)^2 + (v_y)^2}, \qquad v_{t;min} = 0$$
 (3.55)

The minimum shear force is perpendicular to the maximum shear and equals zero. It has been stated that the gradient of the *m*-hill surface is equal to the shear force in corresponding direction. The gradient of the contour lines of the *m*-hill surface is zero, and the steepest direction is perpendicular to the contour lines. Thus the minimum shear forces correspond to the gradient of the contour lines (which equal zero) while the maximum or principal shear forces equal the gradient of the steepest direction which is perpendicular to the minimal one. The principal shear force can thus be written as:

$$v_n = \frac{\partial}{\partial n}(m) \tag{3.56}$$

Where *n* is the steepest direction of the *m*-hill.

3.5. Shell behavior assessment

3.5.1. Ratio normal and bending stress

A well designed shell should carry its load efficiently mainly through normal forces, when it does it shows *shell like* behavior. There are several ways to assess whether a shell carries its load efficiently. One way is to consider the ratio between the normal stress (σ_n) and the total stress caused by bending and normal forces ($\sigma_m + \sigma_n$). If this ratio approaches 100% it indicates more shell like behavior while a ratio closer to 0% indicates more plate like bending behavior (Fig. 3.11).



Fig. 3.11 - Shell assessment based on ratio bending and normal stresses

However determining the ratio of bending and normal stresses is not as straightforward as it may seem. The difficulty in determining this ratio is that the result is a vector quantity and is therefore dependent on which direction is considered. In this thesis it is decided to consider this ratio in the directions of the principal normal forces:

$$R_{1} = \frac{\sigma_{n1}}{\sigma_{m;\alpha_{0}} + \sigma_{n1}} \cdot 100\% = \frac{\left(\frac{n_{1}}{t}\right)}{\left(\frac{6\ m_{(\alpha_{0})}}{t^{2}} + \frac{n_{1}}{t}\right)} \cdot 100\% = \frac{n_{1}}{\frac{6\ m_{(\alpha_{0})}}{t} + n_{1}} \cdot 100\%$$
(3.57)

$$R_{2} = \frac{\sigma_{n2}}{\sigma_{m;(\alpha_{0}+0,5\pi)} + \sigma_{n2}} \cdot 100\% = \frac{\left(\frac{n_{2}}{t}\right)}{\left(\frac{6\,m_{(\alpha_{0}+0,5\pi)}}{t^{2}} + \frac{n_{2}}{t}\right)} \cdot 100\% = \frac{n_{2}}{\frac{6\,m_{(\alpha_{0}+0,5\pi)}}{t} + n_{2}} \cdot 100\%$$
(3.58)

To determine the stress caused by bending in the direction of the principal normal forces it is necessary to determine first the direction α_0 of the principal normal forces themselves.



Fig. 3.12 - Transformation of stresses and principal stresses and direction (Image from Blaauwendraad 2010)

The direction of the principal normal forces is denoted as α_0 and can be calculated from the equation:

$$\tan(2\alpha_0) = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2n_{xy}}{n_{xx} - n_{yy}}$$
(3.59)

$$\Rightarrow \alpha_0 = \frac{1}{2} \cdot \arctan\left(\frac{2n_{xy}}{n_{xx} - n_{yy}}\right) \tag{3.60}$$

Then the bending moments in the corresponding directions can be calculated with the following transformation equations:

$$m(\alpha_0) = m_{xx} \sin^2 \alpha_0 - m_{xy} \sin 2\alpha_0 + m_{yy} \cos^2 \alpha_0$$
(3.61)

$$m(\alpha_0 + \frac{\pi}{2}) = m_{xx} \cos^2 \alpha_0 + m_{xy} \sin 2\alpha_0 + m_{yy} \sin^2 \alpha_0$$
(3.62)

Once these bending moments have been determined the ratios R_1 and $R_2(3.xx)$ can be finally be calculated, then from the values of these ratios insight into the behavior of the shell can be gained.

3.5.2. Load carried by the S-surface and by B-surface

Another way to assess the efficiency of a shell is by considering how much of the total load is carried by the stretching surface and how much is carried by the bending surface (in accordance with the two surface theory explained in 3.2.1). Recall the following equation:

$$p = p_B + p_S = D\nabla^4 w + \Gamma^2 \phi \tag{3.63}$$

It follows that once the displacement field (w) and the Airy stress function field (ϕ) are known one can calculate back the load carried by bending (p_B) and the load carried by stretching (p_S). When most of the load is carried by the stretching surface (and p_S is large compared to p_B) this indicates shell like behavior. With these quantities the following ratio can then also be calculated:

$$R_3 = \frac{p_S}{p_B + p_S}$$
(3.64)

4. Development of the parametric design tool

4.1. Introduction

This chapter will elaborate on the development of the parametric structural design tool for shell structures. Following the list of steps which have been stated in section 1.2, the next steps to be undertaken in this chapter are the following:

- Define which demands have to be fulfilled with respect to functionality and usability by the parametric tool
- Provide a general outline and structure for the parametric tool in accordance with the demands
- Implement the theoretical framework into the structural design tool

Though much of the current developed tool was initially based on the models developed by M. Oosterhuis (2010) and D. Liang (2012), most components of their models had to be modified significantly in the current tool. Moreover, many new components had to be developed from scratch as well. However, the development of the current tool has still greatly benefited from the earlier work by M. Oosterhuis and D. Liang.

4.2. Functionality and usability

It is envisioned that the parametric tool for shell structures can be used and understood by both architects and engineers to gain quantitative and qualitative insight into the structural behavior of shell structures in a conceptual design stage. Based on this objective the following demands can be defined:

- Able to provide real-time results
- Able to choose in a simple way which- and how results are represented
- Able to change design parameters such as geometry, material properties, boundary conditions and loading
- Able to extend functionality further for future development (by adding parametric components and procedures)

With respect to the specific calculation results the tool should be able to give as output the following quantities:

- Airy stress function ϕ
- Transverse displacement *w*
- Internal forces:
 - Normal forces
 - Principal normal forces
 - Sum of normal forces
 - Bending moments
 - Principal moments
 - Sum of bending moments
 - Shear forces
 - Principal shear forces
- Trajectories of the principal shear force (rain flow analogy)
- Measure for assessment shell behavior

4.3. Choice of software

For developing the tool the software application Rhinoceros (or Rhino) was chosen with the parametric plugin Grasshopper (**Error! Reference source not found.**). As been explained in the introduction of this thesis (1.1) a recent development in the field of computational design are parametric associative design tools which capture design information by defining logical relations between (geometrical) components, controlled by parameters. Rhino in combination with Grasshopper is an example of such a tool and offers a powerful approach for creating parametric models.

Moreover, the previous models developed by M. Oosterhuis (2010) and D. Liang (2012) were also developed with Rhino and Grasshopper. Since this thesis can be considered as an extension of their work it is only logical that the same software applications will be used here.

Following Oosterhuis and Liang, also VB.net (Visual Basic) programming is used to create new components within the Grasshopper interface (Fig. 4.1).



Fig. 4.1 - Example of VB.net script

Also for solving matrix equations an external matrix class library for linear algebra computations was used called Mapack for .Net which was developed by Lutz Roeder (Roeder, 2002). Mapack library can be accessed by referencing from within the VB script components to the external library.

4.4. General outline and structure of the tool

The general outline of the tool can be seen in the figure below (Fig. 4.2). It consists of all the main components and the output that they give. The outline shows how the output information of each component is sent to the next component, thus the relation between all the components becomes clear.



Fig. 4.2 - General outline and structure of the tool

4.5. Implementation of the theoretical framework

In this section explanation of all the developed components will be given and how the theoretical framework discussed in chapter 3 is applied in these components. Together these components generate all the results and visualize these in the Rhino interface.

4.5.1. Geometry and mesh component

The first component in the parametric tool is the geometry and mesh component. It is used to specify and visualize the geometry of the shell and to create a mesh on it. Here a shape function for the shell can be given, thereby defining the shape of the shell. Also, the mesh size and the length in x-direction and width in y-direction of the shell can be determined here. The component will then automatically create the surface and mesh.

It is however desirable that the parametric tool can handle more arbitrary shapes which are not merely determined by having a certain shape function as input. Therefore the tool was further developed in such a way that one can give a surface itself as input. In the Rhino / Grasshopper environment this means a NURBS surface can be given as input. The component then automatically calculates the z-coordinates of the shell for a given specified grid.



Fig. 4.3 - Geometry & Mesh component in Grasshopper

Fig. 4.4 - Output of the Geometry & Mesh component

The component creates an organized list of the coordinates and sends these to the other components.

4.5.2. Load component

In this component the loads can be specified. Though only loads perpendicular to the shell can be applied, they do not have to be uniformly distributed. The loads are applied on the grid nodes and visualised as point loads. However each 'point load' should actually be thought of as a distributed load on a small piece of the surface which equals the mesh size squared (Fig. 4.5). Thus with a smaller mesh size, the load specification can be given more precisely.



Fig. 4.5 - Corresponding surface of a load on a grid node

The component visualizes the loads on the shell and also creates a list of all the loads which will then be sent to the shell calculation component.

· Lc	ad co	oqmc	nent	- , -
0 = Uniform loading, 1= ar magnitude uniform loading select points f arbitrary loadin 	bitrary loading 1 0	-34.29 o	free_points loading_type uniform_load loadis1 magnitude_loads1 loads2 magnitude_loads2	out) List_of_Joads1) List_of_Joads2)
magnitude loads2 magnitud Display loads2 magnitud Turn load vector position p Display magnification factor	le uniform loading (RN/m2)	25.0 0		

Fig. 4.6 - Loads component in Grasshopper



Fig. 4.7 - Visualisation of loads on surface

4.5.3. Curvature component

The curvature component calculates the curvatures and the twist of the shell surface (Fig. 4.8).



Fig. 4.8 - Curvature component in Grasshopper

The calculation is based on the following formulas:

$$k_{xx} = \frac{\partial^2 z}{\partial x^2} \approx \frac{\partial^2 \overline{z}}{\partial \overline{x}^2}, \qquad k_{yy} = \frac{\partial^2 z}{\partial y^2} \approx \frac{\partial^2 \overline{z}}{\partial \overline{y}^2}, \qquad k_{xy} = \frac{\partial^2 z}{\partial x \partial y} \approx \frac{\partial^2 \overline{z}}{\partial \overline{x} \partial \overline{y}}$$
(4.1)

It calculates these derivatives with the finite difference method. The previous equations rewritten in finite difference form become:

$$[k_{xx}] = [K_x] \cdot [\overline{z}], \qquad [k_{yy}] = [K_y] \cdot [\overline{z}], \qquad [k_{xy}] = -[K_{xy}] \cdot [\overline{z}]$$
(4.2)

Where $[K_x]$, $[K_y]$, $[K_{xy}]$ stand for the finite difference matrices which use the following operators:

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)_0 = \frac{1}{4\lambda^2} \cdot \underbrace{\begin{array}{c} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{array}}_0$$

The component produces lists of all the values of k_{xx} , k_{yy} and k_{xy} and sends these to the shell calculation component which will is discussed at 4.4.5.

(4.4)

4.5.4. Boundary conditions component

Before any calculations will be performed first the fictitious outer points have to be introduced. These points are only used so the finite difference method can be applied at and near the boundary of the shell as will be seen when calculation procedures are discussed. From a top view the grid of the shell would look as follows (Fig. 4.9):



Fig. 4.9 - Top view of shell grid with imaginary outer points

Now on each boundary edge two stretching boundary conditions and two bending boundary conditions are applied (3.2.9). To implement these boundary conditions, the finite difference method is used. Here the finite difference operators are given for specific boundary conditions for an edge parallel to the y-axis, but can be used as well for an edge parallel to the x-axis by simply rotating the finite difference operators:

Boundary condition type 1:

Stretching boundary condition type 1:	Bending boundary condition type 1:			
$u_y = 0 \Rightarrow \frac{\partial u_y}{\partial y} = \varepsilon_y = 0$	$M_x = 0$			
$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 0$	$\Rightarrow \frac{\partial^2 w}{\partial x^2} = 0$			
$\Rightarrow \left(\frac{\partial^2 \dots}{\partial x^2}\right)_0 = \frac{1}{\lambda^2} \cdot 1 - 2 \cdot 1$				

Boundary condition 2:

Stretching boundary condition type 2:	Bending boundary condition type 2:		
$u_x = 0 \Rightarrow -\frac{\partial \theta}{\partial y} = \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y} = 0$	$f = \frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} = 0$		
$\Rightarrow \left(\frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2}\right) = 0$	$\Rightarrow \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) = 0$		
$\Rightarrow \left(\frac{\partial^3 \dots}{\partial x^3} + \frac{\partial^3 \dots}{\partial x \partial y^2}\right)_0 = \frac{1}{2\lambda^3} \cdot \underbrace{\begin{array}{c} 1\\ -8\\ -8\\ 20\\ -8\\ -4\\ 1\end{array}}^{(1)}$			

Boundary condition 3:

Stretching boundary condition type 3:	Bending boundary condition type 3:	
$N_{xy} = G_y$	$\kappa_{xy} = \kappa_{xy;0}$ (prescribed twist)	
$\Rightarrow \frac{\partial^2 \varphi}{\partial x \partial y} = G_y$	$\Rightarrow \frac{\partial^2 w}{\partial x \partial y} = -\frac{1}{2} \kappa_{xy;0}$	
$\Rightarrow \left(\frac{\partial^2 \dots}{\partial x \partial y}\right)_0 = \frac{1}{4\lambda^2} \cdot \underbrace{\begin{array}{c} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{array}}_0$		

Boundary condition 4:

Stretching boundary condition type 4:	Bending boundary condition type 4:		
$N_x = G_x$	$\kappa_y = \kappa_{y;0}$ (prescribed curvature)		
$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = G_x$	$\Rightarrow \frac{\partial^2 w}{\partial y^2} = -\kappa_{y;0}$		
$\Rightarrow \left(\frac{\partial^2 \dots}{\partial y^2}\right)_0 = \frac{1}{\lambda^2} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$			

With these the finite difference operators matrices can be assembled. In the following equation E stands for the applied finite difference operator matrix which depend on which boundary conditions are applied. For stretching boundary conditions:

$$[E_S] \cdot \left[\phi_{free+bound+out}\right] = [N_S] \tag{4.5}$$

For bending boundary conditions:

$$[E_B] \cdot [w_{free+bound+out}] = [N_B] \tag{4.6}$$

Here $[N_S]$ and $[N_B]$ simply stand for the values that the boundary condition equations have to be equal to (and are in general equal to zero). On each boundary edge two stretching boundary conditions and two bending boundary conditions are applied. Furthermore on each corner three stretching boundary conditions and three bending boundary conditions are applied, therefore:

$$E_{S} = \begin{bmatrix} E_{S;bound;bc1} \\ E_{S;bound;bc2} \\ E_{S;corner;bc1} \\ E_{S;corner;bc2} \\ E_{S;corner;bc2} \end{bmatrix}, \qquad N_{S} = \begin{bmatrix} N_{S;bound;1} \\ N_{S;bound;2} \\ N_{S;corner;1} \\ N_{S;corner;2} \\ N_{S;corner;3} \end{bmatrix}, \qquad E_{B} = \begin{bmatrix} E_{B;bound;bc1} \\ E_{B;bound;bc2} \\ E_{B;corner;bc1} \\ E_{B;corner;bc2} \\ E_{B;corner;bc2} \\ E_{B;corner;bc3} \end{bmatrix}, \qquad N_{B} = \begin{bmatrix} N_{B;bound;1} \\ N_{B;bound;2} \\ N_{B;corner;1} \\ N_{B;corner;2} \\ N_{B;corner;2} \\ N_{B;corner;3} \end{bmatrix}$$
(4.7)

Where:

$$E_{S;bound,j} = \begin{bmatrix} E_{S;boundA-i} \\ E_{S;boundB-i} \\ E_{S;boundD-i} \end{bmatrix}, E_{S;corner,j} = \begin{bmatrix} E_{S;corner0-i} \\ E_{S;corner2-i} \\ E_{S;corner3-i} \end{bmatrix}, E_{B;bound,j} = \begin{bmatrix} E_{B;boundA-i} \\ E_{B;boundB-i} \\ E_{B;boundD-i} \end{bmatrix}, E_{B;corner,j} = \begin{bmatrix} E_{B;corner0-i} \\ E_{B;corner1-i} \\ E_{B;corner3-i} \end{bmatrix}$$
(4.8)
$$i = 1 \text{ for boundary condition 1}$$
$$i = 2 \text{ for boundary condition 2}$$
$$i = 3 \text{ for boundary condition 3}$$
$$i = 4 \text{ for boundary condition 4}$$

The purpose of the boundary condition component is thus to provide the finite difference operator matrices which will then be sent to the shell calculation component which will use it to calculate the Airy stress function and displacements. In (Fig. 4.10) an example can be seen from the boundary condition component with the matrices as output.



Fig. 4.10 - Boundary condition component for bending behavior with matrix E_Bending as output

4.5.5. Shell calculation component

Perhaps the most important component is the "shell calculation"-component. The purpose of this component will be to calculate the transverse displacements w, the Airy stress function ϕ , the load carried by the stretching surface p_S and the load carried by the bending surface p_B . Recall from (3.2.6) the governing coupled-equations for a shallow shell:

$$-\frac{1}{Et}\nabla^4\phi = -\Gamma^2 w = g \tag{4.9}$$

$$\Gamma^2 \phi + D \nabla^4 w = p_S + p_B = p \tag{4.10}$$

Where Γ^2 is the "shell-operator" defined by:

$$\Gamma^{2}(...) = k_{xx} \frac{\partial^{2}}{\partial y^{2}}(...) - 2k_{xy} \frac{\partial^{2}}{\partial x \partial y}(...) + k_{yy} \frac{\partial^{2}}{\partial x^{2}}(...)$$

$$(4.11)$$

Rewriting equations [...] in finite difference method notation:

$$\frac{1}{Et} \cdot [A] \cdot \left[\phi_{free+bound+out}\right] = [G] \cdot \left[w_{free+bound+out}\right] = [-g]$$
(4.12)

$$[G] \cdot [\phi_{free+bound+out}] + D \cdot [A] \cdot [w_{free+bound+out}] = [p]$$
(4.13)

Where:

- [G] stands for the Γ^2 finite difference operator matrix with the factors k_{xx} , k_{xy} and k_{yy} supplied by the curvature component (4.4.4)
- [A] stands for the ∇^4 finite difference method matrix :

$$\nabla^{4}(...) = \left(\frac{\partial^{4}(...)}{\partial x^{4}} + 2\frac{\partial^{4}(...)}{\partial x^{2}\partial y^{2}} + \frac{\partial^{4}(...)}{\partial y^{4}}\right) = \frac{1}{\lambda^{4}} \cdot \underbrace{1}_{2} \cdot \underbrace{8}_{20} \cdot \underbrace{2}_{8} \cdot \underbrace{1}_{1}$$
(4.14)
$$2 \cdot \underbrace{8}_{20} \cdot \underbrace{2}_{8} \cdot \underbrace{1}_{2}$$
(1)



Recall the specification of the generated grid for the shell with imaginary outer points:

Fig. 4.11 - Top view of grid for the shell

Now we need to find a way to express the change of Gaussian curvature [g] in the load [p]. Therefore we first need to express ϕ and w in g. We start with ϕ using the following equation (which is based on equation 4.xx):

$$[A] \cdot \left[\phi_{free+bound+out}\right] = -Et \cdot [g] \tag{4.15}$$

For every interior point we can provide the finite difference operator equations. However, at this point the Airy stress function ϕ cannot yet be expressed in terms of g because matrix A is not a square matrix and is thus not invertible. Therefore extra equations need to be used which are provided by the boundary conditions. This is where matrix E_1 comes in and needs to be used. It stands for the finite difference operator matrix resulting from the boundary condition component for stretching as explained in (4.4.3):

$$[E_s] \cdot \left[\phi_{free+bound+out}\right] = [N_s] \tag{4.16}$$

Adding equations to equations (4.xx) gives:

$$\Rightarrow \begin{bmatrix} A \\ E_s \end{bmatrix} \cdot \begin{bmatrix} \phi_{free+bound+out} \end{bmatrix} = \begin{bmatrix} -Etg \\ N_S \end{bmatrix}$$
(4.17)

Now let:

$$[F_s] = \begin{bmatrix} A \\ E_s \end{bmatrix} \tag{4.18}$$

Then:

$$\Rightarrow [F_s] \cdot \left[\phi_{free+bound+out}\right] = \begin{bmatrix} -Etg\\ N_S \end{bmatrix}$$
(4.19)

Now matrix F_S will be a square invertible matrix and thus the Airy stress function ϕ can now be expressed in terms of g through:

$$\Rightarrow \left[\phi_{free+bound+out}\right] = \left[F_{S}^{-1}\right] \cdot \begin{bmatrix}-Etg\\N_{S}\end{bmatrix}$$
(4.20)

Now let:

$$[F_{\mathcal{S}}^{-1}] = [H_{\mathcal{S}}] = [H_{\mathcal{S}1} \quad H_{\mathcal{S}2}]$$
(4.21)

Then the full expression for ϕ becomes:

$$\Rightarrow \left[\phi_{free+bound+out}\right] = \left[F_s^{-1}\right] \cdot \begin{bmatrix} -Etg\\N_S \end{bmatrix} = \left[H_{s1} \quad H_{s2}\right] \cdot \begin{bmatrix} -Etg\\N_s \end{bmatrix} = -Et \cdot \left[H_{s1}\right] \cdot \left[g\right] + \left[H_{s2}\right] \cdot \left[N_s\right]$$
(4.22)

Now the next step is to express the displacement w in terms of g. For this we use the following equation:

$$[G] \cdot [w_{free+bound+out}] = [-g] \tag{4.23}$$

Now the displacement w cannot yet be expressed in terms of g because matrix G is not a square matrix and is thus not invertible. Therefore again extra equations need to be used which are provided by the boundary conditions. This is where matrix E_b comes in and needs to be used. It stands for the finite difference operator matrix resulting from the applied boundary conditions for bending as explained in (4.5.4):

$$[E_b] \cdot [w_{free+bound+out}] = [N_b] \tag{4.24}$$

Adding equations ... to equations ... gives:

$$\Rightarrow \begin{bmatrix} G \\ E_b \end{bmatrix} \cdot \begin{bmatrix} w_{free+bound+out} \end{bmatrix} = \begin{bmatrix} -g \\ N_b \end{bmatrix}$$
(4.25)

Now let:

$$[F_b] = \begin{bmatrix} G\\ E_b \end{bmatrix} \tag{4.26}$$

Then:

$$\Rightarrow [F_b] \cdot \left[w_{free+bound+out} \right] = \begin{bmatrix} -g \\ N_b \end{bmatrix}$$
(4.27)

Now matrix F_2 will be a square invertible matrix and thus the displacement w can now be expressed in terms of g through:

$$\Rightarrow \left[w_{free+bound+out} \right] = \left[F_b^{-1} \right] \cdot \begin{bmatrix} -g \\ N_b \end{bmatrix}$$
(4.28)

Now let:

$$[F_b^{-1}] = [H_b] = [H_{b1} \quad H_{b2}]$$
(4.29)

Then the full expression for w becomes:

$$\Rightarrow \begin{bmatrix} w_{free+bound+out} \end{bmatrix} = \begin{bmatrix} F_b^{-1} \end{bmatrix} \cdot \begin{bmatrix} -g \\ N_b \end{bmatrix} = \begin{bmatrix} H_{b1} & H_{b2} \end{bmatrix} \cdot \begin{bmatrix} -g \\ N_b \end{bmatrix} = -\begin{bmatrix} H_{b1} \end{bmatrix} \cdot \begin{bmatrix} g \end{bmatrix} + \begin{bmatrix} H_{b2} \end{bmatrix} \cdot \begin{bmatrix} N_2 \end{bmatrix}$$
(4.30)

Now we can substitute the expressions for ϕ and w in equation (4.xx):

$$[G] \cdot \left[-Et \cdot [H_{s1}] \cdot [g] + [H_{s2}] \cdot [N_s] \right] + D \cdot [A] \cdot \left[-[H_{b1}] \cdot [g] + [H_{b2}] \cdot [N_b] \right] = [p]$$
(4.31)

Now we can express g in terms of p:

$$-Et \cdot [G] \cdot [H_{s1}] \cdot [g] + [G] \cdot [H_{s2}] \cdot [N_s] - D \cdot [A] \cdot [H_{b1}] \cdot [g] + D \cdot [A] \cdot [H_{b2}] \cdot [N_b] = [p] \quad (4.32)$$

$$Et \cdot [G] \cdot [H_{s1}] \cdot [g] - D \cdot [A] \cdot [H_{b1}] \cdot [g] = [p] - [G] \cdot [H_{s2}] \cdot [N_s] - D \cdot [A] \cdot [H_{b2}] \cdot [N_b]$$
(4.33)

$$\left[Et \cdot [G] \cdot [H_{s1}] - D \cdot [A] \cdot [H_{b1}]\right] \cdot [g] = [p] - [G] \cdot [H_{s2}] \cdot [N_s] - D \cdot [A] \cdot [H_{b2}] \cdot [N_b]$$
(4.34)

$$\Rightarrow [g] = \left[Et \cdot [G] \cdot [H_{s1}] - D \cdot [A] \cdot [H_{b1}]\right]^{-1} \cdot \left[[p] - [G] \cdot [H_{s2}] \cdot [N_s] - D \cdot [A] \cdot [H_{b2}] \cdot [N_b]\right]$$
(4.35)

At this point the change of Gaussian curvature is determined at every point. Now the Airy stress function ϕ and the displacements w can be determined with the expressions:

$$\begin{bmatrix} \phi_{free+bound+out} \end{bmatrix} = \begin{bmatrix} F_s^{-1} \end{bmatrix} \cdot \begin{bmatrix} -Etg \\ N_s \end{bmatrix} = \begin{bmatrix} H_{s1} & H_{s2} \end{bmatrix} \cdot \begin{bmatrix} -Etg \\ N_s \end{bmatrix} = -Et \cdot \begin{bmatrix} H_{s1} \end{bmatrix} \cdot \begin{bmatrix} g \end{bmatrix} + \begin{bmatrix} H_{s2} \end{bmatrix} \cdot \begin{bmatrix} N_s \end{bmatrix}$$
(4.36)

$$\begin{bmatrix} w_{free+bound+out} \end{bmatrix} = \begin{bmatrix} F_b^{-1} \end{bmatrix} \cdot \begin{bmatrix} -g \\ N_b \end{bmatrix} = \begin{bmatrix} H_{b1} & H_{b2} \end{bmatrix} \cdot \begin{bmatrix} -g \\ N_b \end{bmatrix} = -\begin{bmatrix} H_{b1} \end{bmatrix} \cdot \begin{bmatrix} g \end{bmatrix} + \begin{bmatrix} H_{b2} \end{bmatrix} \cdot \begin{bmatrix} N_2 \end{bmatrix}$$
(4.37)

Which were already derived previously. Now the Airy stress function ϕ and the displacements w are known and it becomes possible to calculate the load carried by the S-surface and the load carried by the B-surface:

$$p_S = \Gamma^2 \phi$$
, $p_B = D \nabla^4 w$ (4.38)

Which is again calculated with the finite difference method:

$$[p_S] = [G] \cdot [\phi_{free+bound+out}], \qquad [p_B] = D \cdot [A] \cdot [w_{free+bound+out}]$$
(4.39)

This information is then sent to the shell assessment component.

4.5.6. Internal forces component

Once the Airy stress function ϕ and the displacements w are determined the internal forces can also easily calculated using the finite difference method by calculating the derivatives of the ϕ - and w field. Recall from (3.2) the following equations:

$$n_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \qquad n_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \qquad n_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$
 (4.40)

$$m_{xx} = -D \frac{\partial^2 w}{\partial x^2}, \qquad m_{yy} = -D \frac{\partial^2 w}{\partial y^2}, \qquad m_{xy} = -D \frac{\partial^2 w}{\partial x \partial y}$$
 (4.41)

Rewrite in finite difference form:

$$[n_{yy}] = [K_x] \cdot [\phi], \qquad [n_{xx}] = [K_y] \cdot [\phi], \qquad [n_{xy}] = -[K_{xy}] \cdot [\phi] \qquad (4.42)$$

$$[m_{xx}] = -D[K_x] \cdot [w], \qquad [m_{yy}] = -D[K_y] \cdot [w], \qquad [m_{xy}] = -D[K_{xy}] \cdot [w] \qquad (4.43)$$

Where $[K_x]$, $[K_y]$, $[K_{xy}]$ stand for the finite difference matrices which use the following operators:

$$\begin{pmatrix} \frac{\partial^2 \dots}{\partial x^2} \end{pmatrix}_o = \frac{1}{\lambda^2} \cdot \underbrace{1}_{O} \underbrace{2}_{O} \underbrace{1}_{O} \underbrace{2}_{O} \underbrace{1}_{O} \underbrace{1}_{O}$$

At this point the normal forces and bending moments are known. The principal normal forces and principal bending moments are then calculated by the component using the formulas:

$$n_{1} = \frac{n_{xx} + n_{yy}}{2} + \sqrt{\left(\frac{n_{xx} - n_{yy}}{2}\right)^{2} + n_{xy}^{2}} \qquad m_{1} = \frac{m_{xx} + m_{yy}}{2} + \sqrt{\left(\frac{m_{xx} - m_{yy}}{2}\right)^{2} + m_{xy}^{2}} \qquad (4.46)$$
$$n_{2} = \frac{n_{xx} + n_{yy}}{2} - \sqrt{\left(\frac{n_{xx} - n_{yy}}{2}\right)^{2} + n_{xy}^{2}} \qquad m_{2} = \frac{m_{xx} + m_{yy}}{2} - \sqrt{\left(\frac{m_{xx} - m_{yy}}{2}\right)^{2} + m_{xy}^{2}}$$

4.5.7. Derivative component (for shear forces)

The derivative component was a component developed by M. Oosterhuis (2010) and is also used in the current tool. The component is meant to visualise and calculate the magnitude and direction of the principal shear forces and its components in x- and y-direction. It is based on the analytical relationship between the sum of bending moments and the shear forces. As been pointed out in paragraph 3.4, the shear force in a particular direction equals the first derivative of the sum of bending moments 'hill' in that particular direction.

The component developed by M. Oosterhuis was developed by combining a set of predefined components which were already provided in Grasshopper. The required input for the derivative component consists of the sum of bending moments surface (which in the current tool is provided by the internal forces component discussed in the previous paragraph) and the grid points.

To determine the magnitude and direction of the principal shear force in a grid point, the derivative (slope) of the sum of bending moments hill has to be calculated in the direction of the steepest descent. As explained by M. Oosterhuis (2010) in his thesis, first, the steepest descent direction is determined by using the surface normal vector. By multiplying this normal vector with the global z-vector the cross vector is obtained. This is then used to rotate the normal vector around over 90° and thereby becomes the steepest descent vector which is tangential to the sum of bending moments surface in the considered point. The resulting vector collection can then be plotted on the sum of the bending moments hill (Fig. 4.12). Finally the magnitude of the shear force is calculated by calculating the slope of the steepest descent vector.



Fig. 4.12 - Direction of the principal shear forces (Image from M. Oosterhuis 2010)

The shear forces in x- and y-direction are calculated by determining the corresponding planar components of the principal shear force. The component is also able to display the shear also as scaled arrows where the magnitude is represented by the thickness of the arrows. Below an example of the shear forces visualised is given from Oosterhuis (2010) for a simply supported plate in bending subjected to a uniformly distributed load (Fig. 4.13).



Fig. 4.13 - vn, vx and vy represented by scaled arrows (image from Oosterhuis 2010)

4.5.8. Rain flow analogy component

The component which uses the rain flow analogy and generates the principal shear trajectories was also developed by M. Oosterhuis (2010) and will be used in the current tool as well. The component generates 'continuous' rain flow trajectories on a surface by using a gradient descent algorithm.

This algorithm starts from a predefined point p_i and iteratively determines the steepest descent vector on the sum of bending moments hill and defines the next point p_{i+1} by moving the starting point over a small distance in this direction (Fig. 4.14). This sequence is repeated until a local (or global minimum) has reached or a predefined stopping condition is met. The stopping conditions are:

- 1) Surface normal vector is almost vertical
- 2) Point p_{i+1} is outside the 3D surrounding box
- Iteration number is larger than predefined maximum number of iterations







An example of the results the component produces are given below. In these images provided by Oosterhuis (2010) one can see the sum of bending moments hill with the principal shear force trajectories of a plate in bending with a uniformly distributed loaded. One can thus clearly see how the tool visualizes the load path for out-of-plane structural mechanic behavior





Fig. 4.15 - 3D view of the *m*-hill of a rectangular plate with principal shear force trajectories (image from Oosterhuis 2010)

Fig. 4.16 - Top view of principal shear force trajectories (image from Oosterhuis 2010)

4.5.9. Shell assessment component

The final component is the shell assessment component which is used to assess whether the shell carries its loads efficiently. It uses as input the list of values of the displacements w, Airy stress function ϕ , the load carried by the S-surface p_S and the load carried by the B-surface p_B which are received from the shell calculation component (4.4.5).

Then it first calculates the two ratios based on normal and bending stress as was discussed in paragraph 3.5.1. To calculate these ratios the following quantities: n_1 , n_2 , n_{xx} , n_{yy} , n_{xy} , m_{xx} , m_{yy} , m_{xy} , m_{xx} , m_{yy} , m_{xy} are needed and calculated in the same way as discussed in the internal forces component (4.4.6) using the finite difference method on the displacement field and Airy stress function field. When these quantities are known, the component calculates the direction α_0 of the principal normal forces with the following equation:

$$\Rightarrow \alpha_0 = \frac{1}{2} \cdot \arctan\left(\frac{2n_{xy}}{n_{xx} - n_{yy}}\right) \tag{4.47}$$

When α_0 is known at every point the component proceeds to calculate the bending moments in the direction the principal normal forces using the following transformation equations:

$$m(\alpha_0) = m_{xx} \sin^2 \alpha_0 - m_{xy} \sin 2\alpha_0 + m_{yy} \cos^2 \alpha_0$$
(4.48)

$$m(\alpha_0 + 0.5\pi) = m_{xx} \cos^2 \alpha_0 + m_{xy} \sin 2\alpha_0 + m_{yy} \sin^2 \alpha_0$$
(4.49)

Finally the two ratio's based on normal and bending stress are calculated by the tool:

$$R_1 = \frac{n_1}{\frac{6 m_{(\alpha_0)}}{t} + n_1} \cdot 100\%, \qquad R_2 = \frac{n_2}{\frac{6 m_{(\alpha_0 + 0.5\pi)}}{t} + n_2} \cdot 100\%$$
(4.50)

The tool is then able to plot the values of these ratios unto the shell surface and also show the direction of the considered principal normal forces with vectors (Fig. 4.17 and Fig. 4.18).



Fig. 4.17 - Results ratio R1 on hypar shell under uniform distributed load

Fig. 4.18 - Results ratio R2 on hyper shell under uniform distributed load

Next the tool also calculates the ratio based on the load carried by the S-surface and the load carried by the B-surface. It simply uses the following equation (where p_S and p_B are received values of input from the shell calculation component):

$$R_3 = \frac{p_S}{p_B + p_S}$$
(4.51)

To obtain even quicker insight into the results the values of the ratio for each point are plotted with a colour. Values closer to 1 will be green and values closer to zero red. Thus green indicates that more load is carried by the stretching surface which is considered a more efficient way of carrying the loads, while red indicates more load is carried by the bending surface. An example of the result of the component is given below (Fig. 4.19):



Fig. 4.19 - Result R3 from shell assessment component for a hypar shell under uniform distributed load

5. Results and validation

5.1. Introduction

As been stated in section 1.2, the next steps to be undertaken in this thesis are the following:

- Choose several test cases and use the tool to analyse them
- Validate the results in a qualitative and quantitative manner by comparing the results to analytical solutions or FEM results

When choosing which shell shapes should be analysed it is presumed that it is best to start from simple to more complex shapes. Thus for testing the developed tool the following shapes are chosen:

- 1. Plate loaded out-of-plane as a limit case of a shell
- 2. Plate loaded in-plane as a limit case of a shell
- 3. Basic shell shapes: elpar, cylindrical paraboloid, hypar
- 4. Modified basic shell shape
- 5. Hallenbad shell shape

For comparing and validating the results we shall mainly use the FEM program *SCLA Engineer*. The solver of SCIA Engineer uses elements which are combined plate-membrane elements (see Appendix B for more information concerning SCIA's type of elements). Besides using SCIA Engineer for comparison, also known analytical solutions provided by Pavlovic (1999) shall be used in the test case of the basic shell shapes (5.4).

5.2. Plate loaded out of plane as a limit case of a shell

In this first test case a plate in bending will be approximated. This is done by providing a shape function for the shell with almost zero curvature³. The 'plate' is chosen to be 9m long and 6m wide. An arbitrary load is applied on the surface. On one half of the surface a load of 1 kN/m² downwards is applied and on the other half a load of 0,5 kN/m² upward is applied. The material is steel with E=210000N/mm² and a thickness of 20mm. Also several different (bending) boundary conditions are applied on the edges: edge A is a free edge, edge B and D are a fully clamped edges and edge C is a hinged edge. In the parametric model a mesh size of 0,5m is chosen. For comparison the plate is modelled with the same properties and loading in SCIA Engineer. We set $\nu = 0,0001$ (in SCIA the Poission's ratio cannot be set exactly to zero and this is the smallest value allowed). Furthermore the mesh size in SCIA is set to 0,5m.



Fig. 5.1 – Results deformations SCIA Engineer



When the first results are compared in a qualitative manner by looking at the deformations a very good correspondence appears. In the figures above the initial results are shown. On the left is the result of the deformed plate calculated in SCIA Engineer, on the right we see the result of the parametric tool. The green surface represents the original shape of the plate/shell with the applied loading visualized. The purple surface underneath is the projected Airy stress function field. As expected the Airy stress function has a value of virtually zero everywhere (which means that the normal forces are also virtually zero everywhere). The orange surface then represents the projected displacement field which corresponds well with the result from SCIA Engineer. Next more precise results will be given for the quantities relating to plates in bending.

³ We cannot simply enter as a shape function z = 0 however, since this will result in a singular matrix. We therefore use $z = 0,0000001*(x^2+y^2)$ which results in a virtually flat shell (i.e. a plate) and does not lead to a singular matrix.

5.2.1. Results displacements w



Fig. 5.4 - Displacements parametric tool close-up

Exact results will now be given along the lines:

x = -1,5m

y = -1,0m

y = -1,0m x = -1,5m

Exact results of the displacements w:



5.2.2. Results bending moments

Results m_{xx} :



Exact results of mxx:



Results m_{yy} :



Fig. 5.11 - myy field SCIA



Exact results of myy:


Results m_{xy} :



Exact results of mxy:



5.2.3. Results shear forces

Results v_x :



Fig. 5.19 - vx field SCIA



Exact results of vx:









Fig. 5.24 - vy field parametric tool

Exact results of vy:



Results v_n :



Exact results vn:



5.2.4. Results sum of bending moments and principal shear forces

Results derivative component and rain flow analogy component:



Fig. 5.31 - Sum of bending moments field with vectors in steepest direction (3d)



Fig. 5.32 - Sum of bending moments field with vectors in steepest direction (top view)



Fig. 5.33 - Principal shear force trajectories (3d)



Fig. 5.34 - Principle shear force trajectories (top view)

5.2.5. Discussion of the results

From the previous results the following can be observed:

- In general, when compared to the results of SCIA the results of the developed tool correspond very well.
- Only at the edges a slight inaccuracy occurs with the calculation of the shear forces. This seems to be explained by the fact that the pre-built Grasshopper sub-components which are used in the derivative component and which have to calculate the slope of the m-hill surface have some trouble dealing with edges of surfaces.
- It is clear that all the applied load is carried by the B surface since the Airy stress function field is virtually zero everywhere, indicating that no stretching forces occur.
- The way in which the results of quantities are presented as projected surfaces give the user a better sense of the relative magnitude of the quantities at every point of the structure.
- Thus, it seems that a plate in bending can be approximated well by the developed parametric design tool for shells.

5.3. Plate loaded in-plane as a limit case of a shell

In this second test case a plate loaded in-plane will be approximated. This is again done by providing a shape function for the shell with almost zero curvature. The 'plate' is chosen to be 12m long and 4m wide. A line load on edge B is applied of 100kN/m. The thickness and Young's modulus can be chosen arbitrarily since only the internal forces are of interest in this investigation and they do not depend on the thickness or the Young's modulus. Different (stretching) boundary conditions are applied on the edges: edge A is a fixed edge, edge B is a free edge and edge C and D are chosen as semi-rigid (meaning that only the displacement parallel to the edge are restrained). For comparison the plate is modelled with the same properties and loading in SCIA Engineer. We set $\nu = 0,0001$ (in SCIA the Poission's ratio cannot be set exactly to zero and this is the smallest value allowed).



Fig. 5.35 - Plate loaded in plane modelled in SCIA

In the parametric model a mesh size of 0,5m is chosen. On the right (

Fig. 5.36) the first results are given. As can be seen, the displacement field is virtually zero everywhere, indicating that no bending action occurs. The Airy stress function field is mainly curved in the x-direction indicating large values for n_{yy} can be expected. Next results for n_{xx} , n_{yy} and n_{xy} will be given as well as graphs of the values along lines of interest.



Fig. 5.36 - 'Plate' loaded in-plane modelled in parametric tool with Airy stress field and displacement field

5.3.1. Results normal forces





Results n_{yy} :





5.3.2. Discussion of the results

From the previous results the following can be observed:

- In general, when compared to the results of SCIA the results of the developed tool correspond very well. In general the results differ less than 5%.
- Only near the upper corners where the load is applied some differences occur. These might be explained by the fact that some inaccuracies will inevitably occur in the solution of the FEM when two edges with different boundary conditions and loadings meet at a corner. Moreover it is common that near the corner stress concentrations occur which would require a very fine mesh to calculate accurately.
- It appears that a plate loaded in-plane was thus successfully approximated by the tool as a limit case of a shell. However, during this research problems relating to the stretching boundary conditions of a plate or shell have been encountered, which will unfortunately have a big impact on the remaining part of this thesis. These problems will be discussed next.

5.3.3. Problems with the stretching boundary conditions

It was shown that the tool was able to successfully calculate a plate loaded in-plane. However, during this research it was discovered that not all plates which are loaded in-plane can be successfully modelled by the tool (nor by the tool developed by D. Liang (2012) for that matter).

Problems occur in the situation where only the displacements in the corners have been restrained while the edges that connect to the corners are not necessarily fixed. It was the intent in this thesis that for all considered shells calculated by the tool, the displacements u_x , u_y and w would be restrained for at least the four corner points, even though for example all the edges were free. To continue with shells with these sorts of boundary conditions, it therefore also had to be possible for the tool to calculate a plate loaded in-plane which only had displacement restrictions at the corners (Fig. 5.46).



Fig. 5.46 - plate in stretching with only corners fixed

In the end, it did not seem possible to implement conditions using the Airy stress function and the finite difference method to ensure the desired restrictions. The problem is caused by the fact that it is not clear which expression in terms of the Airy stress function should be used in the boundary condition component to ensure that the displacements of the corner points will remain zero when the edges themselves are not necessarily fixed. Boundary conditions of the corners need to match with the boundary conditions of the corresponding edges it seems, to produce good results.

Also the expressions found by Zienkiewicz and Gerstner (discussed at 3.2.9) appear to be only applicable for the situation where displacements across a distance are all zero, but not for the situation where the displacements of only a single point (i.e. a corner) is specified as zero. This can be explained as follows. Consider an edge parallel to the y-axis, the condition which has to be fulfilled according to these authors for the displacements to be zero is:

$$\frac{1}{Et}\left(\frac{\partial^3 \phi}{\partial x^3} + 2\frac{\partial^3 \phi}{\partial x \partial y^2}\right) = 0, \qquad \frac{1}{Et}\left(\frac{\partial^2 \phi}{\partial x^2}\right) = 0$$
(5.1)

It turns out that these expression can be rewritten and are actually equal to the derivatives of u_x and u_y :

$$\frac{1}{Et} \left(\frac{\partial^3 \phi}{\partial x^3} + 2 \frac{\partial^3 \phi}{\partial x \partial y^2} \right) = \frac{\partial^2 u_x}{\partial y^2}, \qquad \frac{1}{Et} \left(\frac{\partial^2 \phi}{\partial x^2} \right) = \frac{\partial u_y}{\partial y}$$
(5.2)

When an edge in the y-direction is fixed the displacements will be zero across the edge and so also these derivatives of u_x and u_y to y will be zero. But this does not follow for only a single point on that edge which is fixed while the surrounding points are not, thus the problem remains.

However, it turns out that when all edges have semi-rigid boundary conditions (i.e. only the displacements in the direction parallel to the edge are zero) the condition that the four corner points are restrained will be fulfilled automatically and the boundary conditions can always be successfully expressed in terms of the Airy stress function. Therefore the decision was made that for the remaining part of this thesis only shells with semi-rigid boundary conditions shall be further investigated. The challenge for successfully implementing all types of boundary conditions should be considered as an opportunity for future research.

5.4. Basic shell shapes

5.4.1. Elpar, cylindrical paraboloid and hypar

Next, three basic shell shapes shall be considered which have also been analysed by M.N. Pavlovic (1999), namely: an elpar, a cylindrical paraboloid and a hypar.



Fig. 5.47 - Basic shell shapes (images from Bouma 1959)

The shape functions that were used for each shape are given above. All edges were considered hinged and semi-rigid, implying: $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y^2} = 0$ at all edges. Furthermore, the following properties were chosen for all three shells:

a = b = 18m (plan dimensions) $E = 40kN/mm^2 \text{ (concrete)}$ $p = -2,22kN/m^2 \text{ (uniformly distributed on whole surface)}$ t = 100mm

Pavlovic used a rather elaborate method employing an eight-order differential equation⁴ and double Fourier series. In this thesis the same shell shapes will be analysed with the same properties and loading and will be compared to the results of Pavlovic. Note that the Young's modulus used by Pavlovic is relatively high for concrete, a more realistic value would be $E = 10kN/mm^2$, however since this is mainly a theoretical investigation and we want to compare the results for this test case using the same properties as much as possible $E = 40kN/mm^2$ will still be used in the tool. In his article Pavlovic does use a non-zero Poisson ratio of v = 0,15 however, thus some differences in results can be expected upfront. The mesh size in the parametric tool is set to 1m.

⁴ As shown by Vlasov (1964), one can combine the coupled equations (3.2.8) for shallow shells to one single equation.

In his article the results for the transverse displacements, normal forces n_{xx} and n_{yy} , moments m_{yy} and m_{xy} are given. Also Pavlovic provides graphs indicating how much load is carried by the S-surface and by the B-surface. These quantities will also calculated with the developed tool.

5.4.2. First results parametric tool

After specifying the shape function, boundary conditions, loading and material properties the tool shows the first results presenting the shape of the shell, the Airy stress function field and the displacement field:



By looking at the geometry of the shell, the Airy stress function field and the displacement field one can already establish that:

- The shell shapes Pavlovic chose are in fact very shallow.
- The hyper shows a displacement field with large displacements and curvatures and a very shallow Airy stress function field, indicating plate like behavior.
- The elpar shape seems to have an Airy stress function field and displacement field with smaller magnitudes and curvatures when compared to the cylindrical paraboloid, indicating smaller moments and normal forces.

Next more precise results will be given by comparing graphs including results produced from the tool with graphs from Pavlovic his article. It will be seen whether the tool can provide accurate results for these basic shell shapes.

5.4.3. Results displacements w

(along the diagonal from corner to centre):



Fig. 5.51 - Results Pavlovic: displacements along diagonal

Fig. 5.52 - Results parametric design tool: displacements along diagonal

5.4.4. Results normal forces

Results n_{xx} (from the middle of the edge to the centre):



Fig. 5.53 - Results Pavlovic: nxx

Fig. 5.54 - Results parametric design tool: nxx



Results n_{yy} (along the diagonal from corner to centre):

5.4.5. Results moments

Results m_{yy} (from the middle of the edge to the centre):



Fig. 5.57 - Results Pavlovic: myy

Fig. 5.58 - Results parametric design tool: myy





Fig. 5.59 - Results Pavlovic: mxy

Fig. 5.60 - Results parametric tool: mxy

5.4.6. Results Ps and Pb

Load carried by stretching surface along the diagonal:



Fig. 5.61 - Results Pavlovic: P_s along the diagonal





Load carried by bending surface along the diagonal:



Fig. 5.63 - Results Pavlovic: P_B along the diagonal

Fig. 5.64 - Results parametric tool: P_B along diagonal

5.4.7. Results shell assessment

Results shell assessment component elpar (top view):





Results shell assessment component cylindrical paraboloid (top view):



Results shell assessment component hypar (top view):

5.4.8. Discussion of the results

From the results the following can be observed:

- In general, when compared to the results of Pavlovic the results of the developed tool correspond very well.
- The differences in the results with respect to m_{yy} and m_{xy} can be fully explained by the fact that Pavlovic takes into account a Poisson's ratio while in the tool it assumed to be zero (the Poisson's ratio should not affect the normal forces however since these do not appear in the derivatives of the Airy stress function (3.2.6).
- Bending action is predominant in the hypar and thus shows similar behavior to that of a plate in bending, this confirms what authors like Beranek (Beranek W., 1979) have written with respect to hypar shells. A further investigation hypar shells compared to a flat plate is given in Appendix C.
- Stretching action is predominant in the centre zones of the elpar and cylindrical paraboloid shells. The displacements, normal forces and bending moments in the cylindrical paraboloid are larger compared to the elpar. It seems the elpar performs best.
- Edge moments that occur in the elpar are considerably damped out as one moves into the interior of shell, unlike the other two shapes.
- The load carried by the bending or stretching surface can at some points exceed the original applied load (i.e. $p_S > p$ or $p_B > p$) so that for equilibrium the corresponding load on the other surface becomes 'negative' which can be thought of as a sort of suction that occurs.
- The results from the shell assessment component for the hyper show that the ratios R_1 and R_2 are at every point close to zero percent. Also the value of the ratio R_3 at every point seem to be near zero thereby also indicating predominant bending behavior which is in line with the calculated quantities.

5.5. Moified basic shell shape

In the previous section three basic shell shapes have been analysed, these shapes were however very shallow, perfectly symmetrical (square projected floorplan) and had only a uniform distributed load over the whole surface. In this section a basic shape is chosen but modified with respect to its shallowness, floorplan and loading. It can then be checked whether the tool still provides accurate results. The shell is also calculated with SCIA Engineer for comparison of results.

5.5.1. Modified elpar

The shape that is chosen is an elpar with the following shape function and other properties:



A mesh size of 0,5m shall be used in the tool. An arbitrary loading is chosen such that on the left half of the shell a downward distributed load of 50 kN/m² is applied and on the right half an upward distributed load of 25kN/m² is applied (Fig. 5.81).



Fig. 5.81 - Shell surface with non-uniform loading

Next the results will be given for: w, n_{xx} , n_{yy} , n_{xy} , m_{xx} , m_{yy} , m_{xy} , v_x , v_y , v_n given as well as graphs of the values along lines of interest.

5.5.2. Results displacements w



5.5.3. Results normal forces





Results n_{yy} :



Results n_{xy} :



5.5.4. Results bending moments





Results m_{yy} :



Results m_{xy} :



5.5.5. Results Shear forces

Results v_x :







Results v_n :



Results shell assessment



5.5.6. Discussion of the results

From the previous results the following can be observed:

- In general, when compared to the results of SCIA the results of the developed tool correspond well.
- The tool can handle non-uniform distributed loads and still provide sufficiently accurate results.
- Bending action becomes more dominant near the edges and in the middle where the applied loads change in magnitude and direction.
- Slight differences occur in the results for the shear forces v_n , this might be explained by the fact that changes of v_n occur rapidly along the considered line and so a smaller mesh size is required for more accuracy. The finite difference method works better for smooth varying quantities.
- From the results from SCIA it follows that on the edges near the corners the bending moments and normal forces are not zero, even though theoretically they should be. By contrast the results from the parametric tool do show the bending moments and normal forces to be zero since these are 'enforced' to be zero through the boundary condition component. This also shows a fundamental difference between the methods of the finite element method and the finite difference method used by the parametric tool. The finite difference method is much less straightforward in determining and presenting the results. For example, with the FEM there are discontinuities with respect to the forces and stresses at the nodes of the elements (this is caused by the fact that the FEM needs to extrapolate quantities from the integration points which are located inside the element).

5.6. Hallenbad shell shape

In the previous sections several basic shell shapes have been analysed and discussed. The geometry of the surfaces of those basic shell shapes were determined by shape functions. In this section a more arbitrary shell shape shall be analysed. This shape is not determined through the use of a shape function, but rather a NURBS surface is provided as input.

5.6.1. Hallenbad shell analysis

In the following test case a shell shape will be analysed which is based on a shell designed by Heinz Isler, namely the Hallenbad (Fig. 5.118) in the city Brugg in Switzerland. The shell is 35m long and 35m wide and made of concrete.



Fig. 5.118 - Hallenbad shell designed by Heinz Isler (Brugg, 1981)

Using a special 3D scanner, A. Borgart and P. Eigenraam (2012) were able to scan scale models that were originally made and used by Heinz Isler. The results of such a scan, called "cloud points", were then converted to NURBS surfaces (Fig. 5.119).



Fig. 5.119 - 3D scanning proces (image from Borgart and Eigenraam 2012)
One of the scale models that were scanned was a scale model of the Hallenbad (Fig. 5.120). The NURBS surface that was obtained will now be used as input in the parametric tool. There might be a worry that the shape of the shell does not at every point fulfil the conditions to qualify as a shallow shell. For example, it can be observed that near the corners the shell has a quite large slope. If the shell does not qualify as a shallow shell the results produced cannot be expected to be accurate. Therefore a second shell surface based on the original is made and also analysed. This second shell is the same as the original but made more shallow by scaling it a factor 0,5 in z-direction (Fig. 5.121).



Fig. 5.120 - Original shell shape

Fig. 5.121 - Flattened shell shape

For both shell shapes the following properties are chosen for the analysis:

a = b = 35m (plan dimensions) $E = 40kN/mm^2$ (concrete) $p = -10kN/m^2$ (uniformly distributed on whole surface) t = 100mm

Furthermore we specify the boundary conditions on the edges to be semi rigid and hinged implying that: $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y^2} = 0$ on all edges. Both shell shapes will be analysed in the parametric tool with a mesh size of 2,5m and 1,25m so that it can be seen how mesh size is related to the accuracy of the results. For comparison both shell shapes are again calculated with SCIA Engineer.

First results parametric tool

When the shapes are given as input in the parametric tool one can gets the following first results from the shell calculation component:



By looking at the displacement field and Airy stress function field the following can be already observed: the displacement field of the flattened shell shape shows larger displacements and curvatures than that of the original shell shape. Thus more bending action can be expected for the flattened shell. Also the Airy stress function field seems to have larger values for the flattened shell. Indicating that at least at some points the normal forces will be bigger for the flattened shell as well. Next more precise results are given for: w, n_{xx} , n_{yy} , n_{xy} , n_1 , n_2 , m_{xx} , m_{yy} , m_{xy} , m_1 , m_2 v_x , v_y , v_n along lines of interest.

5.6.2. Results displacements w



Exact results displacements w:





5.6.3. Results normal forces







Exact results n_{xx} :



Results n_{yy} :





Exact results n_{yy} :



Results n_{xy} :









Results n_1 :





Exact results n_1 :



Results n_2 :





Exact results n_2 :



5.6.4. Results bending moments







Exact results m_{xx} :



Results m_{yy} :





Exact results m_{yy} :



Results m_{xy} :





Exact results m_{xy} :



Results m_1 :





Exact results m_1 :



Results m_2 :





Exact results m_2 :



5.6.5. Results shear forces

Results v_x :





Exact results v_x :



Results v_y :





Exact results v_y :



Results v_n :





Exact results v_n :



5.6.6. Results sum of bending moments and principle shear forces



Sum of bending moments:

Principal shear trajectories for the flattened shell:



5.6.7. Discussion of results

From the results the following can be observed:

- In general the results from the parametric tool match the results from SCIA better in the case of the flattened shell shape than for the original shell shape.
- In the case of the original shell shape, the results are often not accurate. When compared to the results from SCIA, one can see relative differences in magnitude up to 40% for certain quantities. This indicates that the original shell shape is most likely not shallow enough to be properly modeled by the tool. This is not to say however that for every quantity this is the case, the results from the parametric tool and SCIA for n_{xx} and n_1 seem to correspond quite well. Moreover when the results are plotted, the form of the plot or graph is often still in line with results from SCIA (i.e. when the results of a certain quantity in SCIA varies over a specific region, the results produced by the parametric tool will vary in the same way over that region). In this sense the tool still provides qualitative insight into behavior of the shell.
- In the case of the flattened shell the results from the parametric tool with a mesh of 2,5m x 2,5m in general produces results that differ between 0% and 10% from the results from SCIA. For the calculations of the shear forces these differences becomes even larger and cannot be said to produce accurate results. This might be explained by the fact that more rapid changes in the magnitude of the shear forces occur and the mesh size is simply to big to calculate these changes accurately.
- With a mesh of 1,25m x 1,25m the results are in general better and differ between 0% and 5% from the results from SCIA. However at the edges or corners bigger differences occur for the quantities n_{xy} , v_x , v_y and v_n between the results from SCIA and from the parametric tool. At least with respect to the shear forces this can be explained by the fact that in the parametric tool the derivative component has difficulties with calculating the correct slope of the sum of bending moments hill at the edges.

6. Conclusion

6.1. Introduction

In this chapter it will be evaluated whether the main objective and secondary objectives have been satisfied. Recall from chapter 1 that the main objective was the following:

"Develop a parametric structural design tool for shell structures that can be used by architects and engineers, which is based on simple analytical methods, which gives both quantitative and qualitative (real time) insight in the flow and magnitude of forces during a conceptual design stage."

In order to achieve this goal several secondary objectives had to be satisfied which are listed in 1.2 and can be summarized as: providing the theoretical framework, developing the tool and validating results. In the next paragraph each of these will be shortly evaluated to see whether the secondary objectives have indeed been satisfied. After this a list of recommendations is provided. In the end the final conclusion will be given and determined if the main objective has been achieved.

6.2. Conclusions

6.2.1. Theoretical framework

With respect to the theoretical framework a two-surface theory proposed by Calladine for understanding shell structures has been laid out. It was shown that the idea of separating the behavior of a shell into two distinct parts affords the possibility of thinking separately about two different aspects of shell behavior while still allowing for the actual interaction between them.

The finite difference method has also been provided as a method which quite literally goes back to the analytical equations which it needs to solve. It is a more intuitive and straightforward method which affords the possibility to obtain qualitative insight into the structural problems it needs to solve, this is especially the case because it can be applied to surfaces. This is important because working with the results and visualizing the results as surfaces provides more and faster insight compared to say a table with only numbers. Moreover the finite difference method is very suitable for computational applications.

Furthermore, the rain flow analogy appears to be a powerful and insightful way of thinking about certain structural behavior. It is a very suitable method for obtaining qualitative insight in a direct manner in the way it is used to understand and visualize the principal shear forces trajectories.

With respect to shell behavior assessment methods, two methods of assessments have been provided. It can be said that these methods together provide a very insightful way of thinking about the efficiency of a shell. The first method concerning the ratio of bending and normal stress can be thought of as a very concrete way in which the shell can be said to perform well, its physical interpretation is very clear because the stress caused by bending and the stress caused by stretching are real physical quantities. The second method determines shell efficiency based on the ratio of load carried by the stretching surface and the load carried by the bending surface. This is a bit more abstract way of thinking about it since the bending surface and stretching surface are not really two distinct surfaces but are conceptual in nature. Still the two surface concept by Calladine provides an insightful way of thinking about shells, thus this method of assessing shells is very much in accordance with that.

6.2.2. Development of the tool

For the development of the tool use of several computational applications has been made. Rhino & Grasshopper were chosen and provided the right environment for parametric modelling. Also VB (Visual Basic) scripting was used to create new components which offers more possibilities and flexibility.

With respect to the demands which have to be fulfilled concerning functionality and usability the following can be concluded

- The tool is able to provide real-time results, however with bigger models and higher number of nodal points a decrease in speed can occur. For shell shapes where a small mesh size is not necessary per se the tool works fast and results are obtained in a matter of seconds.
- Options in the tool have been developed to be able to choose in a simple way which- and how results are presented. When results are presented as surfaces, great qualitative insight can be gained. Moreover, in the tool it easy to obtain the magnitude of the quantities as well.
- In the tool parametric modelling becomes possible. Design parameters such as geometry, material properties and loading can be easily changed and results change accordingly. However some problems have been encountered with the implementation of the (stretching) boundary conditions (see 5.3.3). This caused the analysis of shells in this thesis to be limited to shells with semi-rigid boundary conditions. Still it was shown that a plate in bending and a plate in stretching with mixed boundary conditions could be approximated well with the tool.
- Extending the functionality of the tool for future development is certainly possible. This can be done by adding new parametric components and extending current ones.
It was envisioned that the tool was able to give as output the following quantities:

- Airy stress function ϕ
- Transverse displacement *w*
- Internal forces:
 - Normal forces
 - Principal normal forces
 - Sum of normal forces
 - Bending moments
 - Principal moments
 - Sum of bending moments
 - Shear forces
 - Principal shear forces
- Trajectories of the principal shear force (rain flow analogy)
- Measure for assessment shell behavior

It can be said that by implementing the theoretical framework into the components of the tool, the tool is able to produce all mentioned quantities. A general outline and structure of the tool has been given in paragraph (4.4). There it can be seen which components produces which results.

The finite difference method played an important role in many of the components used, these components were the curvature-, boundary conditions-, shell calculation-, internal forces-, and shell assessment component. In the shell calculation component the coupled differential equations for shallow shells are really solved and is considered the most important component. The rain flow analogy is off course used in the rain flow analogy component and in the derivative component which were already developed by M. Oosterhuis (2010). Overall, it can be concluded that the theoretical framework was implemented well within the current developed tool.

6.2.3. Results

As was stated in chapter (5.1), in choosing which shell shapes should be analysed it was presumed that it was best to start from simple to more complex shapes. The following shapes were chosen:

- 1. Plate loaded out-of-plane as a limit case of a shell
- 2. Plate loaded in-plane as a limit case of a shell
- 3. Basic shell shapes: elpar, cylindrical paraboloid, hypar
- 4. Modified basic shell shape
- 5. Arbitrary shell shape

Though most of the conclusions concerning these results can be read at the end of each paragraph of each shell shape, the most important will be summarized here:

- In general the results from the parametric tool provide accurate quantitative as well as qualitative insight into the structural behavior of shell structures. When the considered shell shape qualifies as a shallow shell and the mesh size is not chosen too big, one can obtain results with an accuracy of less than 5% deviation compared to analytical en FEM solutions.
- The tool is able to approximate flat plates in bending with mixed boundary conditions very well.
- Also a flat plate loaded in-plane with mixed boundary conditions was approximated well by the tool, however problems with other types of boundary conditions were discovered (see 5.3.3)
- When stretching boundary conditions in terms of displacements for only the corners of a plate or shell have been specified, this unfortunately cannot yet be handled by the tool. To ensure the corner points remain fixed it was chosen to further only consider semi-rigid boundary conditions on all edges.
- Results for basic shell shapes (elpar, cylindrical paraboloid and hypar) correspond well with results given by Pavlovic (1999). The elpar is shown to perform best, unlike the hypar which shows plate like behavior.
- It is shown that the load carried by the bending or stretching surface can sometimes exceed the magnitude of the original applied load (i.e. $p_S > p$ or $p_B > p$) so that for equilibrium the corresponding load on the other surface becomes 'negative' which can be thought of as a sort of suction that occurs.

- From the results from FEM solutions it sometimes follows that on the edges (especially near the corners) the bending moments and normal forces are not zero, even though theoretically they should be zero there because of the boundary conditions. By contrast the results from the parametric tool do show the bending moments and normal forces to be zero in those cases since these are 'enforced' to be zero through the boundary condition component. This shows a fundamental difference between the nature of the finite element method and the finite difference method used by the parametric tool.
- When analysing a shell shape which is not shallow enough, the results are often not accurate. This seems to be the case of the original Hallenbad shell shape (5.6.1) When compared to the results from SCIA, one can see relative differences in magnitude up to 40% for certain quantities. This indicated that the original shell shape is most likely not shallow enough to be properly modeled by the tool. This is not to say however that for every quantity this is the case, the results from the parametric tool and SCIA for some quantities correspond quite well. Moreover when the results are plotted, the form of the plot or graph is often still in line with results from the FEM programm (i.e. when the results of a certain quantity in SCIA varies over a specific region, the results produced by the parametric tool will vary in the same way over that region). In this sense the tool still provides qualitative insight into behavior of the shell.
- In the case of shell made more flat (5.6.1) and having dimension of 35m by 35m the results from the parametric tool with a mesh of 2,5m x 2,5m in general produces results that differ between 0% and 10% from the results from a FEM program. For the calculations of the shear forces these differences become even larger and cannot be said to produce accurate results. This might be explained by the fact that more rapid changes in the magnitude of the shear forces occur and the mesh size is simply to big to calculate these changes accurately.
- With a mesh of 1,25m x 1,25m in the above mentioned case the results are in general better and differ between 0% and 5% from the results from the FEM program. However at the edges bigger differences occur for the quantities n_{xy} , v_x , v_y and v_n . Some of these differences are explained by the fact that at certain specific points there is a difference between what the FEM approximates and what the theoretical value should be. The parametric tool tries to approximates this theoretical value. It must be kept in mind however that the theoretical results not always describe physical reality at these special points of interest.

6.2.4. Some other cautions and limitations

The theory of (shallow) shells is based, like other areas of the mechanics of solids, on several simplifying assumptions, so that it can only describe the behaviour of shells with a greater or smaller degree of error. Moreover, many errors and uncertainties stem from considering the material as continuous without cracks, homogeneous and isotropic, which is not at all true of reinforced concrete shells. Other uncertainties stem from the assessment of expected loads on a shell, inaccuracies in the erection (e.g. deviations in curvature and thickness), chemical and physical influences (e.g. thermal effects, shrinkage, creep etc.). With the aid of advanced computational methods the accuracy of the analysis can be improved, limited only by the capacity of the computer, but it is important to remember that the results cannot be more exact than the computation model serving as a basis, which, due to unavoidable simplifications, more or less deviates from physical reality.

6.3. Recommendations

Since the research in this thesis is only one step towards a complete structural analysis tool several recommendations can be made for future research and extension of the tool. The following recommendations can be made:

- The tool should be extended by:
 - including the possibility for tangential loading, such an inclusion will change some of the equations used, it should be explored whether these changes can be implemented in the program
 - application of finite difference method for arbitrary meshes so the tool is not limited to only rectangular shallow shells
 - application for non-shallow shells
 - including the possibility for buckling analysis of shells
- Exploring use of the so called displacement potential function (Ahmed, 1998) instead of the Airy stress function, this might solve problems with certain boundary conditions in terms of displacements

6.4. Final conclusion

Reflecting on the main objective of this thesis it can be said that it was achieved with reasonable success. A parametric structural design tool for shell structures has been developed which gives architects and engineers both qualitative and quantitative insight into the behavior of shell structures. Some limitations and problems remain however, especially with respect to the boundary conditions, these provide new opportunities for future research.

7. Bibliography

Ahmed, S. (1998). Investigation of stresses at the fixed end of deep cantilever beams. *Computers and structures*, 329-338.

Beranek, W. (1976). Berekenen van platen, Deel 1, Theoretische grondslagen (Lecture notes). Delft: TH Delft.

- Beranek, W. (1979). K3 dictaat ruimtelijke constructies, deel 2. Delft: TH Delft.
- Blaauwendraad, J. (2010). Plates and FEM. Dordrecht: Springer.
- Borgart, A., & Eigenraam, P. (2012). Scanning in 3D and analysing the models of Heinz Isler, the preliminary results. *LASS-APCS 2012: From spatial structures to space structures*. Seoul: IASS.
- Borgart, A., Hoogenboom, P., & Leeuw, M. &. (2005). The relationship of form and force in (irregular) curved surfaces. *Proceedings of the 5th International Conference on Compution of Shell and Spatial Structures*.
- Bouma, A. (1959). Some applications of the bending theory regarding doubly curved shells. *Proceedings of the Symposium on the Theory of Thin Elastic Shells*, (pp. 202-232). Delft.
- Calladine, C. (1977). The static-geometric analogy in the equations of thin shell structures. Mathematical Proceedings of the Cambridge Philosophical Society / Volume 82 / Issue 02, 335-351.
- Calladine, C. (1983). Theory of Shell structures. Cambridge: Cambridge University Press.
- Coenders, J. (2006). Aproximating intelligent structures: Embedded design intelligence in systems for the early phases of design. *LASS-APCS 2006 Symposium: New Olympics New Shell and Spatial Structures.* Beijng: IASS.
- Csonka, P. (1987). Theory and practice of membrane shells. Budapest: Akadémiai Kiadó.
- Donnell, L. (1933). Stability of Thin Walled Tubes under Torsion. Washington: NACA.
- Flügge, W. (1960). Stresses in shells. Berlin: Springer Berlin Heidelberg.
- Gauss, K. (1827). General investigation of curved surfaces.
- Gol'denveizer, A. (1961). Theory of Elastic Shells. New York: Pergamon Press.
- Liang, D. (2012). A parametric structural (Grasshopper Interface) design tool for plate structures. Delft: Delft University of Technology.
- Lur'e, A. (1940). General theory of elastic shells. Prikl Mat Mekh, 7-34.
- Mushtari, K. M. (1938). Certain generalizations of the theory of thin shells.
- Novozhilov, V. (1964). Thin Shell Theory. Groningen: Noordhoff.
- Oosterhuis, M. (2010). A parametric structural design tool for plate structures. Delft: Delft University of Technology.
- Pavlovic, M. (1999). Interaction of bending and stretching actions in shallow translational shells with various Gaussian curvature types. *Engineering Structures*, 538-553.

Roeder, L. (2002). http://www.aisto.com/roeder/dotnet.

- Ventsel, E., & Krauthammer, T. (2001). Thin Plates and Shells theory analysis and application. CRC Press.
- Vlasov, V. (1964). General Theory of Shells and its Applications in Engineering. Washington: NASA.
- Zienkiewicz, O., & Gerstner, R. (1960). A stress function approach to A stress-function approach to interface and mixed boundary-condition problems. *International Journal of Mechanical Sciences*, 93-101.

Appendix A - Derivation strain compatibility equation

Change of Gaussian curvature in terms of strains ε_{xx} , ε_{yy} and γ_{xy} First and second derivative of ε_{xx} to y:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2$$
$$\frac{\partial \varepsilon_{xx}}{\partial y} = \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y}$$
$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial w}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$
$$= A + B + C$$

First and second derivative of ε_{yy} to x:

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2$$
$$\frac{\partial \varepsilon_{yy}}{\partial x} = \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y}$$
$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y} + \frac{\partial w}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$
$$= D + E + C$$

Derivatives of γ_{xy} to x and y:

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$
$$\frac{\partial^2 \gamma_{xy}}{\partial x} = \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial w}{\partial x} \cdot \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x^2} \cdot \frac{\partial w}{\partial y}$$
$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y} + \left(\frac{\partial w}{\partial x} \cdot \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^2 w}{\partial x \partial y^2} \frac{\partial^2 w}{\partial x \partial y}\right) + \left(\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y}\right)$$
$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y} + \frac{\partial w}{\partial x} \cdot \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$
$$= A + D + B + E + F + C$$

From the previous derivations it follows that:

$$-\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} =$$
$$-(A + B + C) + (A + D + B + E + F + C) - (D + E + C) =$$
$$F - C = \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

This last expression can be recognized as the change in Gaussian curvature (2.xx) from an initial flat surface. In this thesis we are concerned with *shallow* shells. Over the region of interest the surface makes such small angles with a particular tangent plane that the x, y coordinate system for the plane (with origin at the point of tangency) may be used for the surface without significant loss of accuracy. Therefore, for shallow shells it follows:

$$g = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

Change of Gaussian curvature in terms of displacements w

Recall from 2.xx the definition of Gaussian curvature

$$k_G = k_1 \cdot k_2 = \frac{1}{R_1} \cdot \frac{1}{R_2}$$
(2.xx)

Let κ_1 and κ_2 be the change of curvatures after deformation in principal directions 1 and 2 respectively. Then by differentiating the previous equation for the Gaussian curvature one obtains the equation for the change of Gaussian curvature:

$$g = \frac{1}{R_1}\kappa_2 + \frac{1}{R_2}\kappa_1 = \frac{\kappa_2}{R_1} + \frac{\kappa_1}{R_2}$$

Thus two expressions for the change of Gaussian curvature have been obtained which lead to the geometric compatibility equation for shells:

$$g = \frac{\kappa_2}{R_1} + \frac{\kappa_1}{R_2} = -\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

When one is not working in the principal directions one starts with the following equation for the Gaussian curvature (2.xx):

$$k_G = k_{xx} \cdot k_{yy} - k_{xy}^2$$

Let κ_{xx} , κ_{yy} be the change of curvatures in the *x* and *y* direction respectively and κ_{xy} the change of the twist of the surface. Then by differentiating the previous equation for the Gaussian curvature one obtains the equation for the change of Gaussian curvature:

$$g = k_{xx} \cdot \kappa_{yy} + k_{yy} \cdot \kappa_{xx} - 2k_{xy} \kappa_{xy}$$

Next κ_{xx} , κ_{yy} and κ_{xy} can be expressed in terms of the transverse displacements *w*:

$$\kappa_{xx} = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}$$

Substitution gives:

$$g = k_{xx}\frac{\partial^2 w}{\partial y^2} + k_{yy}\frac{\partial^2 w}{\partial x^2} - 2k_{xy}\frac{\partial^2 w}{\partial x \partial y}$$

Thus the compatibility equation can also be formulated as:

$$k_{xx}\frac{\partial^2 w}{\partial y^2} - 2k_{xy}\frac{\partial^2 w}{\partial x \partial y} + k_{yy}\frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

Appendix B - SCIA Engineer elements

Advanced Concept Training - FEM

The elements in Scia Engineer

Type of Element – Solver

The solver of SCIA ENGINEER uses the same element for plates as for the bending behaviour of shell members. Analogously, the same element is used for walls and for their functioning, namely the wall inner forces.

In a General XYZ environment there are 6 degrees of freedom for each node. Physically, these 6 degrees of freedom represent the following: the displacements u, v, w and the rotations φ_x , φ_y , φ_z . The components of displacement are given in the local axis of the element. So u, v, φ_z represent the plane stress/strain state, w, φ_x , φ_y indicate bending/shear force.

The element used in Scia Engineer for the calculation of membrane forces includes a 3-nodes triangle and a 4-nodes quadrilateral with 3 degrees of freedom per node.



For the bending behaviour there are 2 types of elements implemented:

- The Mindlin element including shear force deformation
- The Kirchhoff element without shear force deformation

For more information see ref. [5]. A detailed description of the element used for bending/shear force is given in ref. [7].

[7] Lynn-Dhillon, 2nd Conf. Matrix Meth. JAPAN - USA, Tokyo, 1971.

Appendix C - Hypar vs flat plate

In chapter 5.4 several basic shell shapes were analysed one of which was a hyper shell shape. In this appendix a further investigation is shown with respect to this shape. From the previous results in 5.4 it followed that bending action was predominant in the hyper shell, now it will be investigated to what degree a hyper shell really behaves like a flat plate. The same loading, properties and dimensions as the hyper from 5.4 are chosen but now only the shallowness will be changed. Thus the original shape function of the hyper was:

$$z_{EP} = -\frac{1}{300}(x^2 - y^2)$$

Now a shape function which equals (almost) zero is analysed to approximate a flat plate in bending, as well as two shape functions which are steeper than the original hyper:



From the first results produced by the tool it can be seen that the transverse displacements do not differ that much from each other, and also the moments seem to differ only little. Still it can be seen directly that when the shape function is chosen as virtually zero (approximating a flat plate in bending), the membrane action is also zero. By contrast, some membrane forces still appear to develop in the hypar shell shapes.



Displacements w:





From these more exact results it follows that hypar shells show similar behavior to that of a flat plate in bending. However it seems that some differences do occur. The displacements w and bending moments m_{yy} around the centre increase slightly when the hypar is chosen steeper. However between the edges and the centre area the opposite appears to be true.

With respect to the twisting moments m_{xy} , near the corner the flat plate seems to have the largest value while choosing steeper hypar shapes lead to a slightly smaller value near the corner. Between the centre area and the corners again the opposite appears to be true.

Results ratio R3 (= Ps / (Ps + Pb))



From the results for the ratio R3 (= Ps / (Ps + Pb)) of the shell assessment component (4.5.9) it follows that in the case of the flat plate approximation, the ratio R3 equals zero everywhere indicating that no part of the external load is carried by the stretching surface and is carried by the bending surface only, which is what one ought to expect from a flat plate in bending.

When the hyper is very shallow $(z_{EP} = -\frac{1}{300}(x^2 - y^2))$ non-zero values for the ratio R3 appear. Thus some part of the load is carried by the stretching surface but since ratio R3 is closer to zero at most points, most of the load is still carried by the bending surface. Negative values for ratio R3 around the centre indicate the load carried by the bending surface exceeds the external applied load at those points. It follows that the load carried by the stretching surface works in the opposite direction of the external applied load, this is to ensure equilibrium.

When the hyper becomes steeper $(z_{EP} = -\frac{1}{150}(x^2 - y^2))$ and $z_{EP} = -\frac{1}{75}(x^2 - y^2))$ it can be seen that near the edges the load carried by the stretching surface increases In the centre, the load carried by the bending surface becomes larger and exceeds the external applied load even further. The load carried by the stretching surface in that area therefore also has to increase, though in the opposite direction.

From these results it can be concluded that hyper shells which are quite shallow behave similar to flat plates (bending action is dominant) but still not completely. Some membrane forces still develop while in a flat plate this is not the case. For steeper hyper shells more membrane forces develop and the differences in bending quantities compared to a flat plate slightly become larger.