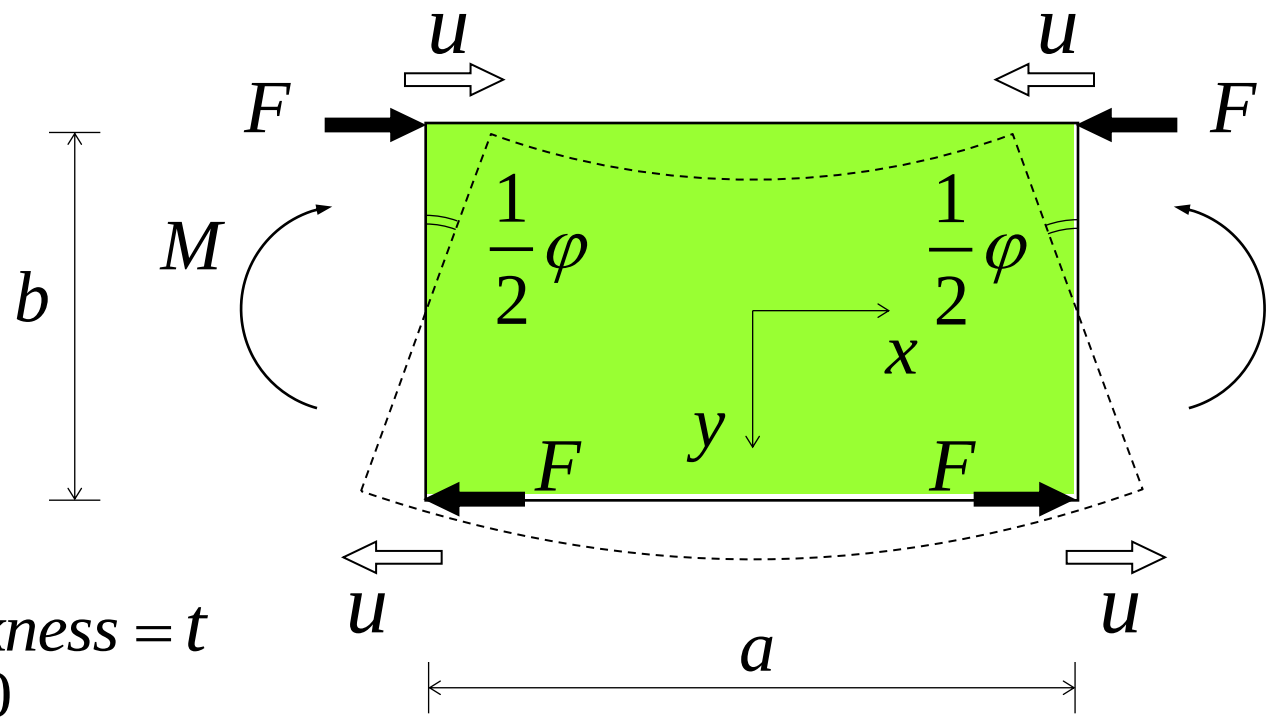
Bending theory



$$M = \frac{EI}{a}\varphi$$

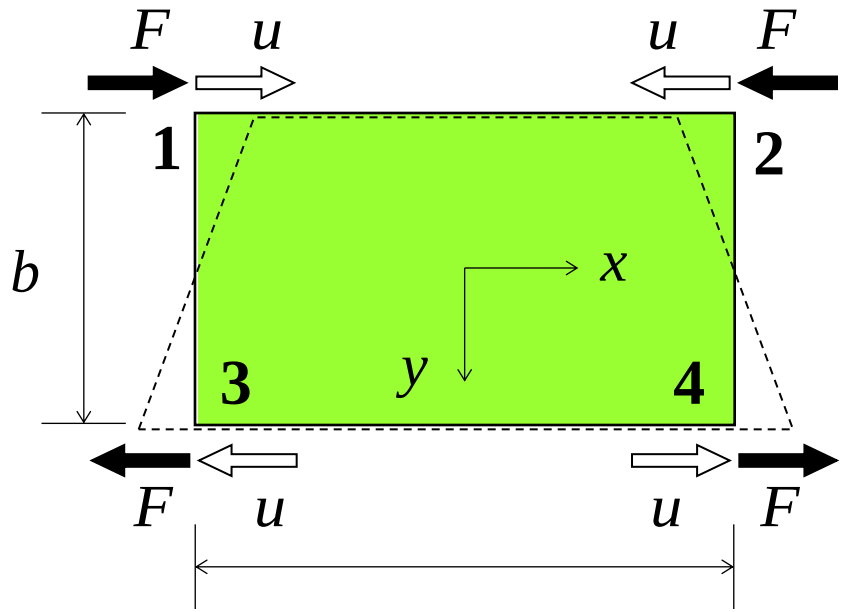
$thickness = t$

$\nu = 0$

Using: $M = bF$, $\varphi = 4u/b$ and $I = t(b)^3/12$

$$F = \frac{1}{3}E\frac{tb}{a}u \quad \rightarrow \quad K_{exact} = \frac{1}{3}E\frac{tb}{a}$$

Rectangular Element :



$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{g^+}{a}u_{x1} + \frac{g^+}{a}u_{x2} - \frac{g^-}{a}u_{x3} + \frac{g^-}{a}u_{x4}$$

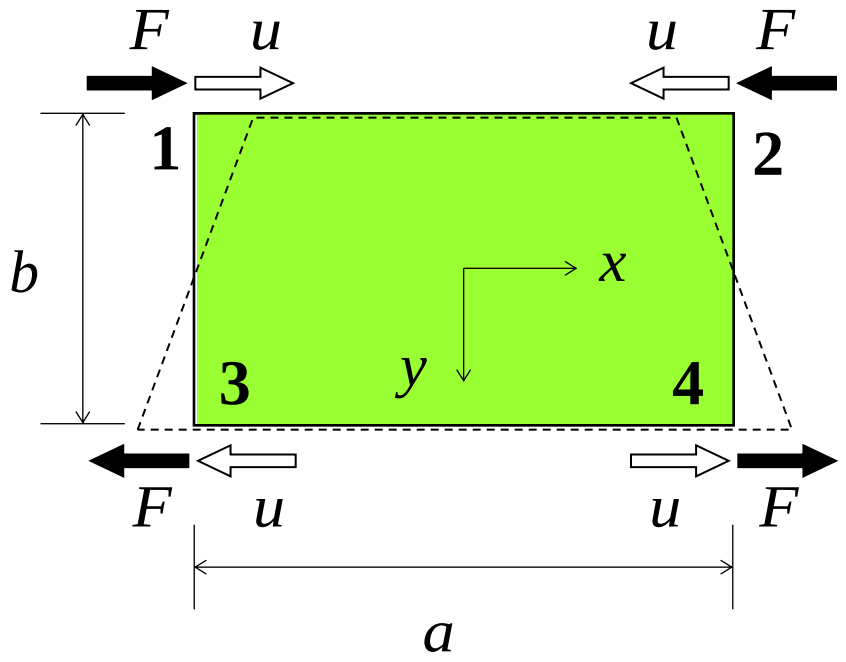
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = -\frac{f^+}{b}\overset{0}{u_{y1}} - \frac{f^-}{b}\overset{0}{u_{y2}} + \frac{f^+}{b}\overset{0}{u_{y3}} + \frac{f^-}{b}\overset{0}{u_{y4}}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{f^+}{b}u_{x1} - \frac{f^-}{b}u_{x2} + \frac{f^+}{b}u_{x3} + \frac{f^-}{b}u_{x4} +$$

$$-\frac{g^+}{a}\overset{0}{u_{y1}} + \frac{g^+}{a}\overset{0}{u_{y2}} - \frac{g^-}{a}\overset{0}{u_{y3}} + \frac{g^-}{a}\overset{0}{u_{y4}}$$

Rectangular Element :

$$u_x(x,y) = 4\frac{x}{a}\frac{y}{b}u$$
$$u_y(x,y) = 0$$



$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{4y}{a\,b}u$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{4x}{a\,b}u$$

$$\boldsymbol{B}(x,y) = \begin{bmatrix} \frac{4y}{a\,b} \\ 0 \\ \frac{4x}{a\,b} \end{bmatrix}$$

Rectangular Element

$$\boldsymbol{K} = t \iint \boldsymbol{B}^T \boldsymbol{D}\, \boldsymbol{B}\, dx\, dy$$

$$\boldsymbol{K} = t \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \begin{bmatrix} \frac{4y}{ab} & 0 & \frac{4x}{ab} \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} \frac{4y}{ab} \\ 0 \\ \frac{4x}{ab} \end{bmatrix} dy\, dx$$

$\upsilon = 0$

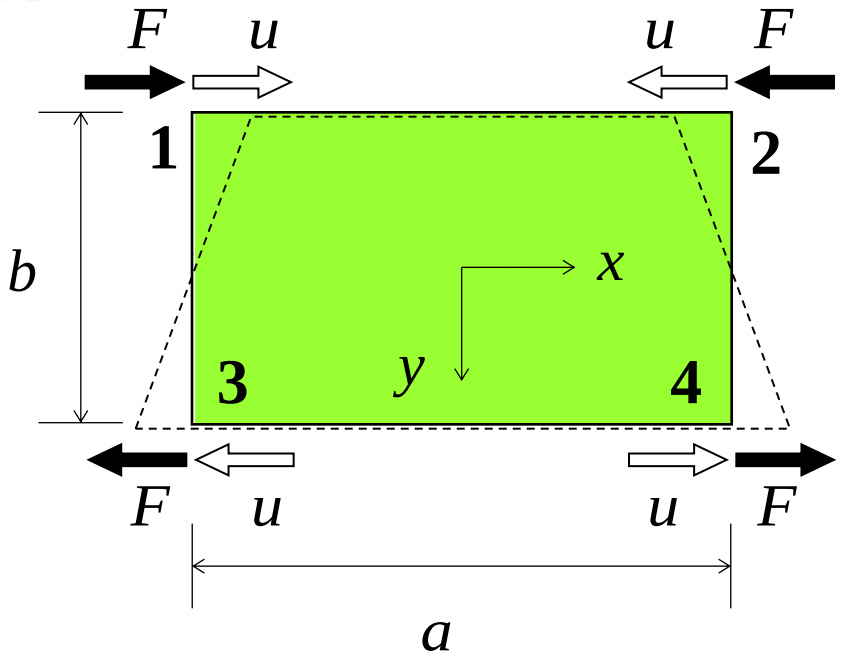
$$\boldsymbol{K} = \frac{4}{3} E \frac{tb}{a} \left(1 + \frac{1}{2} \frac{a^2}{b^2} \right)$$

Beam stiffness

- the virtual work equation:

$$4F\,\delta u = \delta u\,\boldsymbol{K}\,u$$

Internal forces



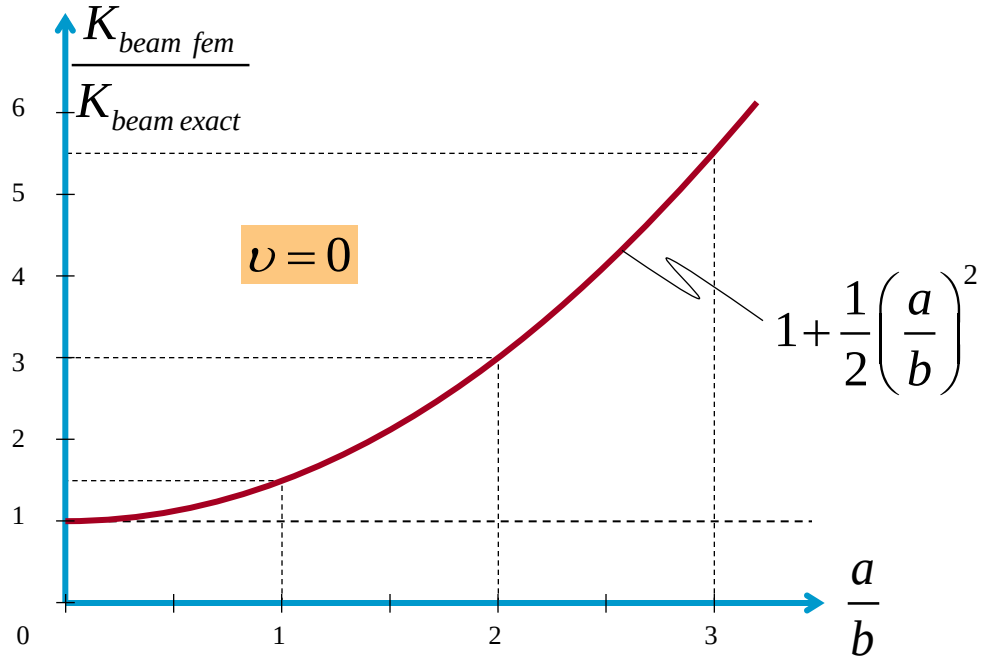
$$\boldsymbol{K} = \frac{4}{3} E \frac{t\,b}{a}\left(1 + \frac{1}{2}\frac{a^2}{b^2}\right)$$

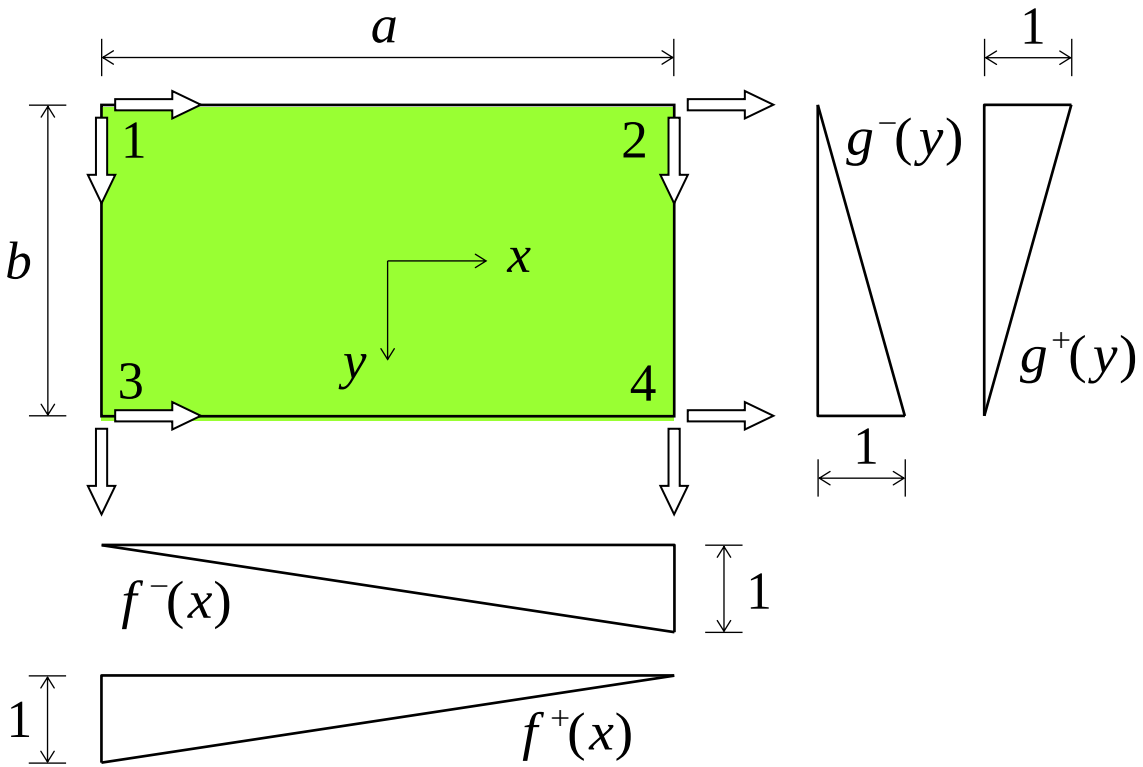
$$F = \underbrace{\frac{1}{3} E \frac{t\,b}{a}\left(1 + \frac{1}{2}\frac{a^2}{b^2}\right)}_{K_{beam\ fem}} u$$

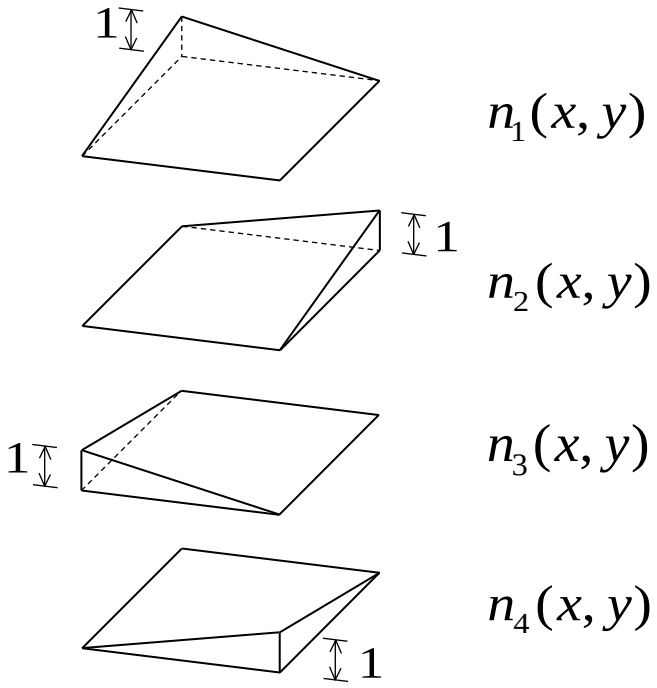
Rectangular Element

$$K_{beam\ fem} = \frac{1}{3} E \frac{t\,b}{a} \left(1 + \frac{1}{2} \frac{a^2}{b^2} \right)$$

$$K_{beam\ exact} = \frac{1}{3} E \frac{t\,b}{a}$$







$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$

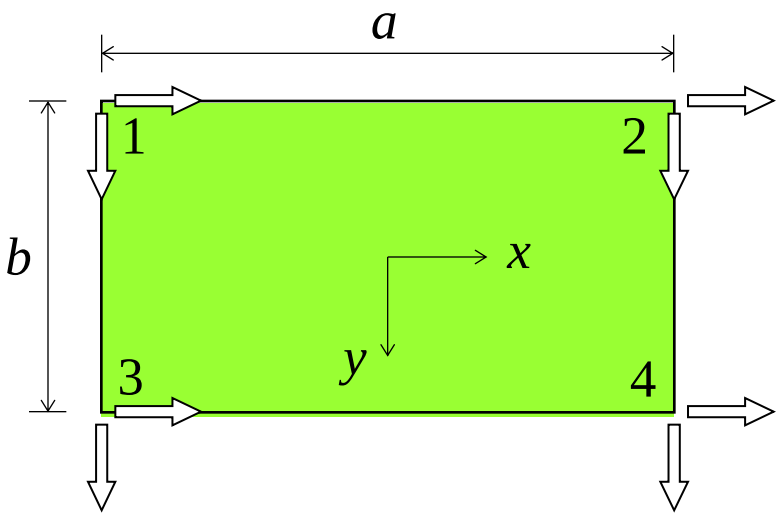
$$n_1(x,y) = f^+(x) \cdot g^+(y)$$

$$n_2(x,y) = f^-(x) \cdot g^+(y)$$

$$n_3(x,y) = f^+(x) \cdot g^-(y)$$

$$n_4(x,y) = f^-(x) \cdot g^-(y)$$

Constant Shear Element



$$@ \quad x = y = 0 \quad \Rightarrow$$

$$f^- = f^+ = g^- = g^+ = \frac{1}{2}$$

$$n_1(x,y) = f^+(x) \cdot g^+(y)$$

$$n_2(x,y) = f^-(x) \cdot g^+(y)$$

$$n_3(x,y) = f^+(x) \cdot g^-(y)$$

$$n_4(x,y) = f^-(x) \cdot g^-(y)$$

Constant Shear Element

$$@ \quad x = y = 0 \quad \Rightarrow \quad f^- = f^+ = g^- = g^+ = \frac{1}{2}$$

$$\iint_A \left(f^+\right)^2 dA = \iint_A \left(f^-\right)^2 dA = \iint_A \left(g^+\right)^2 dA = \iint_A \left(g^-\right)^2 dA = \frac{1}{4} a\, b$$

$$\iint_A f^- f^+ dA = \iint_A g^- g^+ dA = \frac{1}{4} a\, b$$

$$\iint_A f^- g^+ dA = \iint_A f^- g^- dA = \iint_A f^+ g^- dA = \iint_A f^+ g^+ dA = \frac{1}{4} a\, b$$

$$f^- = f^+ = g^- = g^+ = \frac{1}{2}$$

$$\boldsymbol{K} = \iint_A \boldsymbol{B}^T(x,y)\,\boldsymbol{D}(x,y)\,\boldsymbol{B}(x,y)\,dA$$

$$\begin{bmatrix}
\frac{(g^+)^2}{a^2} + \frac{1-\upsilon}{2}\frac{(f^+)^2}{b^2} & \frac{\upsilon f^+ g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^+}{ab} & -\frac{(g^+)^2}{a^2} + \frac{1-\upsilon}{2}\frac{f^- f^+}{b^2} & \frac{\upsilon f^- g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^+}{ab} & \frac{g^- g^+}{a^2} + \frac{1-\upsilon}{2}\frac{(f^+)^2}{b^2} & -\frac{\upsilon f^+ g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^-}{ab} & -\frac{g^- g^+}{a^2} + \frac{1-\upsilon}{2}\frac{f^- f^+}{b^2} & -\frac{\upsilon f^- g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^-}{ab} \\
 & \frac{(f^+)^2}{b^2} + \frac{1-\upsilon}{2}\frac{(g^+)^2}{a^2} & -\frac{\upsilon f^+ g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^+}{ab} & \frac{f^- f^+}{b^2} + \frac{1-\upsilon}{2}\frac{(g^+)^2}{a^2} & \frac{\upsilon f^+ g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^+}{ab} & -\frac{(f^+)^2}{b^2} + \frac{1-\upsilon}{2}\frac{g^- g^+}{a^2} & -\frac{\upsilon f^+ g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^+}{ab} & -\frac{f^- f^+}{b^2} + \frac{1-\upsilon}{2}\frac{g^- g^+}{a^2} \\
 & & \frac{(g^+)^2}{a^2} + \frac{1-\upsilon}{2}\frac{(f^-)^2}{b^2} & -\frac{\upsilon f^- g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^+}{ab} & -\frac{g^- g^+}{a^2} + \frac{1-\upsilon}{2}\frac{f^- f^+}{b^2} & \frac{\upsilon f^+ g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^-}{ab} & \frac{g^- g^+}{a^2} + \frac{1-\upsilon}{2}\frac{(f^-)^2}{b^2} & \frac{\upsilon f^- g^+}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^-}{ab} \\
 & & & \frac{(f^-)^2}{b^2} + \frac{1-\upsilon}{2}\frac{(g^+)^2}{a^2} & \frac{\upsilon f^- g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^+}{ab} & -\frac{f^- f^+}{b^2} + \frac{1-\upsilon}{2}\frac{g^- g^+}{a^2} & -\frac{\upsilon f^- g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^+}{ab} & -\frac{(f^-)^2}{b^2} + \frac{1-\upsilon}{2}\frac{g^- g^+}{a^2} \\
 & & & & \frac{(g^-)^2}{a^2} + \frac{1-\upsilon}{2}\frac{(f^+)^2}{b^2} & -\frac{\upsilon f^+ g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^-}{ab} & -\frac{(g^-)^2}{a^2} + \frac{1-\upsilon}{2}\frac{f^- f^+}{b^2} & \frac{\upsilon f^- g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^+ g^-}{ab} \\
 & & & & & \frac{(f^+)^2}{b^2} + \frac{1-\upsilon}{2}\frac{(g^-)^2}{a^2} & \frac{\upsilon f^+ g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^-}{ab} & \frac{f^- f^+}{b^2} + \frac{1-\upsilon}{2}\frac{(g^-)^2}{a^2} \\
 & & & & & & \frac{(g^-)^2}{a^2} + \frac{1-\upsilon}{2}\frac{(f^-)^2}{b^2} & \frac{\upsilon f^- g^-}{ab} + \frac{1-\upsilon}{2}\frac{f^- g^-}{ab} \\
 & & & & & & & \frac{(f^-)^2}{b^2} + \frac{1-\upsilon}{2}\frac{(g^-)^2}{a^2}
\end{bmatrix}$$

symmetric

Each term must be multiplied by: $\frac{Et}{1-\upsilon^2}$

Constant Shear Element

$$\boldsymbol{K} = \frac{E\,t}{1-\upsilon^2}\begin{bmatrix}
\frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{8} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{8} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & \frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{8} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{8} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} \\
 & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{8} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & \frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{8} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{8} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{8} \\
 & & \frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{8} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{8} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & \frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{8} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} \\
 & & & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{8} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{8} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{8} \\
 & & & & \frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{8} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{8} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} \\
 & & & & & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{8} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & \frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{8} \\
 & & & & & & \frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{8} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} \\
 & & & & & & & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{8}
\end{bmatrix}$$

symmetric

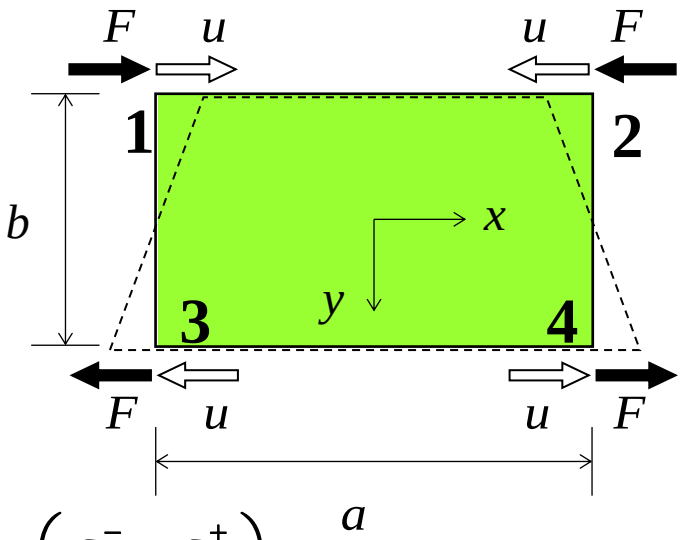
Constant Shear Element

$\sigma = D\varepsilon$ ⇒

$$\begin{Bmatrix} n_{xx_1} \\ n_{yy_1} \\ n_{xy_1} \\ \hline n_{xx_2} \\ n_{yy_2} \\ n_{xy_2} \\ \hline n_{xx_3} \\ n_{yy_3} \\ n_{xy_3} \\ \hline n_{xx_4} \\ n_{yy_4} \\ n_{xy_4} \end{Bmatrix} = \frac{Et}{1-\upsilon^2} \left[\begin{array}{cccccccc} -\frac{1}{a} & -\frac{\upsilon}{b} & \frac{1}{a} & 0 & 0 & \frac{\upsilon}{b} & 0 & 0 \\ -\frac{\upsilon}{a} & -\frac{1}{b} & \frac{\upsilon}{a} & 0 & 0 & \frac{1}{b} & 0 & 0 \\ -\frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & -\frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} \\ \hline -\frac{1}{a} & 0 & \frac{1}{a} & -\frac{\upsilon}{b} & 0 & 0 & 0 & \frac{\upsilon}{b} \\ -\frac{\upsilon}{a} & 0 & \frac{\upsilon}{a} & -\frac{1}{b} & 0 & 0 & 0 & \frac{1}{b} \\ -\frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & -\frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} \\ \hline 0 & -\frac{\upsilon}{b} & 0 & 0 & -\frac{1}{a} & \frac{\upsilon}{b} & \frac{1}{a} & 0 \\ 0 & -\frac{1}{b} & 0 & 0 & -\frac{\upsilon}{a} & \frac{1}{b} & \frac{\upsilon}{a} & 0 \\ -\frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & -\frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} \\ \hline 0 & 0 & 0 & -\frac{\upsilon}{b} & -\frac{1}{a} & 0 & \frac{1}{a} & \frac{\upsilon}{b} \\ 0 & 0 & 0 & -\frac{1}{b} & -\frac{\upsilon}{a} & 0 & \frac{\upsilon}{a} & \frac{1}{b} \\ -\frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & -\frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & -\frac{1-\upsilon}{4a} & \frac{1-\upsilon}{4b} & \frac{1-\upsilon}{4a} \end{array}\right] \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

TU Delft

Constant Shear Element



$$-u_{x2} = -u_{x3} = u_{x1} = u_{x4} = u$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{g^+}{a}u_{x1} + \frac{g^+}{a}u_{x2} - \frac{g^-}{a}u_{x3} + \frac{g^-}{a}u_{x4} = 2\left(\frac{g^- - g^+}{a}\right)u$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = -\frac{f^+}{b}u_{y1} - \frac{f^-}{b}u_{y2} + \frac{f^+}{b}u_{y3} + \frac{f^-}{b}u_{y4} = 0$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{f^+}{b}u_{x1} - \frac{f^-}{b}u_{x2} + \frac{f^+}{b}u_{x3} + \frac{f^-}{b}u_{x4}$$

$$-\frac{g^+}{a}u_{y1} + \frac{g^+}{a}u_{y2} - \frac{g^-}{a}u_{y3} + \frac{g^-}{a}u_{y4} = 2\left(\frac{f^- - f^+}{b}\right)u$$

$$\mathbf{B} = 2\begin{bmatrix} \dfrac{g^- - g^+}{a} \\ 0 \\ \dfrac{f^- - f^+}{b} \end{bmatrix}$$

$g^- - g^+$ and $f^- - f^+$ should be zero

Constant Shear Element

$$\boldsymbol{K} = t \iint \boldsymbol{B}^T \boldsymbol{D}\, \boldsymbol{B}\, dx\, dy \qquad \upsilon = 0$$

$$\boldsymbol{K} = t \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} 4 \begin{bmatrix} \frac{g^- - g^+}{a} & 0 & \frac{f^- - f^+}{b} \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} \frac{g^- - g^+}{a} \\ 0 \\ \frac{f^- - f^+}{b} \end{bmatrix} dy\, dx$$

$$\int_{-a/2}^{a/2} \int_{-b/2}^{b/2} 4 \left(\frac{g^- - g^+}{a} \right)^2 dxdy = \frac{4}{3} \frac{b}{a} \quad ???$$

$$\int_{-a/2}^{a/2} \int_{-b/2}^{b/2} 2 \left(\frac{f^- - f^+}{b} \right)^2 dxdy = 0$$

$$K_{fem} = \frac{1}{3} E \frac{t\, b}{a}$$