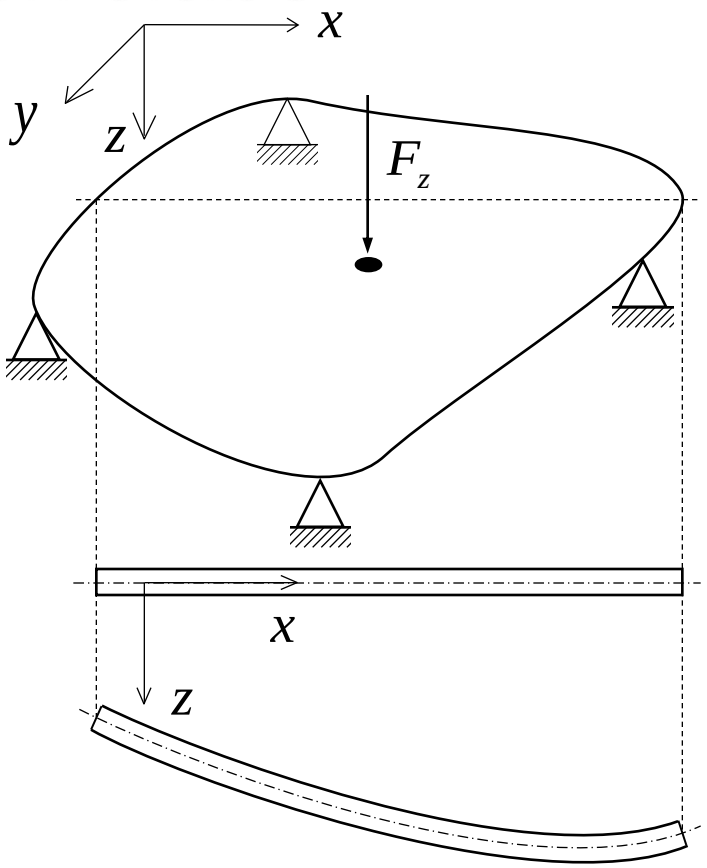
Isotropic thick slabs

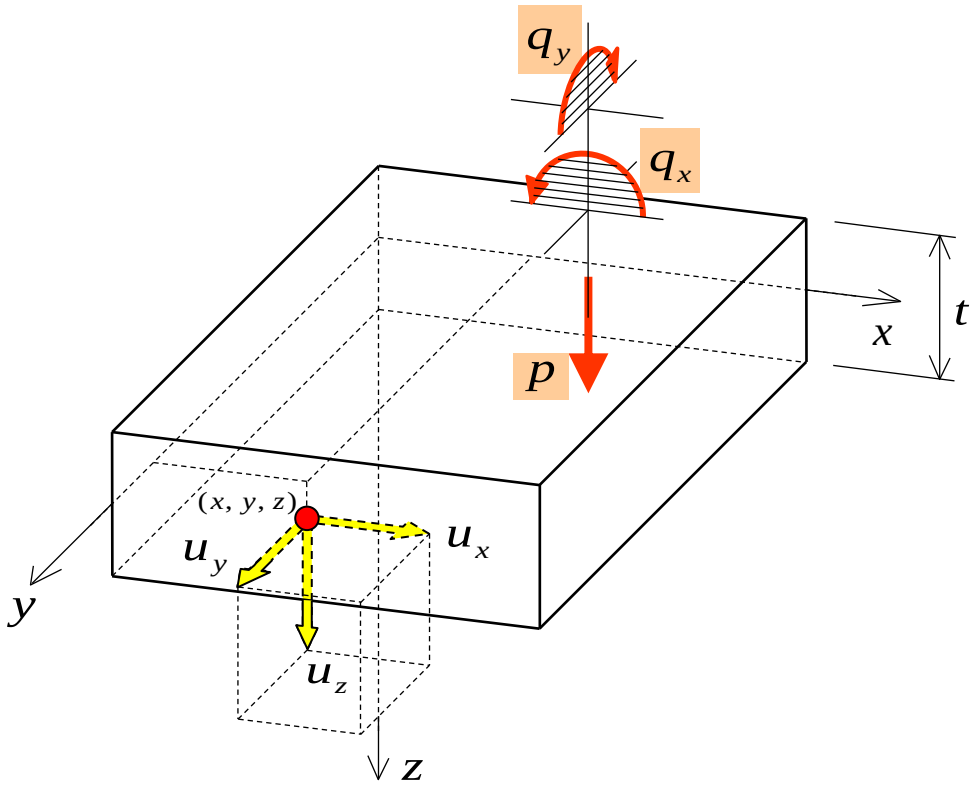


"Plate" with transverse force

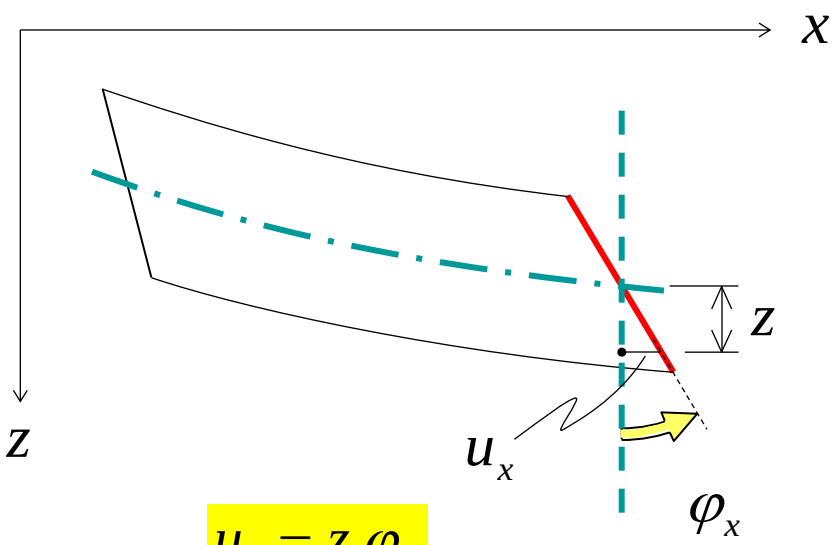
Thick isotropic slabs

degrees of freedom **deformations** **stress resultants** **loads**

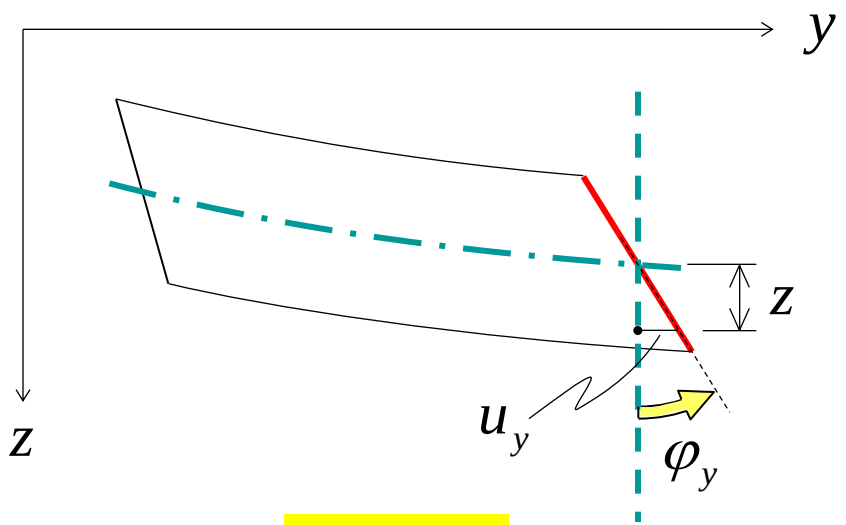
kinematic equations **constitutive equations** **equilibrium equations**



• positive rotations & actions definition

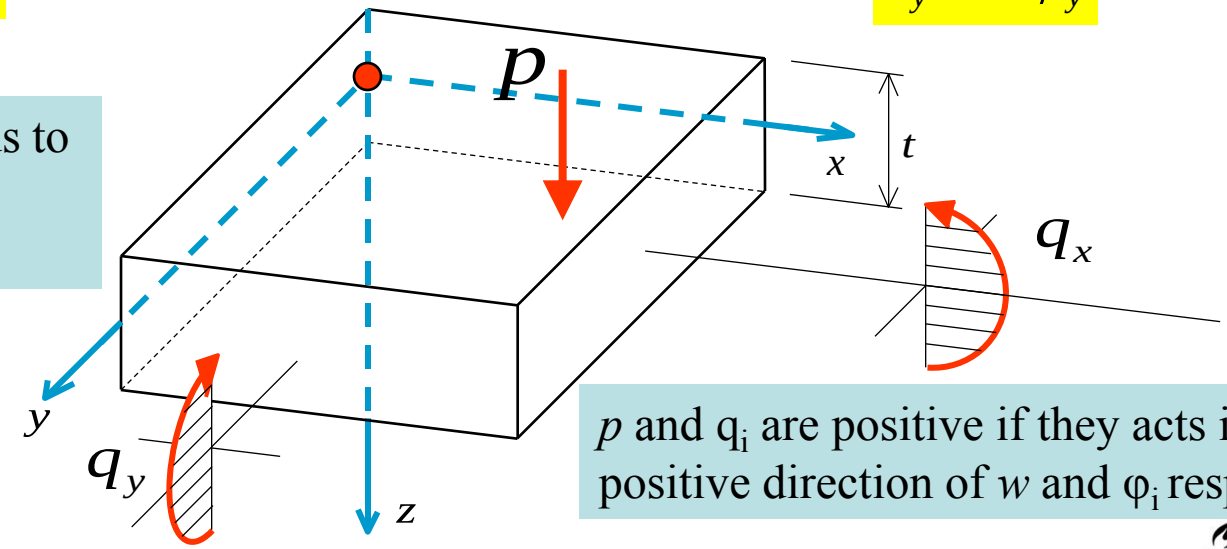


$u_x = z\,\varphi_x$



$u_y = z\,\varphi_y$

φ_i is positive if it leads to positive u_i on the positive z-side

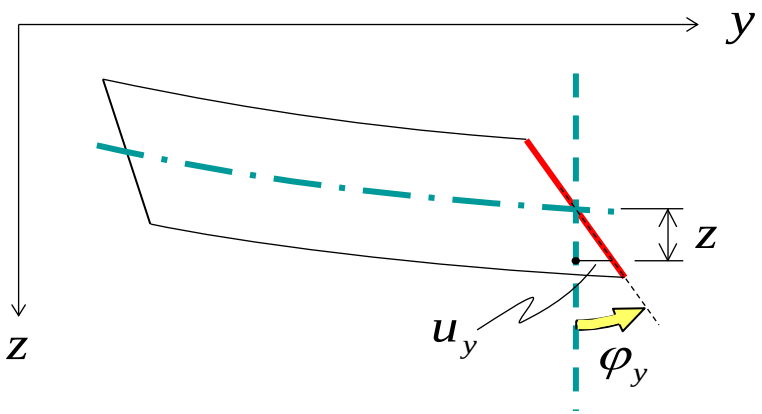


p and q_i are positive if they acts in the positive direction of *w* and φ_i respectively

Suppositions

•No membrane forces occur, so the mid-plane will remain strainless after loading. This is valid if $w << t$;

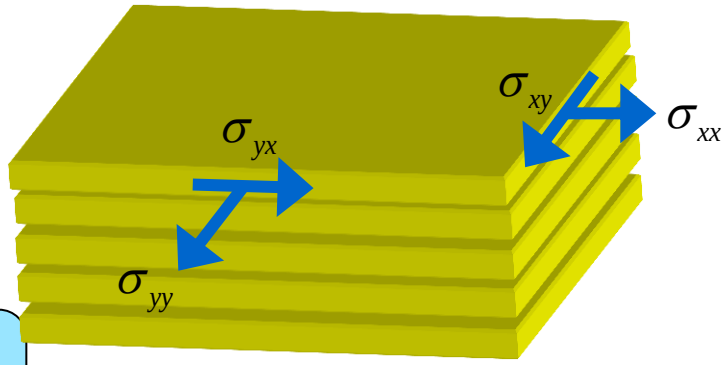
• The "needle hypothesis" is applied on the plane cross-section of the plate;

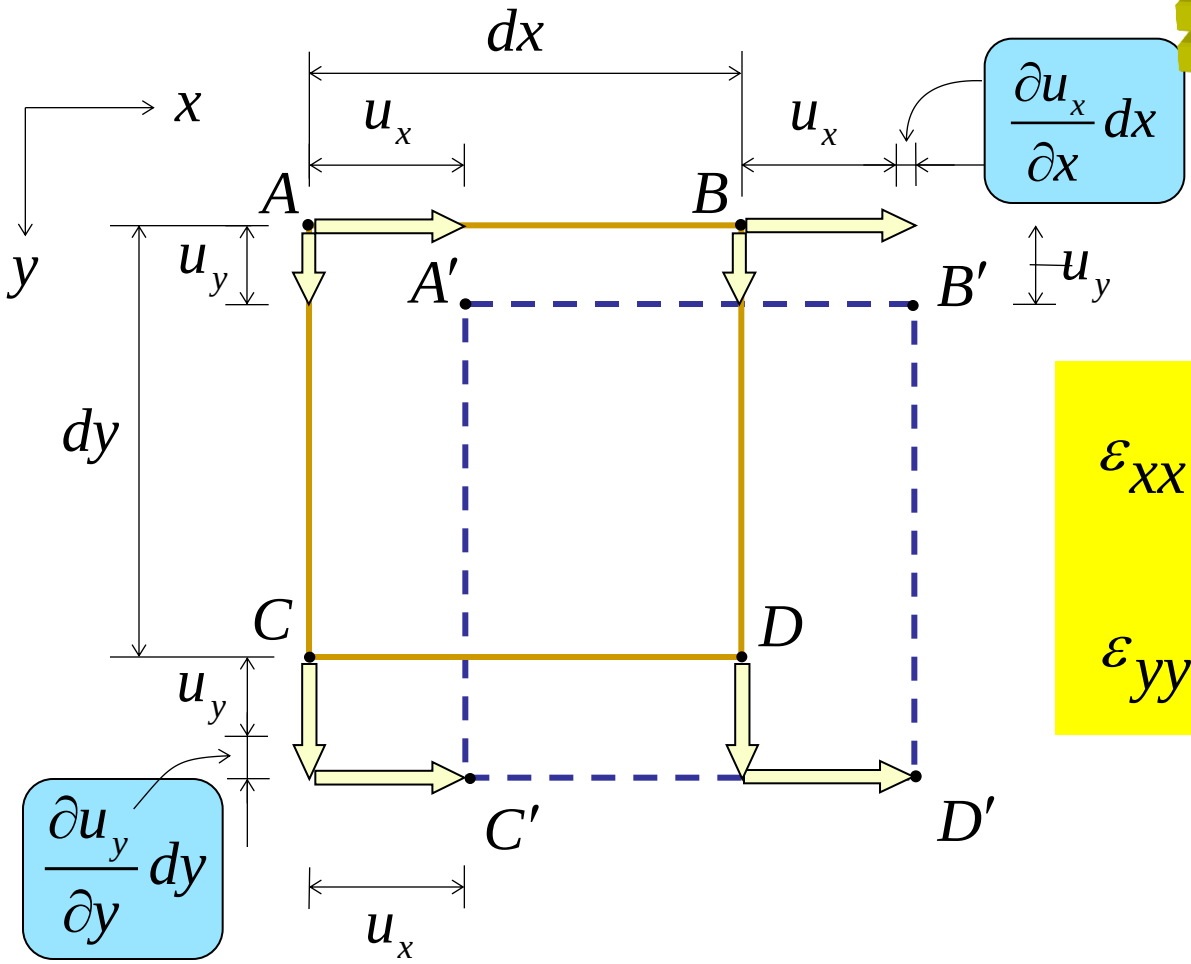


• $\sigma_{zz} = 0$ and ε_{zz} is ignored, no variation of the vertical displacement occur over the thickness of the slab;

• The above three suppositions are valid both for thin and thick plates. For thin plates, the shear deformation is neglected.

Slab in-plane normal strains

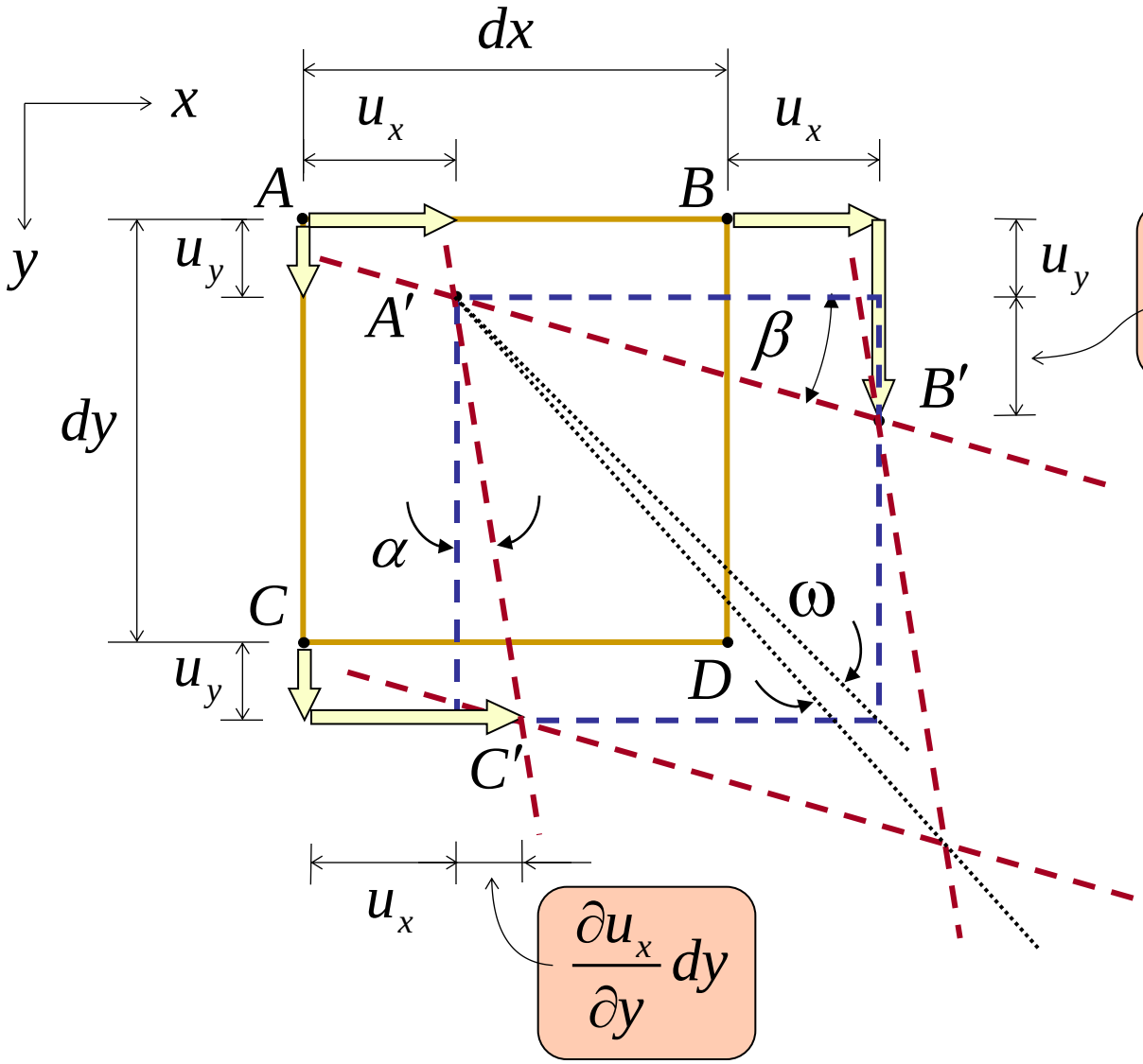




$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

Slab in-plane shear strains

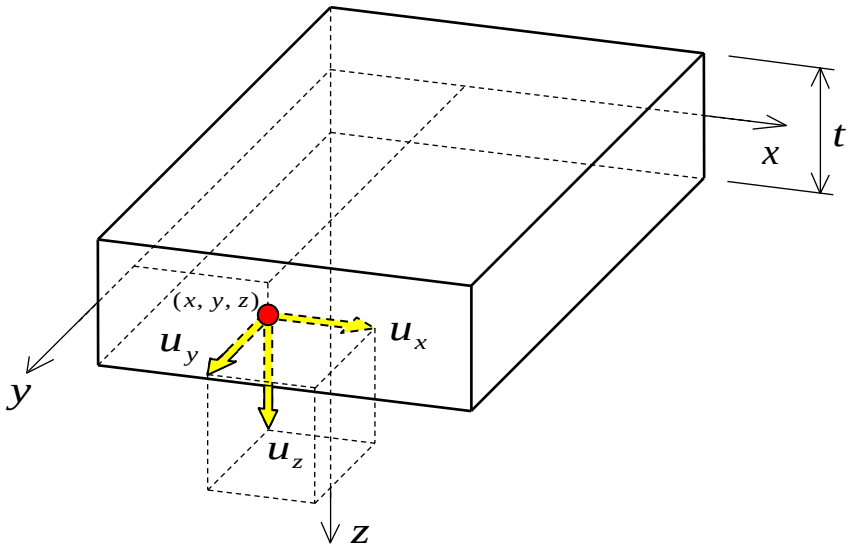


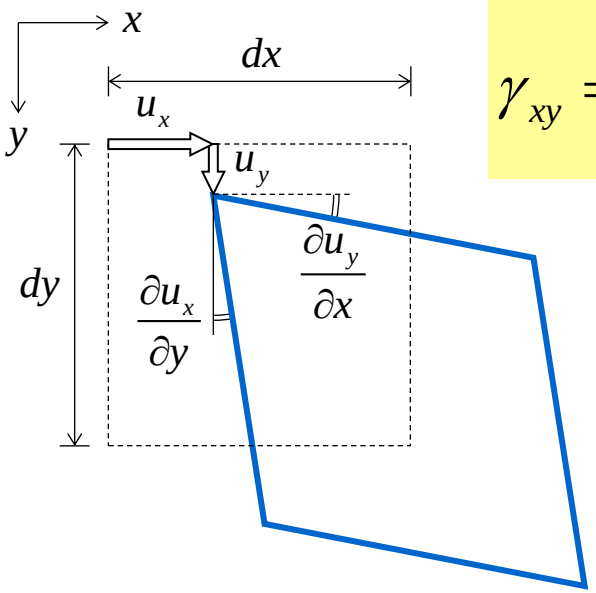
$$\alpha = \frac{\partial u_x}{\partial y}$$

$$\beta = \frac{\partial u_y}{\partial x}$$

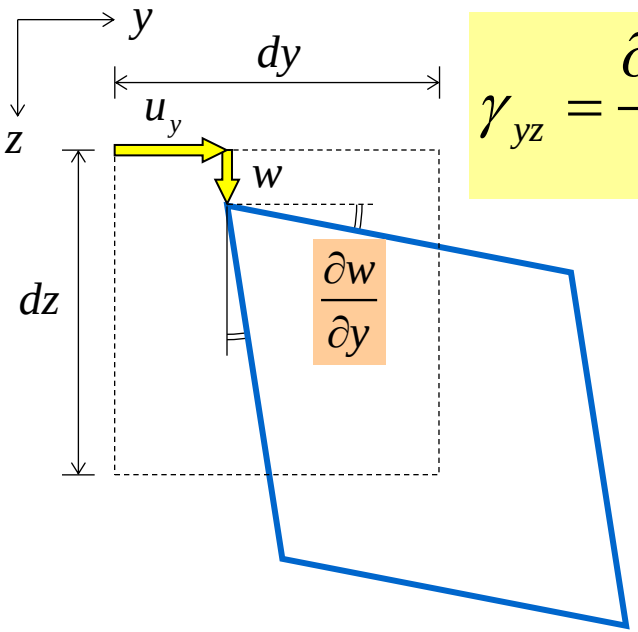
$$\gamma_{xy} = \alpha + \beta$$

Slab shear deformations

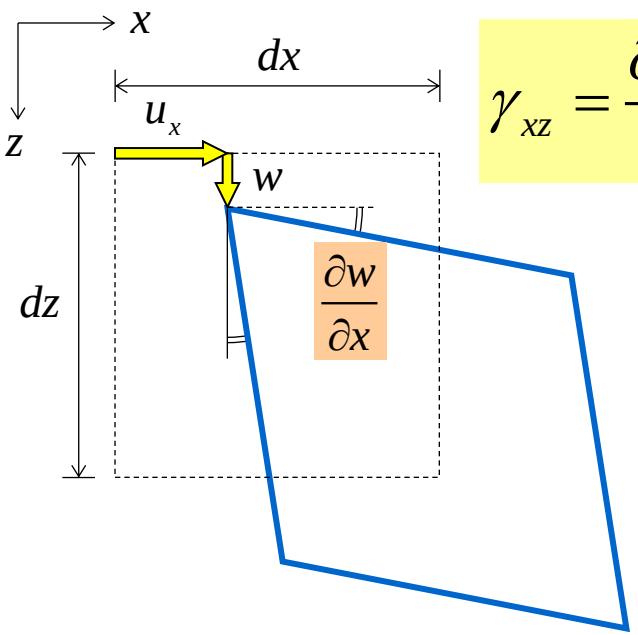




$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$



$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}$$

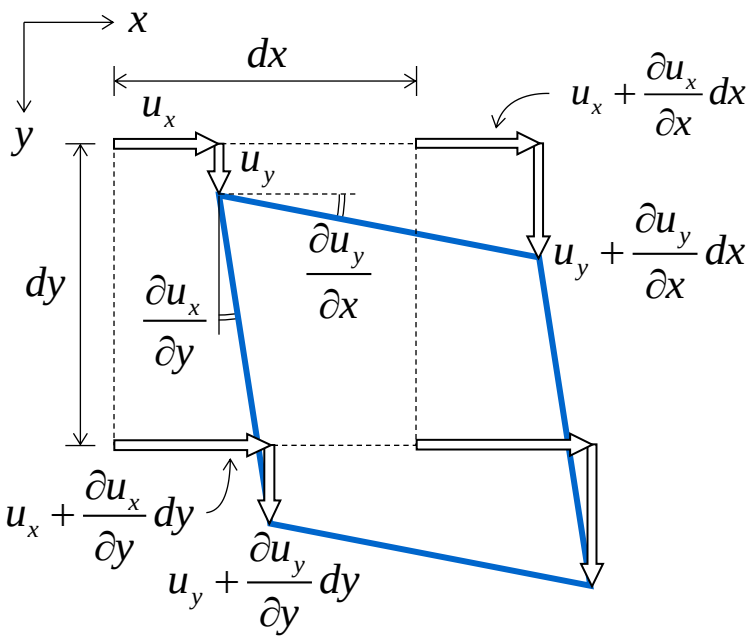


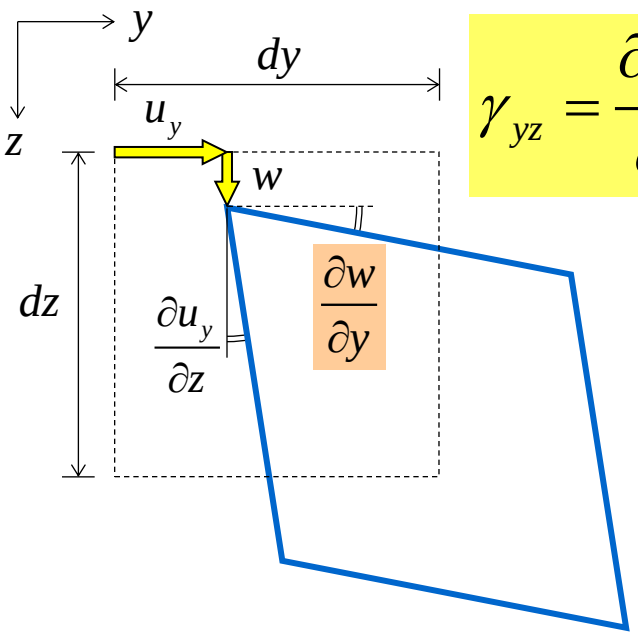
$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial w}{\partial x}$$

Slab total strains:

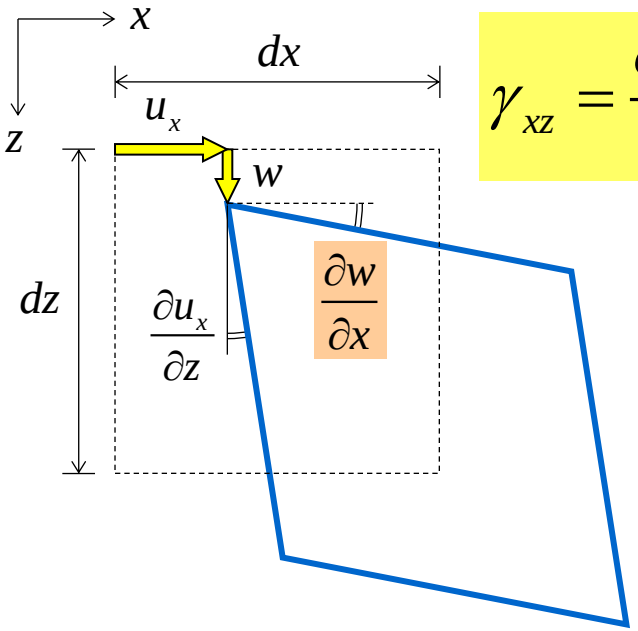
$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \quad ; \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$



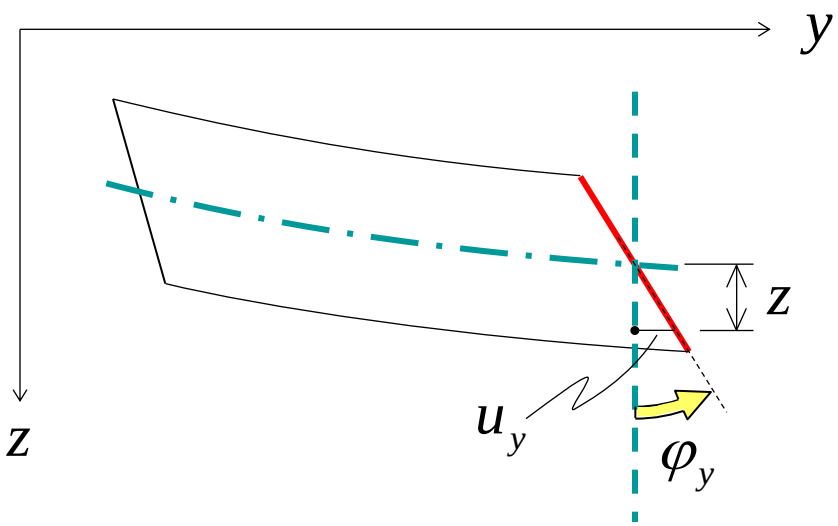


$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}$$

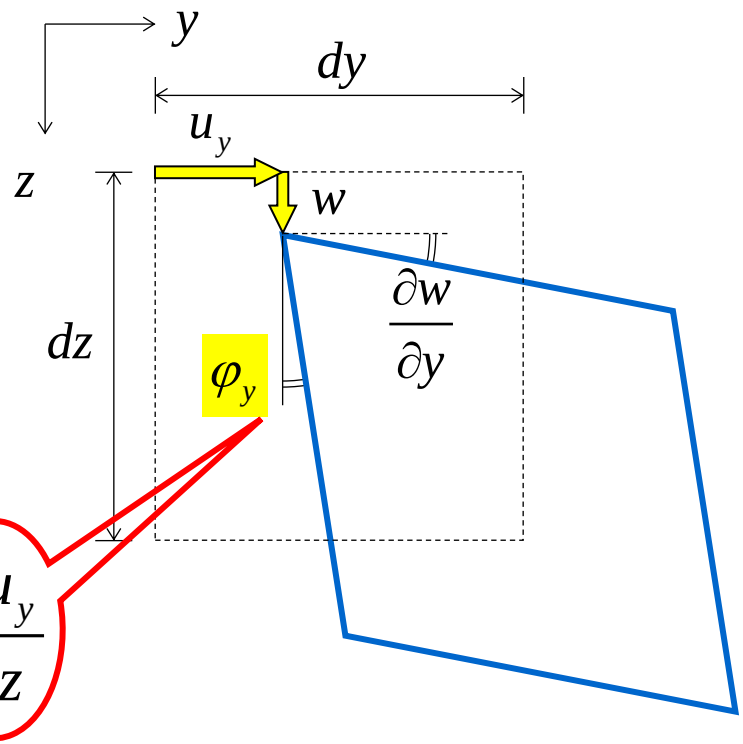


$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial w}{\partial x}$$

Slab kinematic equations



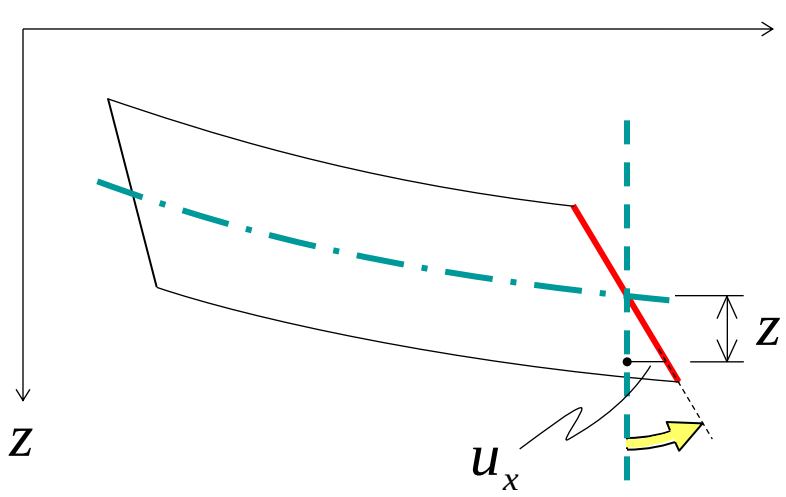
$$u_y = z\,\varphi_y$$



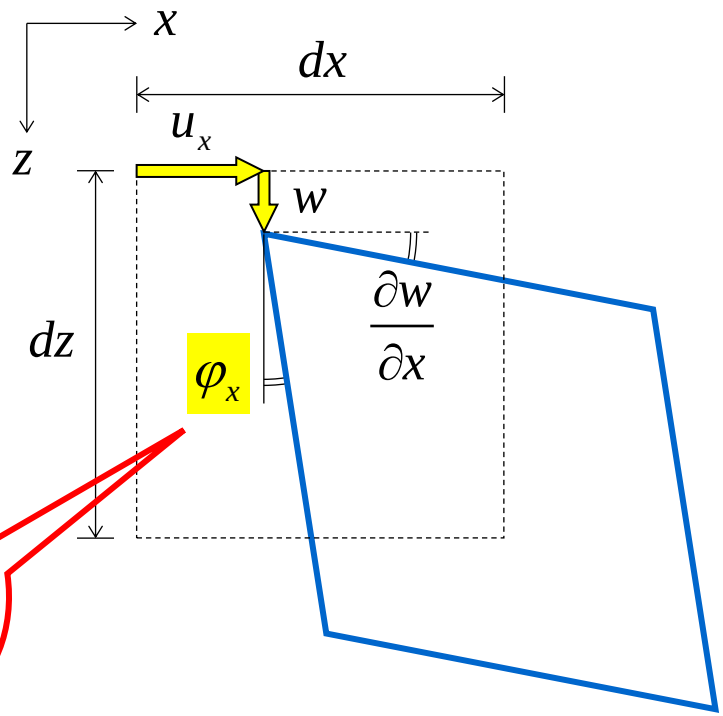
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = z\,\frac{\partial \varphi_y}{\partial y} = z\,\kappa_{yy}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial u_y}{\partial z} = \frac{\partial w}{\partial y} + \varphi_y$$

Slab kinematic equations



$$u_x = z\,\varphi_x$$



$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = z\,\frac{\partial \varphi_x}{\partial x} = z\,\kappa_{xx}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z} = \frac{\partial w}{\partial x} + \varphi_x$$

Slab kinematic equations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}$$



$$\varepsilon_{xx} = z\,\frac{\partial \varphi_x}{\partial x} = z\,\kappa_{xx}$$

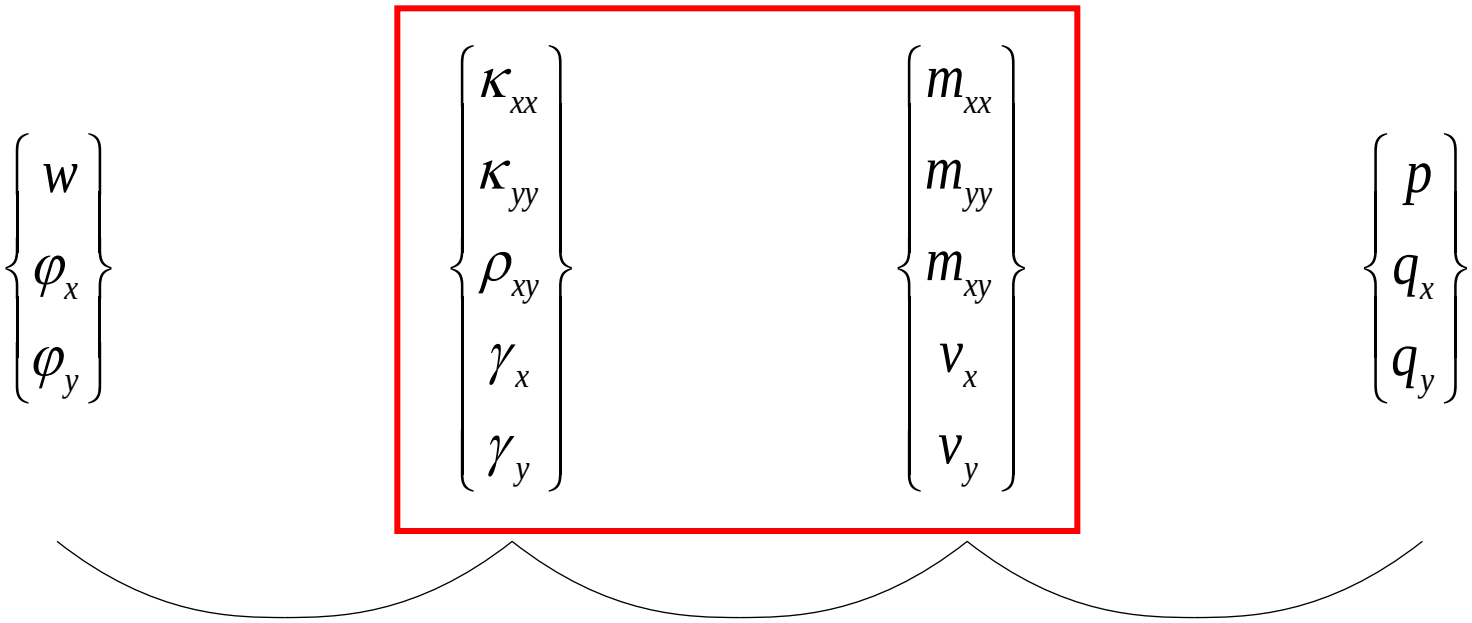
$$\varepsilon_{yy} = z\,\frac{\partial \varphi_y}{\partial y} = z\,\kappa_{yy}$$

$$\gamma_{xy} = z\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right) = z\,\rho_{xy}$$

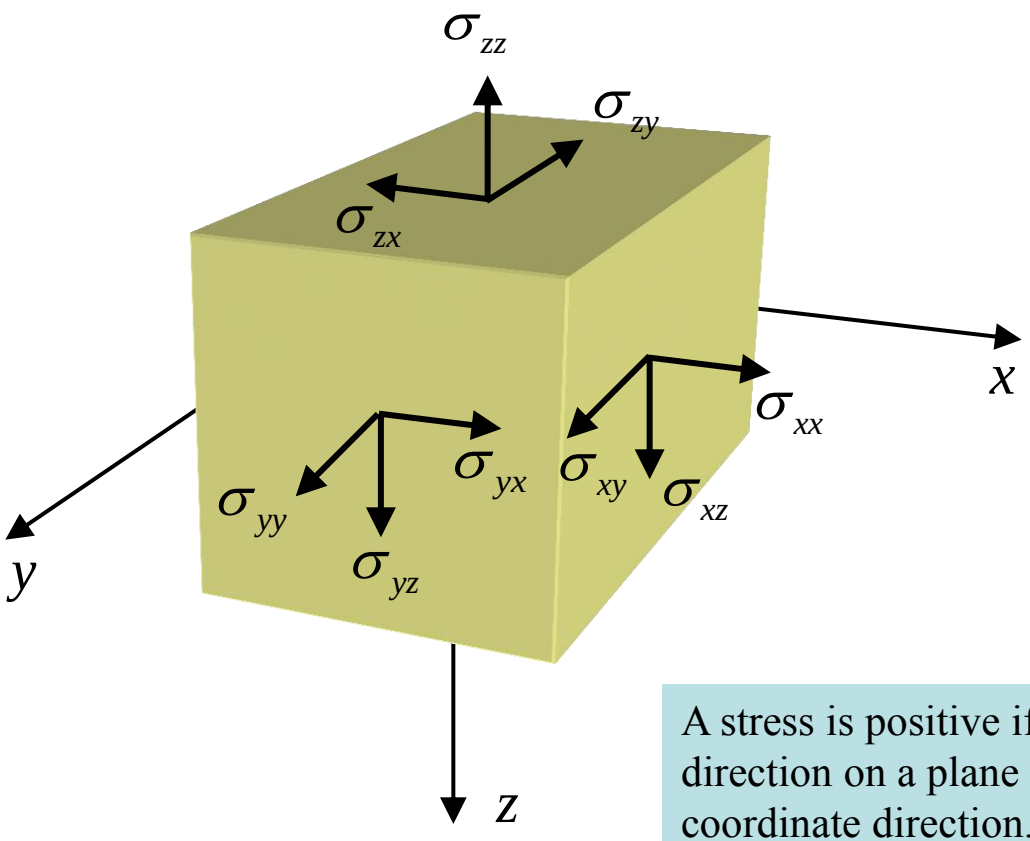
$$\gamma_{xz} = \varphi_x + \frac{\partial w}{\partial x} = \gamma_x$$

$$\gamma_{yz} = \varphi_y + \frac{\partial w}{\partial y} = \gamma_y$$

Thick isotropic slabs



Thick isotropic slabs



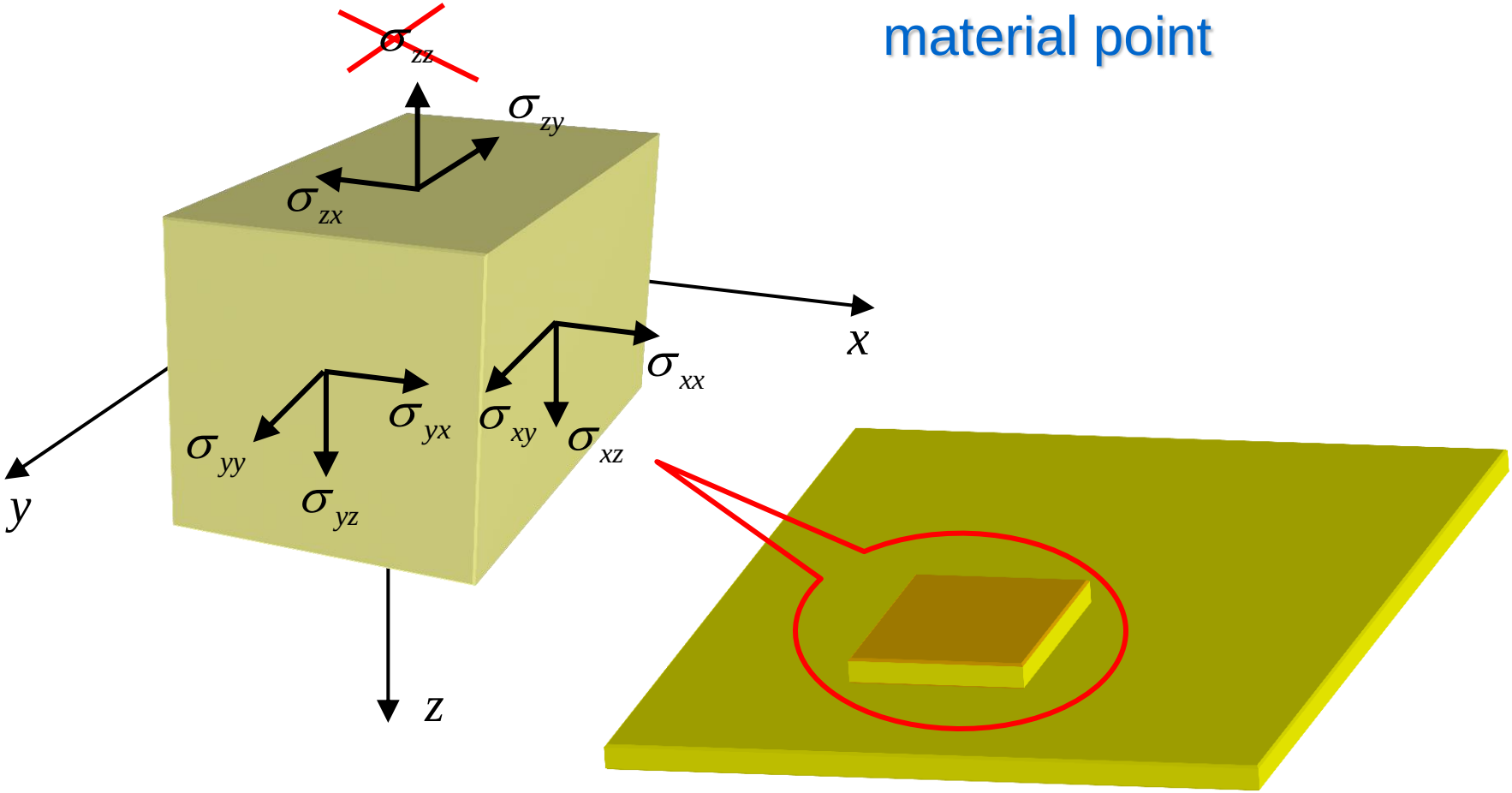
material point

- stress notation
- stress sign convention

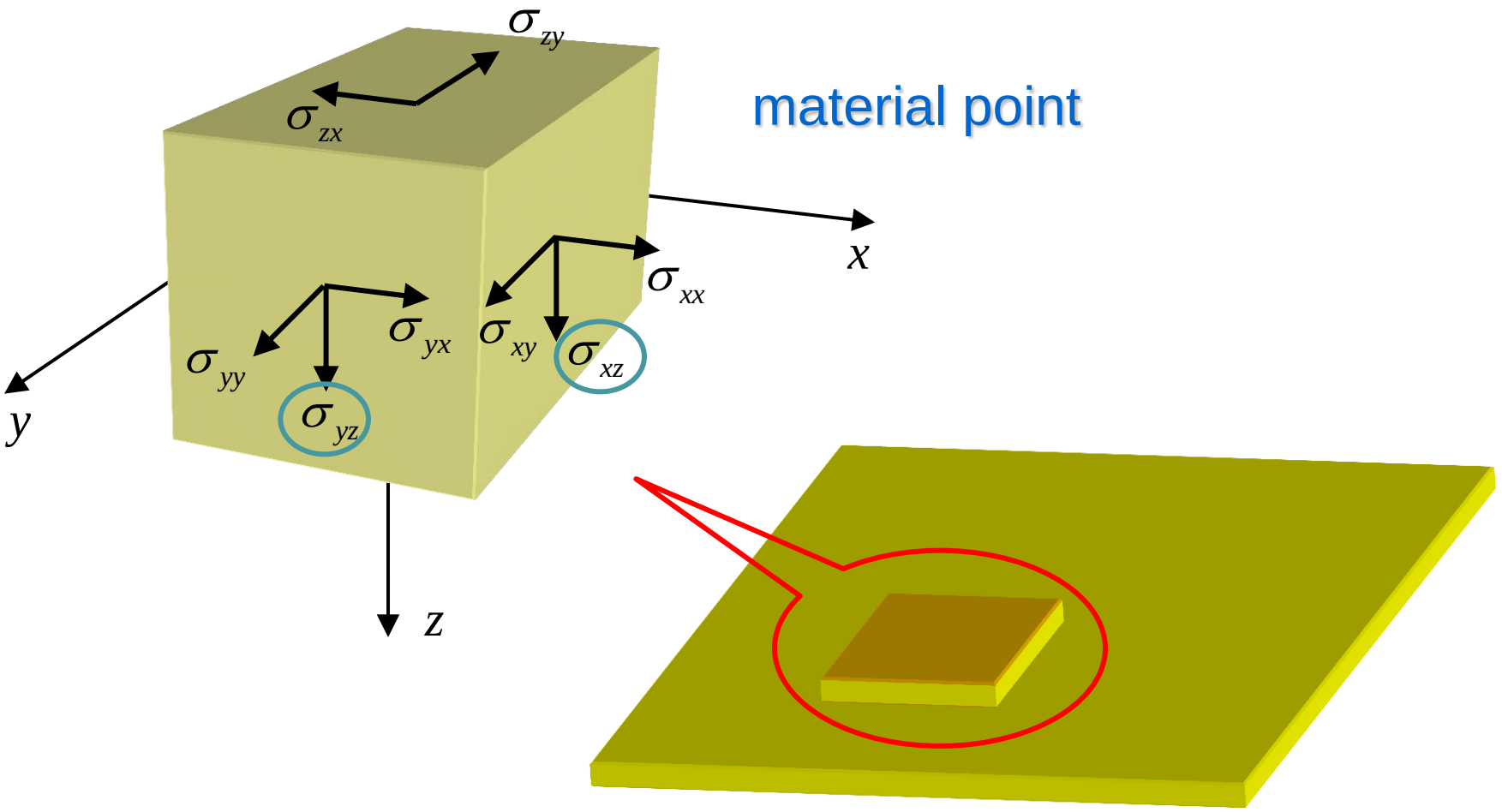
A stress is positive if it acts in the positive coordinate direction on a plane with the normal vector in the positive coordinate direction. Correspondingly, a stress is positive if it acts in the negative coordinate direction on a plane with its normal in the negative coordinate direction.

Thick isotropic slabs

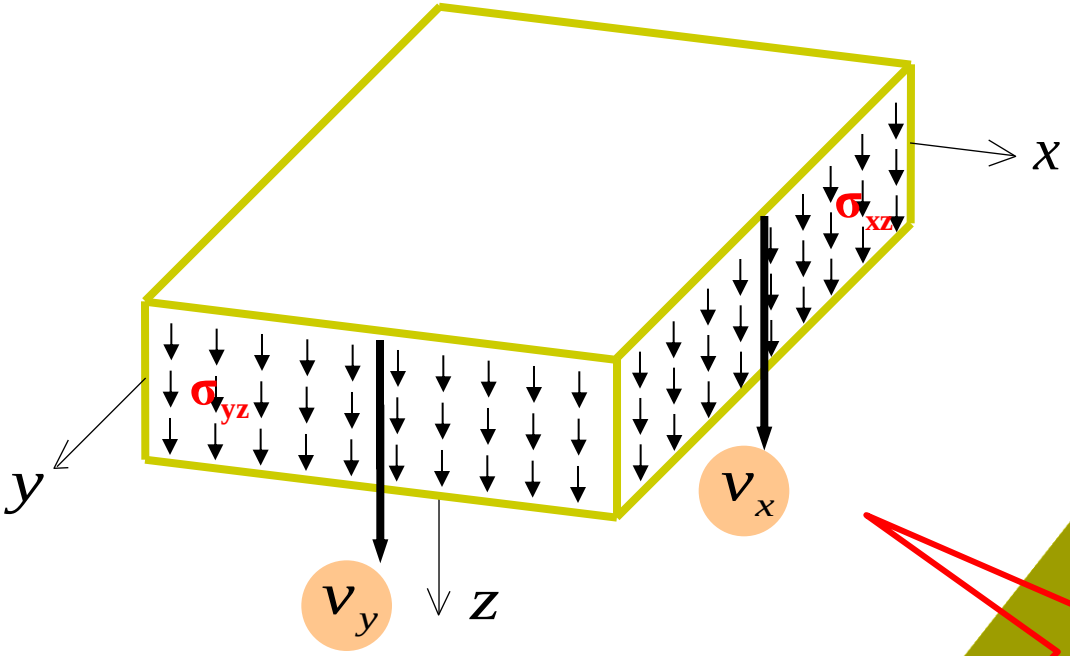
material point

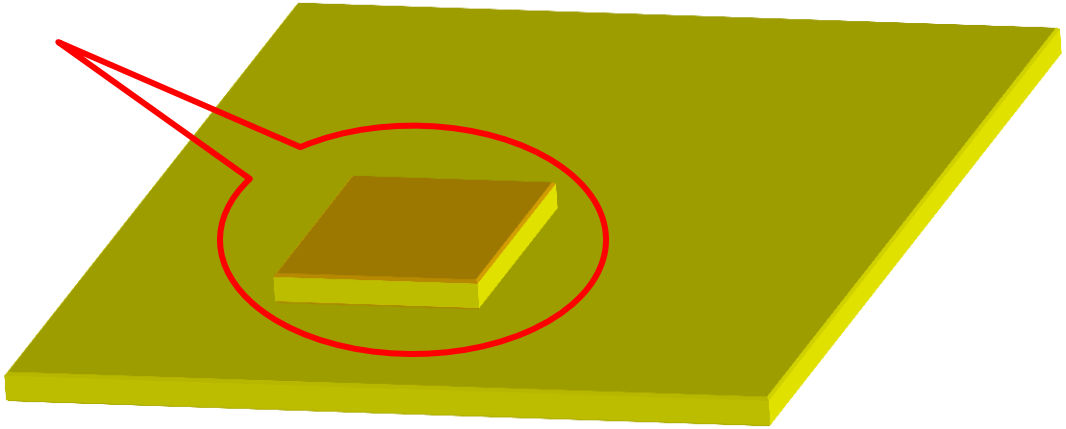


Internal actions in slabs



Internal actions in slabs

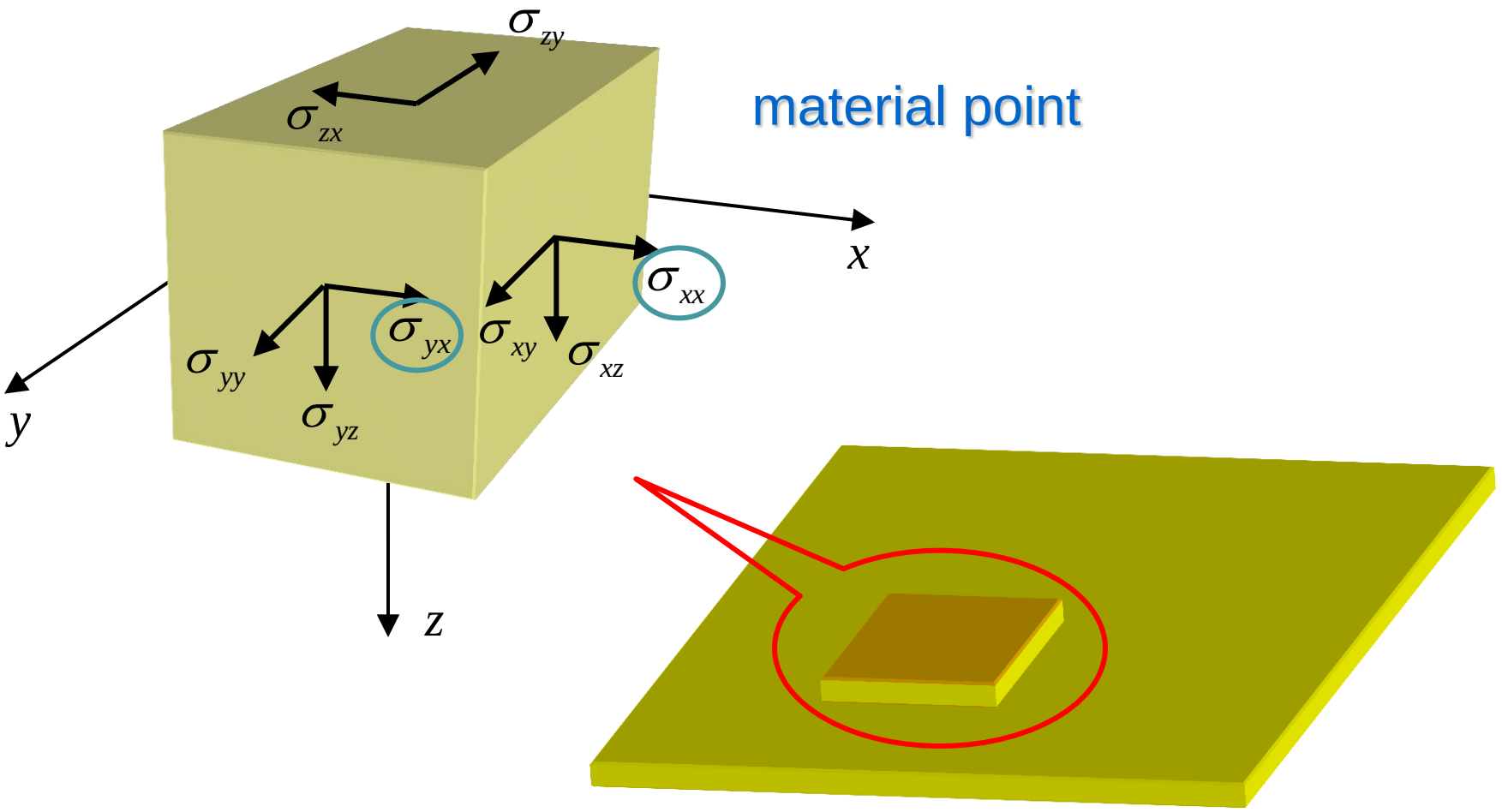




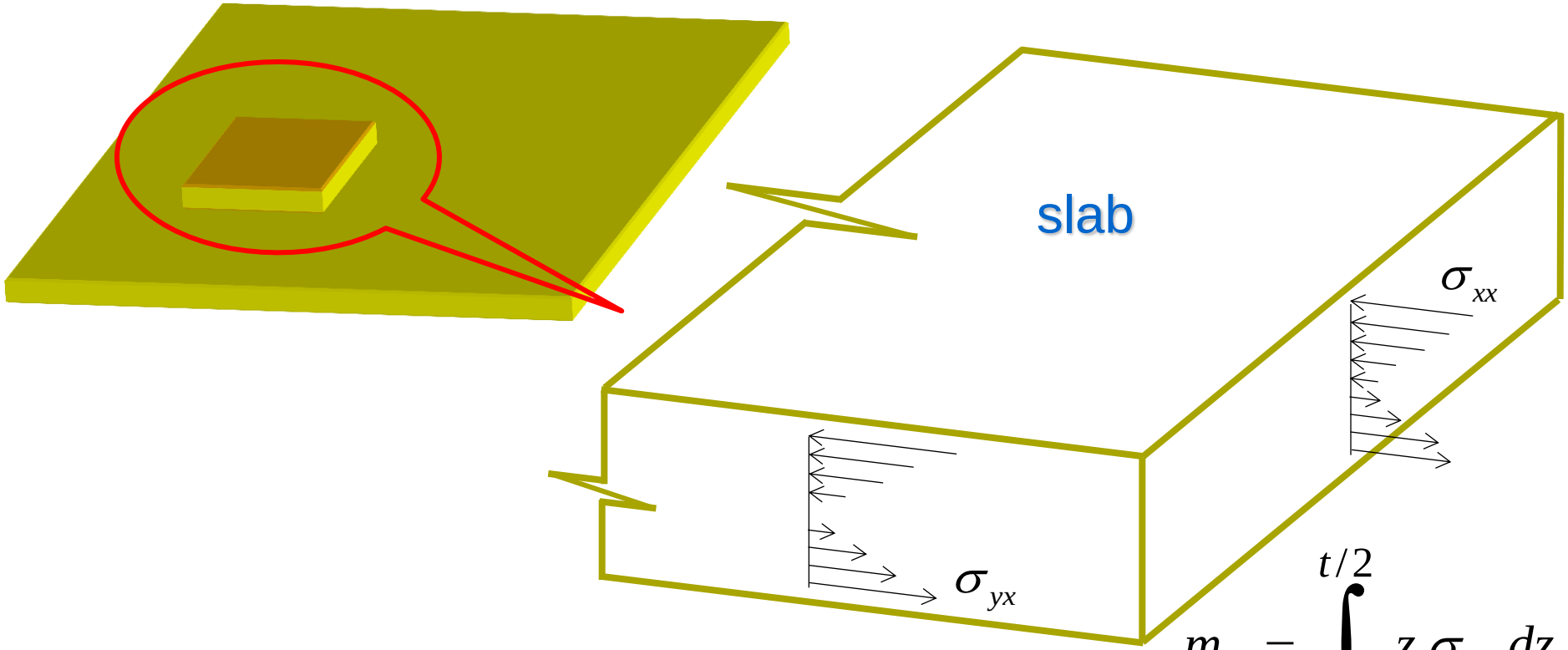
$$v_x = \int_{-t/2}^{t/2} \sigma_{xz} dz$$

$$v_y = \int_{-t/2}^{t/2} \sigma_{yz} dz$$

Internal actions in slabs

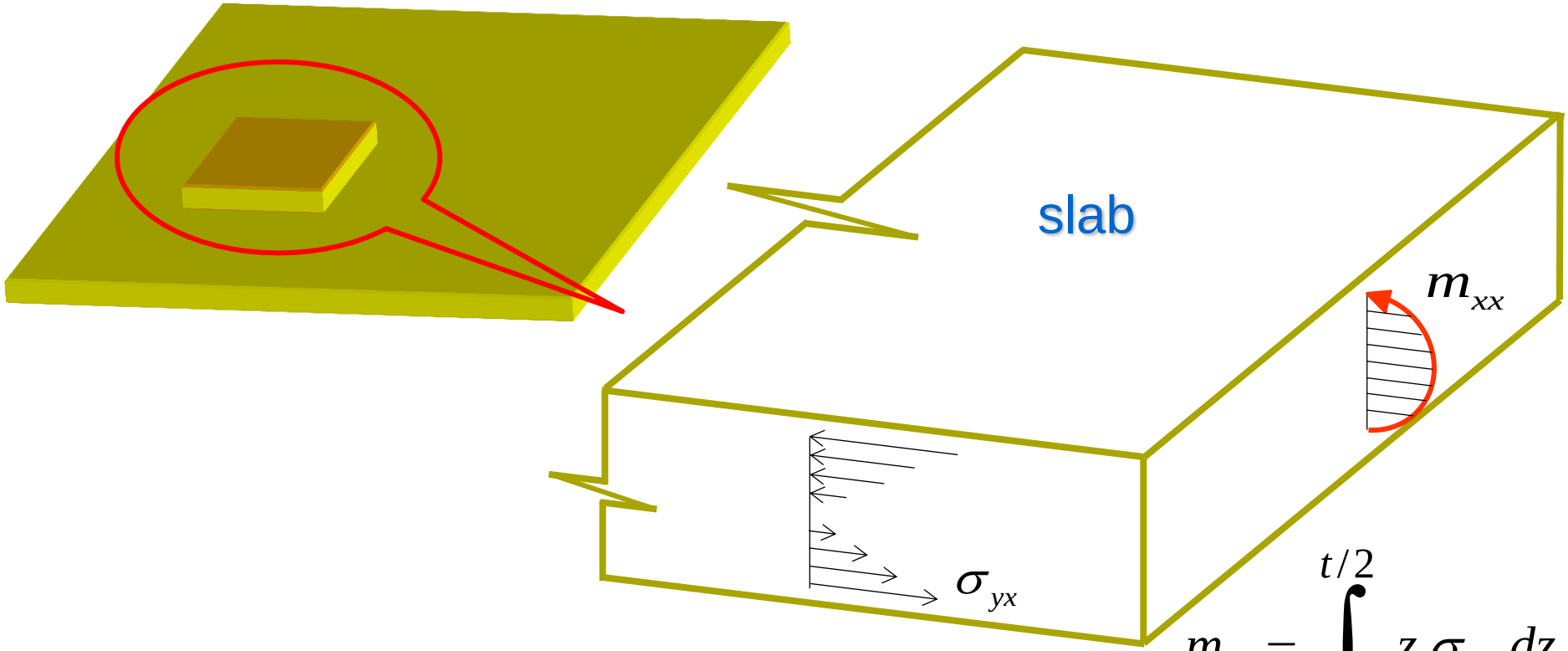


Internal actions in slabs



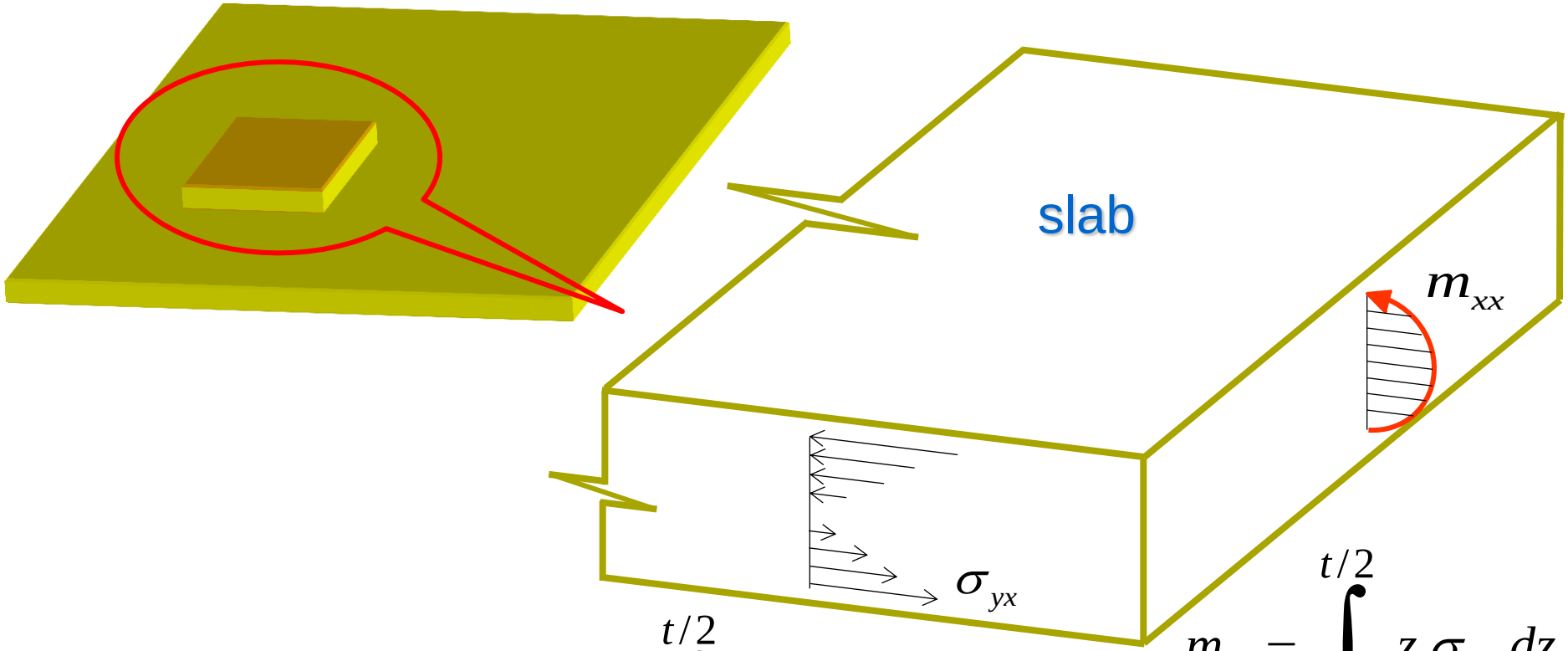
$$m_{xx} = \int_{-t/2}^{t/2} z\,\sigma_{xx}\,dz$$

Internal actions in slabs



$$m_{xx} = \int_{-t/2}^{t/2} z\,\sigma_{xx}\,dz$$

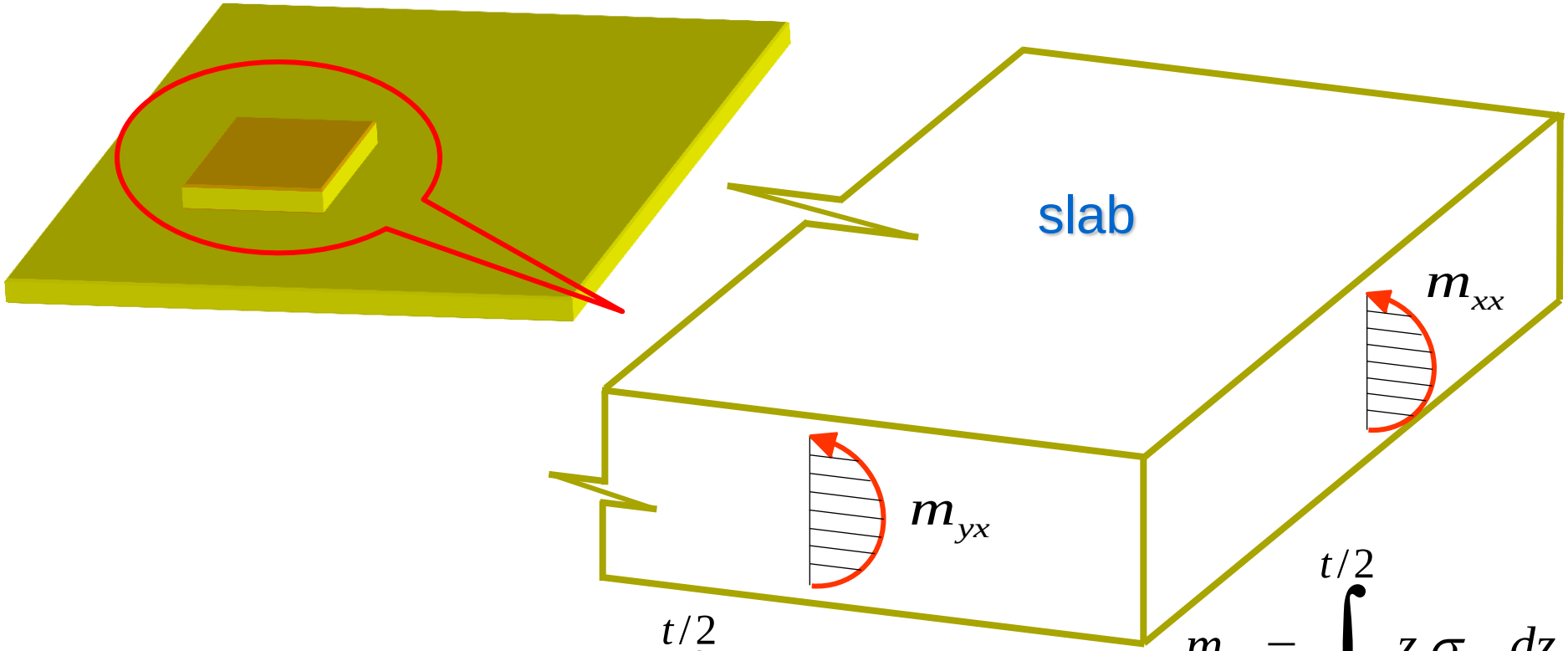
Internal actions in slabs



$$m_{yx} = \int_{-t/2}^{t/2} z\,\sigma_{yx}\,dz$$

$$m_{xx} = \int_{-t/2}^{t/2} z\,\sigma_{xx}\,dz$$

Internal actions in slabs



$$m_{yx} = \int_{-t/2}^{t/2} z\,\sigma_{yx}\,dz$$

$$m_{xx} = \int_{-t/2}^{t/2} z\,\sigma_{xx}\,dz$$

Internal actions in slabs

m_{ij} is positive if it leads to a positive stress in layers on the positive z-side of the mid-plane.

v_x and v_y are positive if they directed in the positive z-direction on a section with positive normal.

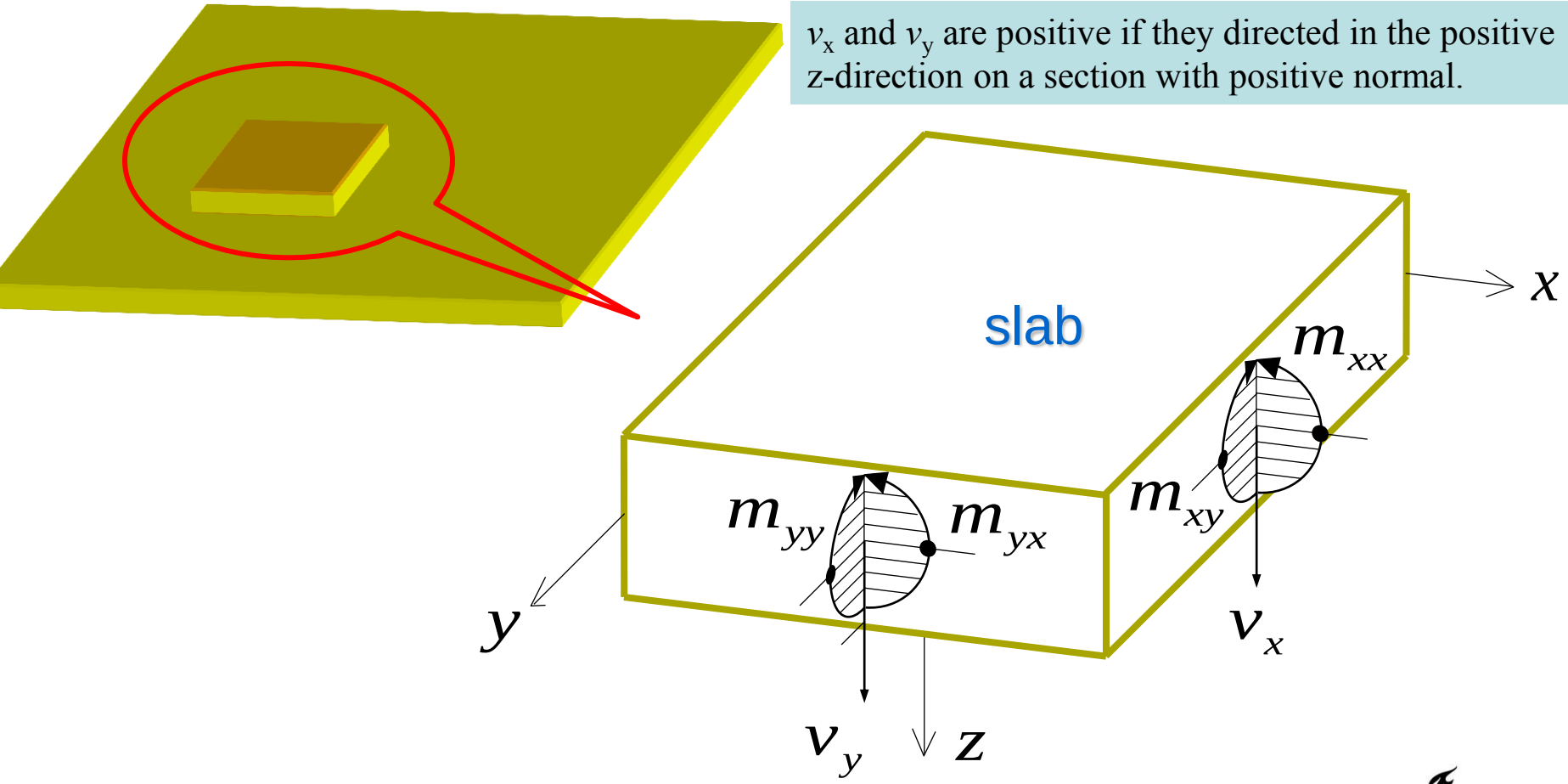


Plate constitutive equations

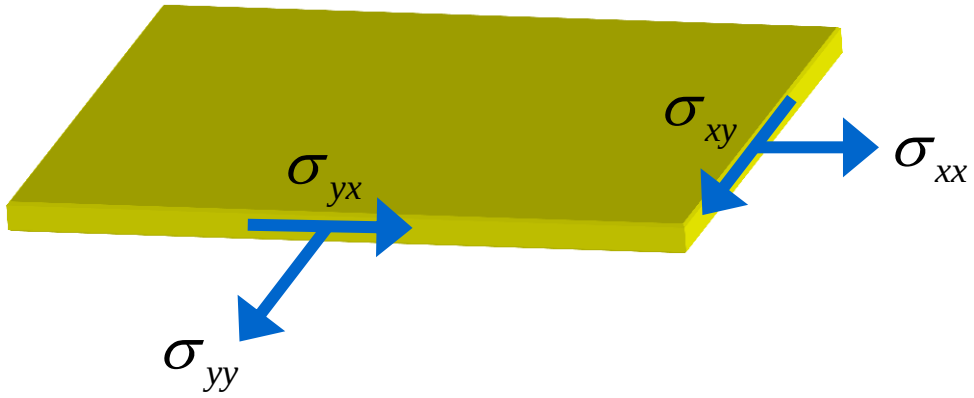
$$\varepsilon_{xx} = z\frac{\partial \varphi_x}{\partial x}$$

$$\varepsilon_{yy} = z\frac{\partial \varphi_y}{\partial y}$$

$$\gamma_{xy} = z\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right)$$

$$\gamma_{xz} = \varphi_x + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \varphi_y + \frac{\partial w}{\partial y}$$

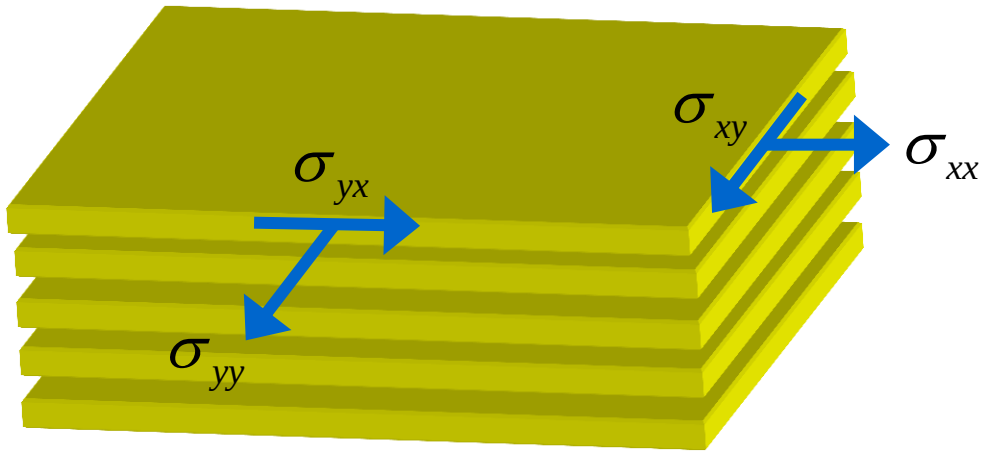


$$\sigma_{xx} = \frac{E}{1-\upsilon^2}(\varepsilon_{xx} + \upsilon\,\varepsilon_{yy})$$

$$\sigma_{yy} = \frac{E}{1-\upsilon^2}(\varepsilon_{yy} + \upsilon\,\varepsilon_{xx})$$

$$\sigma_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy}$$

Slab constitutive equations



$$m_{xx} = \int_{-t/2}^{t/2} z\,\sigma_{xx}\,dz$$

$$m_{yy} = \int_{-t/2}^{t/2} z\,\sigma_{yy}\,dz$$

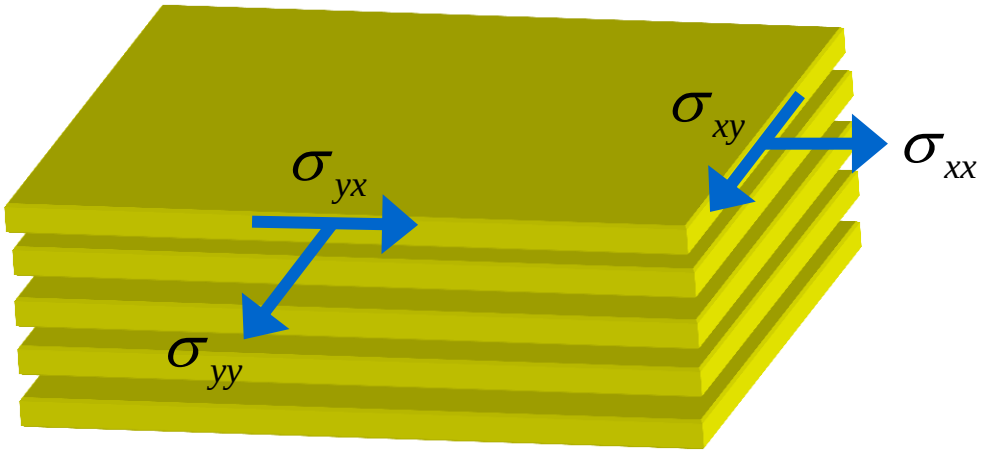
$$m_{xy} = \int_{-t/2}^{t/2} z\,\sigma_{xy}\,dz$$

$$\sigma_{xx} = \frac{E}{1-\upsilon^2} z\,(\kappa_{xx} + \upsilon\,\kappa_{yy})$$

$$\sigma_{yy} = \frac{E}{1-\upsilon^2} z\,(\kappa_{yy} + \upsilon\,\kappa_{xx})$$

$$\sigma_{xy} = \frac{E}{2(1+\upsilon)} z\,\rho_{xy}$$

Slab constitutive equations

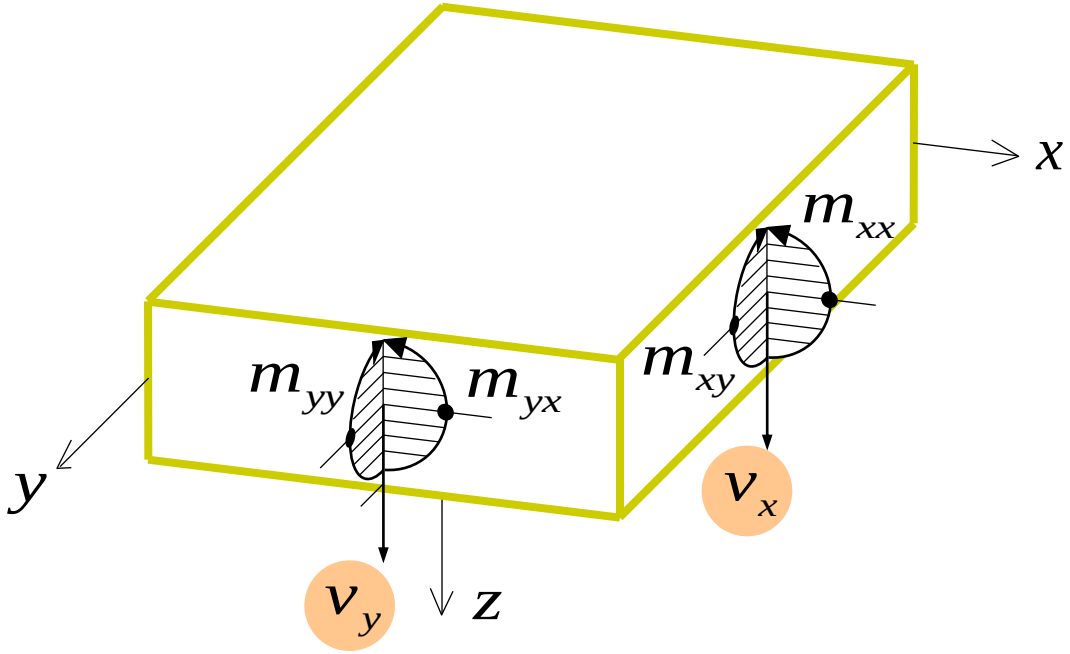


$$\begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix}$$

$$D = \frac{E\,t^3}{12\left(1-\upsilon^2\right)}$$

plate stiffness

Slab constitutive equations



$$D_\gamma = Gt/\eta$$

$$m_{xx} = D\left(\kappa_{xx} + \upsilon\,\kappa_{yy}\right)$$

$$m_{yy} = D\left(\kappa_{yy} + \upsilon\,\kappa_{xx}\right)$$

$$m_{xy} = \frac{1}{2} D\left(1-\upsilon\right)\rho_{xy}$$

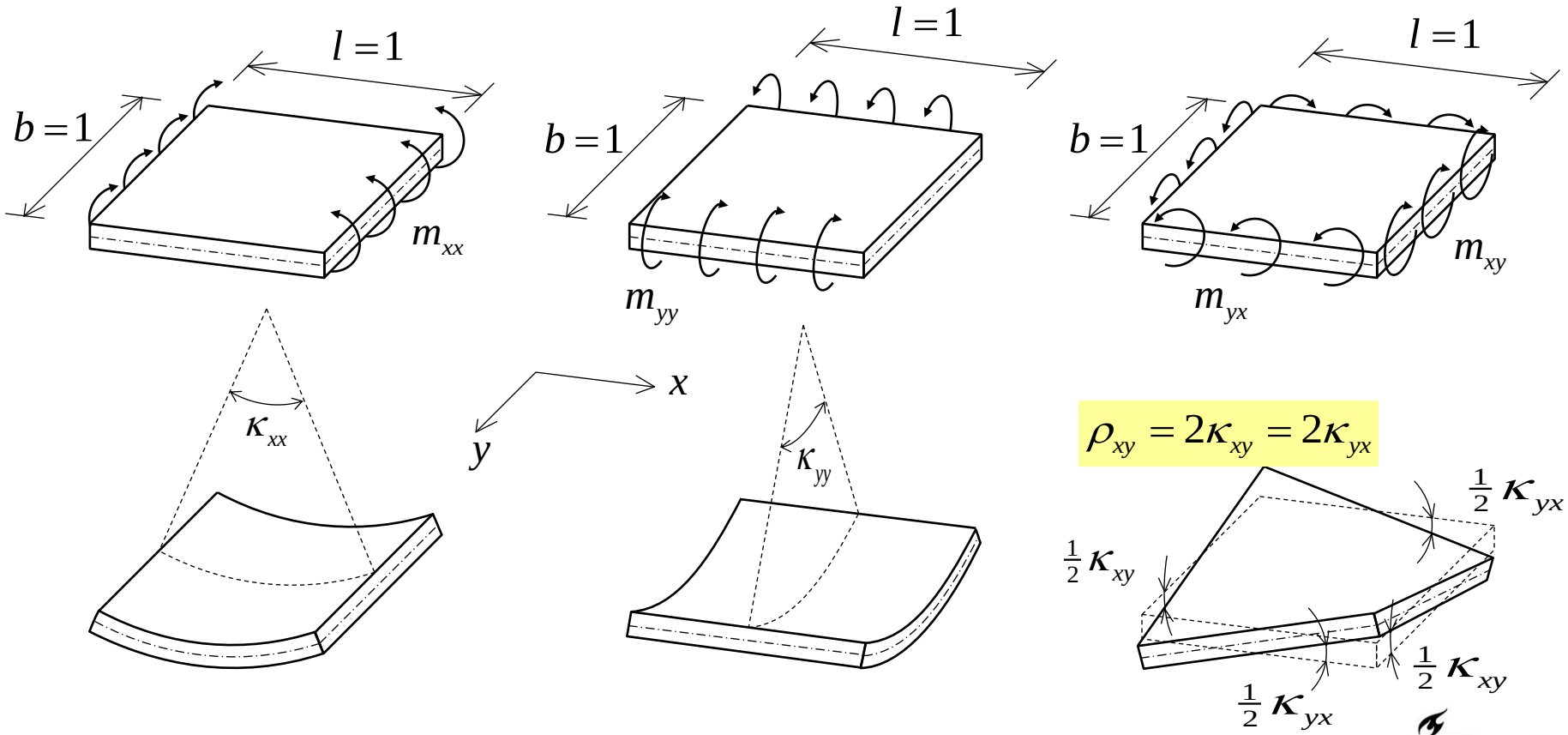
$$v_x = D_\gamma\,\gamma_x$$

$$v_y = D_\gamma\,\gamma_y$$

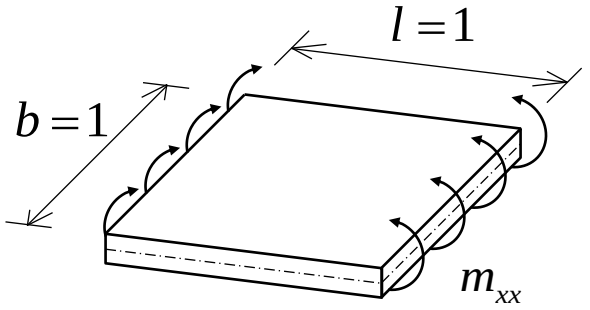
Slab constitutive equations

$$\upsilon = 0$$

$$\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} = D^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix}$$

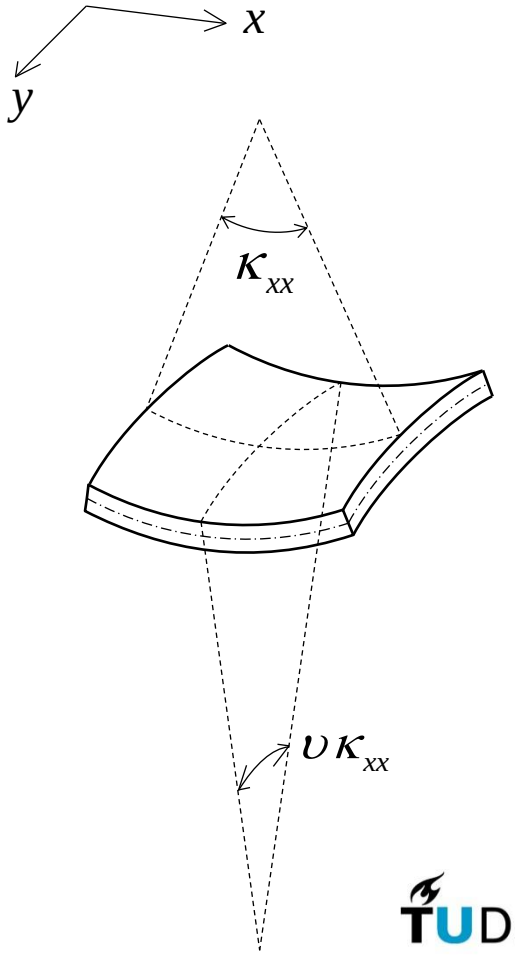


Slab constitutive equations



$$\upsilon \neq 0$$

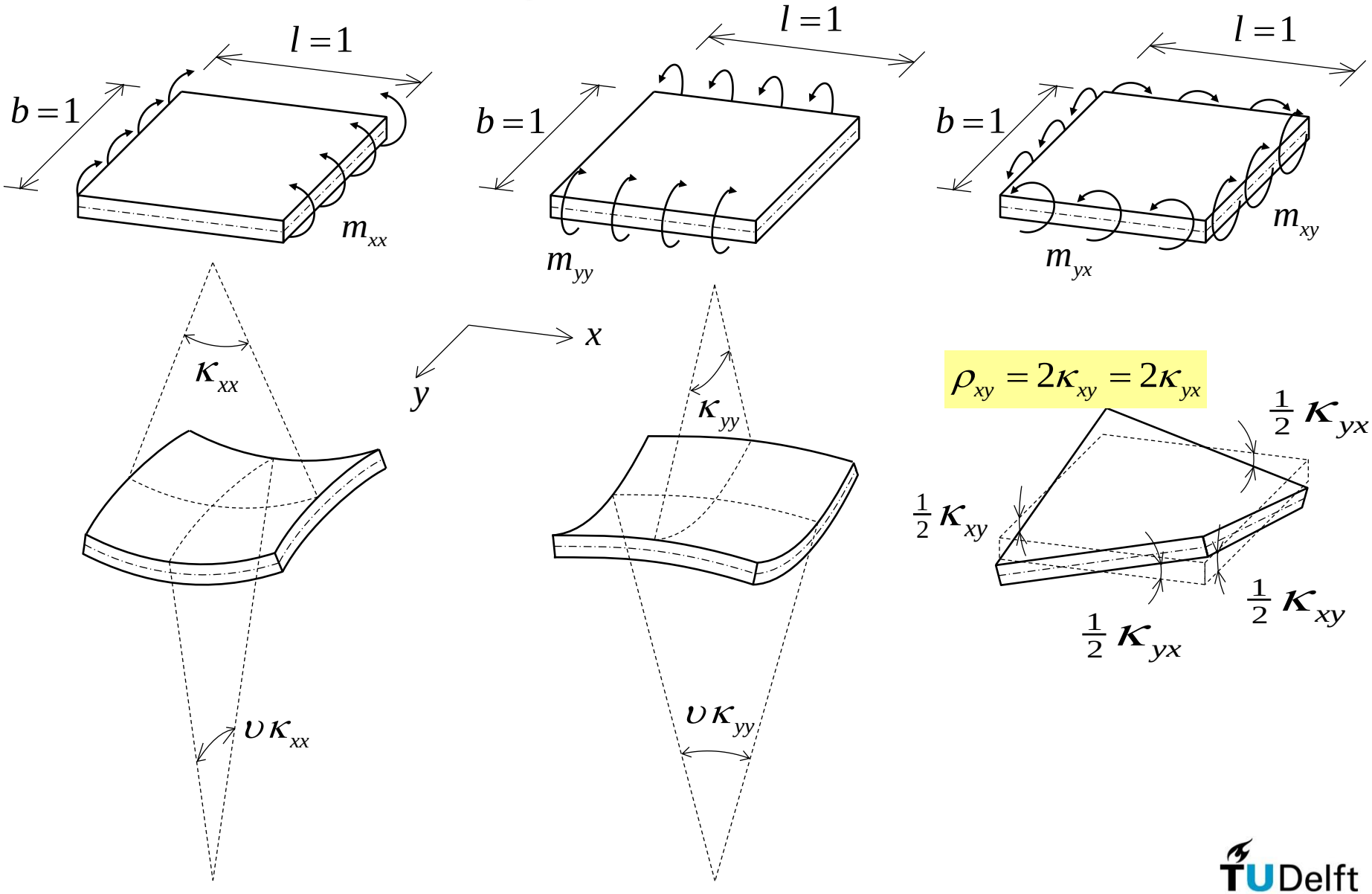
$$\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} = \frac{D^{-1}}{1-\upsilon^2} \begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix} \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix}$$



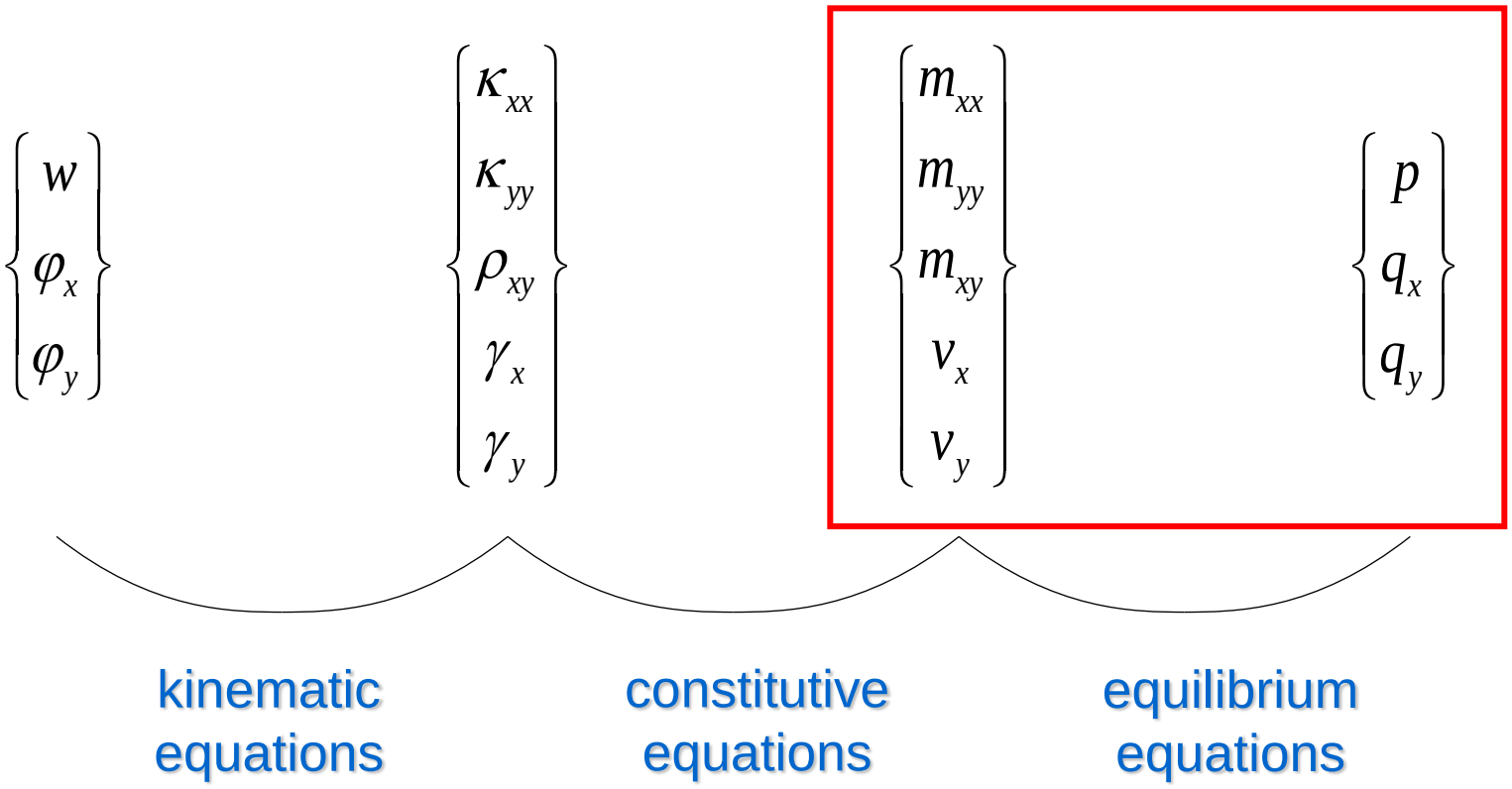
$$\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} = \frac{D^{-1}}{1-\upsilon^2} \begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix} \begin{Bmatrix} m_{xx} \\ 0 \\ 0 \end{Bmatrix}$$

Slab constitutive equations

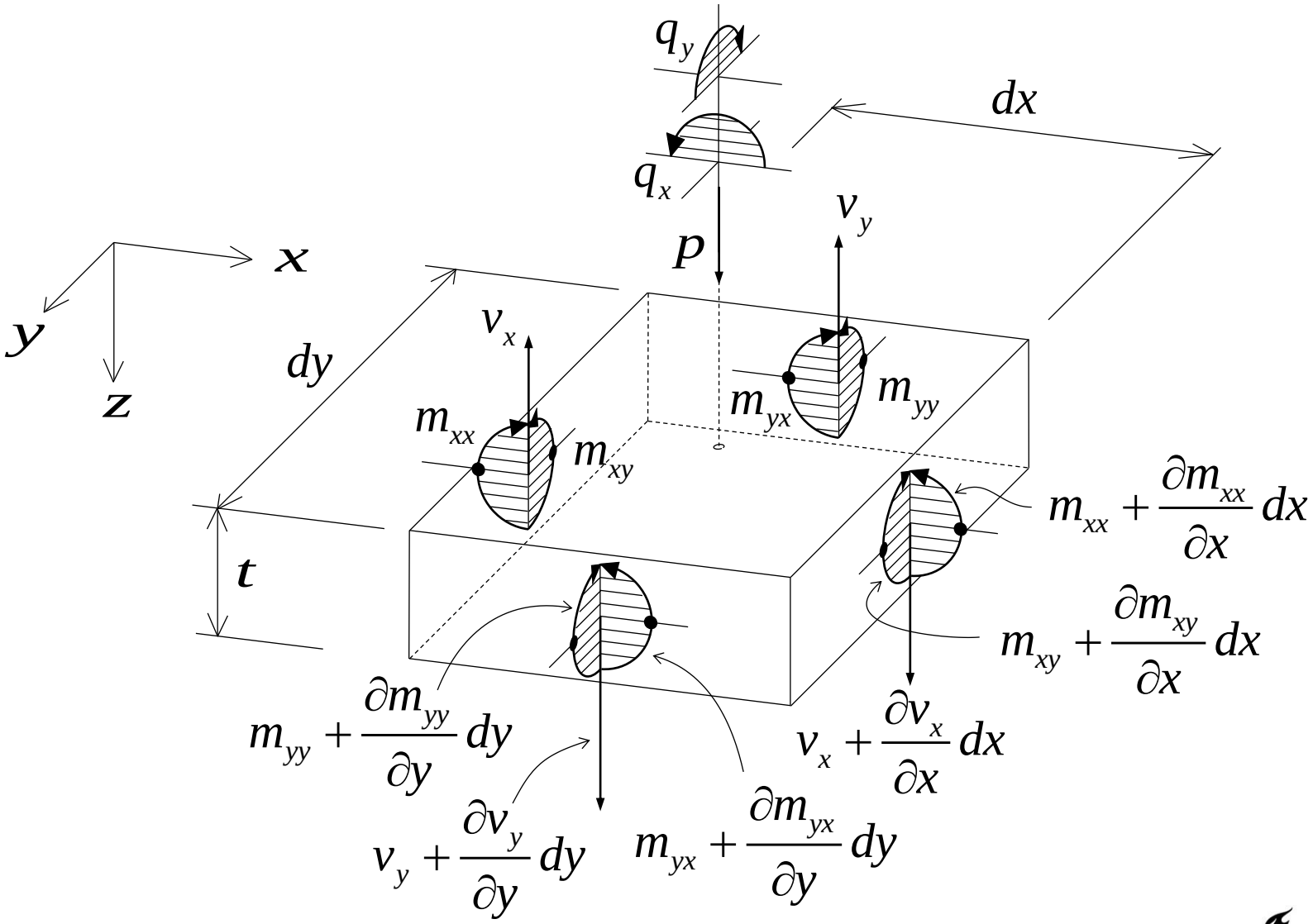
$\upsilon \neq 0$



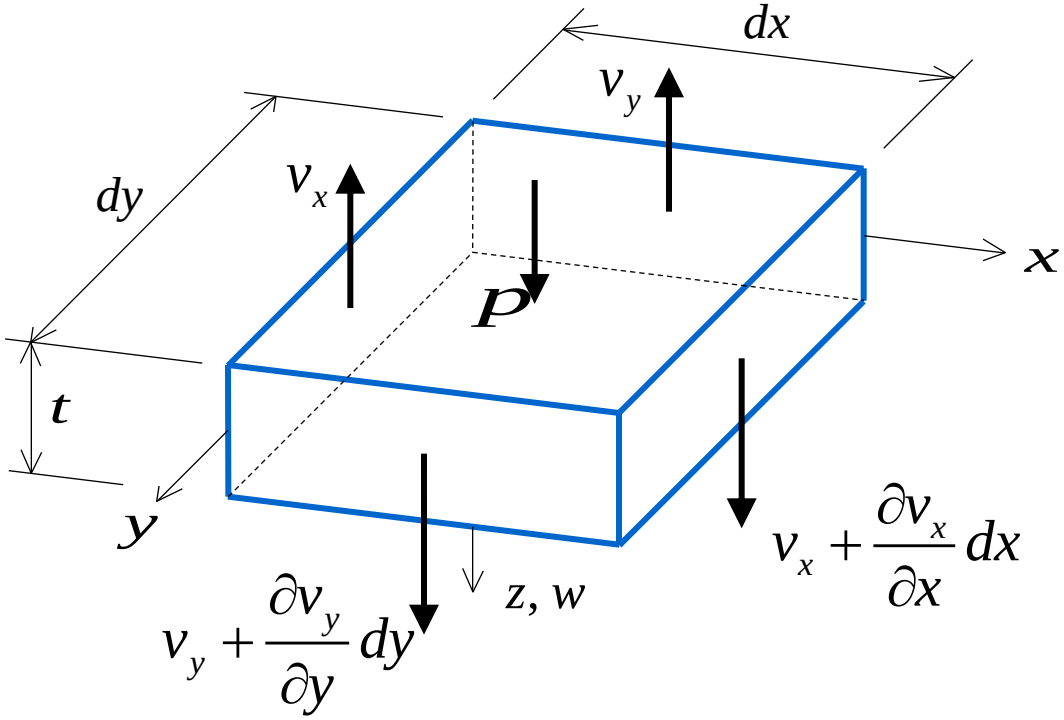
Thick isotropic slabs



Slab equilibrium equations

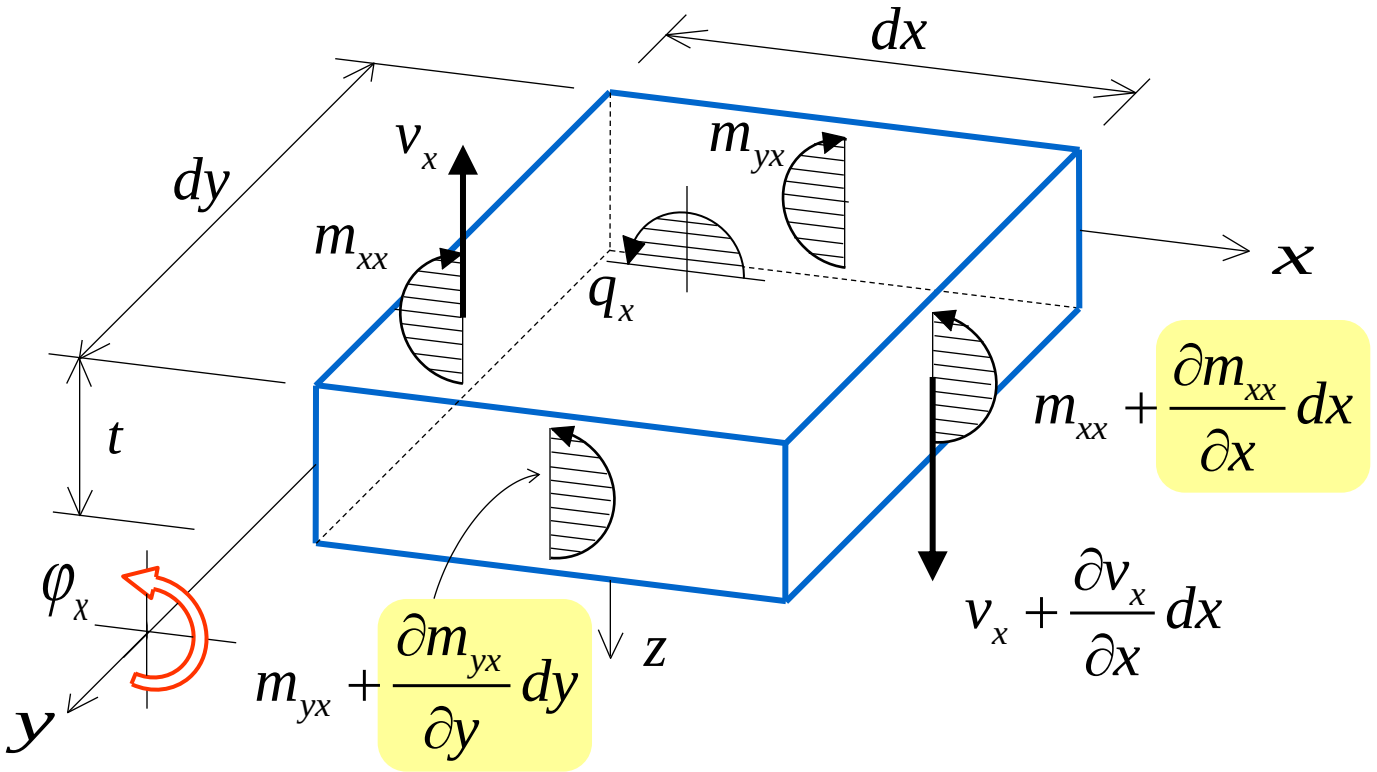


Slab equilibrium equations



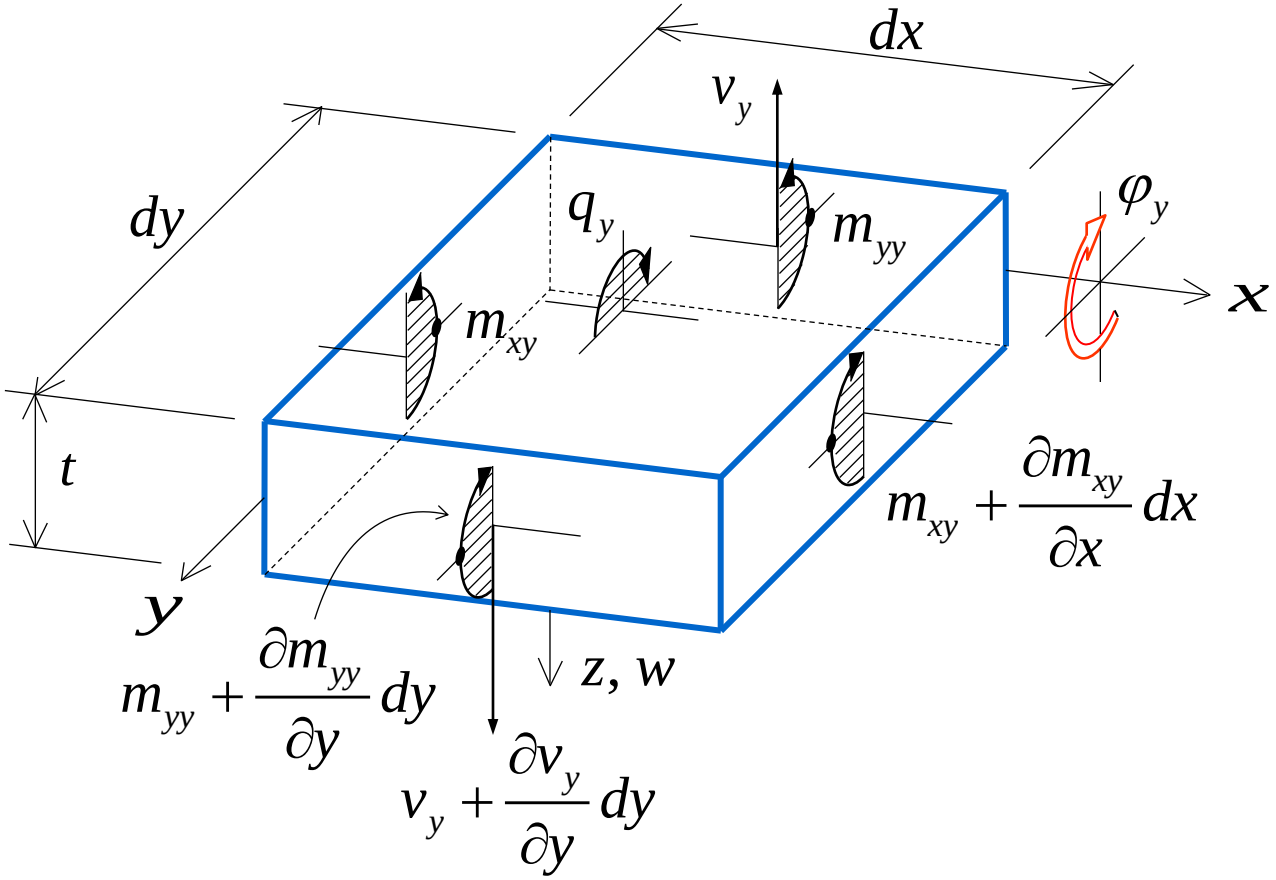
$$\sum f_z = 0 \quad : \quad \left(\frac{\partial v_x}{\partial x} dx\right) dy + \left(\frac{\partial v_y}{\partial y} dy\right) dx + p\, dx\, dy = 0$$

Slab equilibrium equations



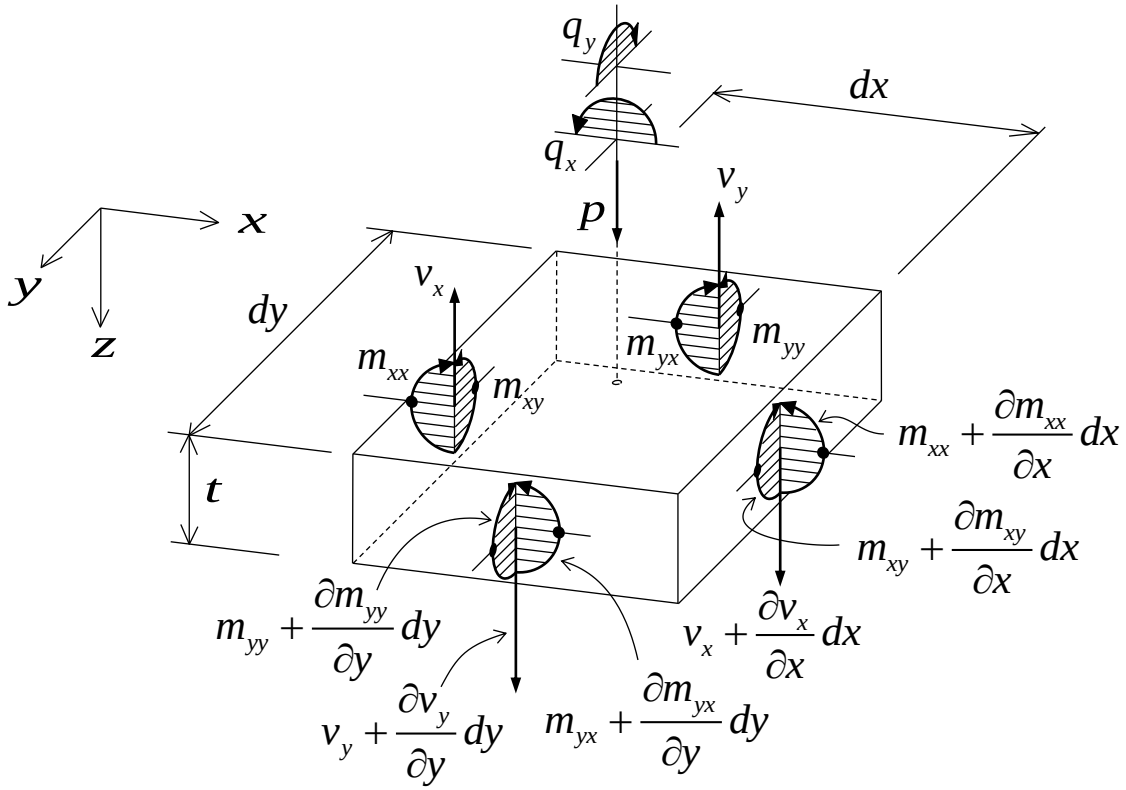
$$\sum m_y = 0 \quad : \quad \left(\frac{\partial m_{xx}}{\partial x} dx\right) dy + \left(\frac{\partial m_{yx}}{\partial y} dy\right) dx - \left(v_x dy\right) dx + q_x \; dx \, dy = 0$$

Slab equilibrium equations



$$\sum m_x = 0 \quad : \quad \left(\frac{\partial m_{yy}}{\partial y} dy\right) dx + \left(\frac{\partial m_{xy}}{\partial x} dx\right) dy - \left(v_y dx\right) dy + q_y \, dx \, dy = 0$$

Slab equilibrium equations



$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + p = 0$$

$$\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} - v_x + q_x = 0$$

$$\frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x} - v_y + q_y = 0$$

Isotropic plates

- finally...

$$\begin{Bmatrix} w \\ \varphi_x \\ \varphi_y \end{Bmatrix} \quad \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} \quad \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \\ v_x \\ v_y \end{Bmatrix} \quad \begin{Bmatrix} p \\ q_x \\ q_y \end{Bmatrix}$$

kinematic equations

constitutive equations

equilibrium equations

....

$$\kappa_{xx} = \frac{\partial \varphi_x}{\partial x} \quad ; \quad \kappa_{yy} = \frac{\partial \varphi_y}{\partial y} \quad ; \quad \rho_{xy} = \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}$$

$$\gamma_x = \varphi_x + \frac{\partial w}{\partial x} \quad ; \quad \gamma_y = \varphi_y + \frac{\partial w}{\partial y}$$

$$m_{xx} = D\left(\kappa_{xx} + \upsilon\, \kappa_{yy}\right)$$

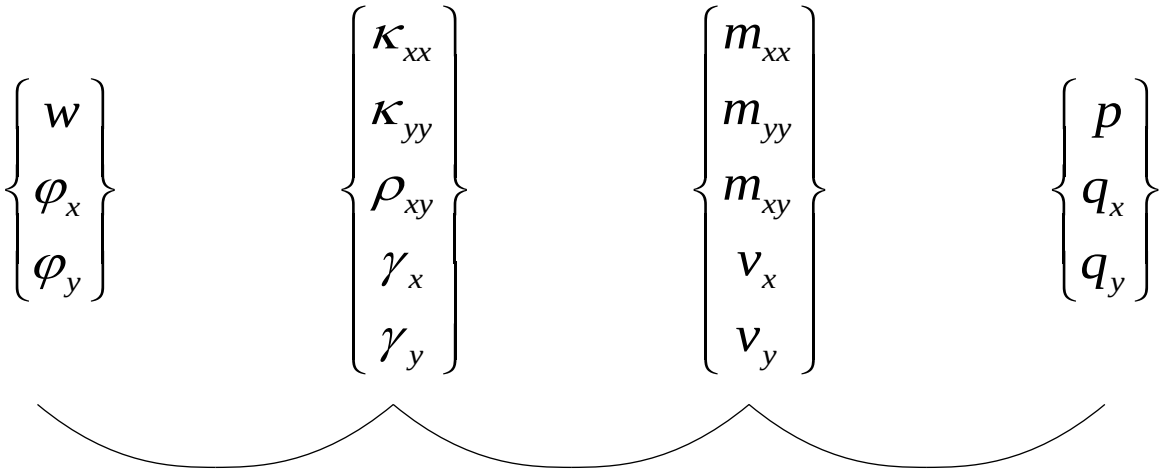
$$m_{yy} = D\left(\kappa_{yy} + \upsilon\, \kappa_{xx}\right) \qquad v_x = D_\gamma\, \gamma_x$$

$$m_{xy} = \frac{1}{2} D\left(1-\upsilon\right) \rho_{xy} \qquad v_y = D_\gamma\, \gamma_y$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + p = 0$$

$$\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} - v_x + q_x = 0$$

$$\frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x} - v_y + q_y = 0$$



$$\begin{bmatrix} -D_\gamma\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) & -D_\gamma\frac{\partial}{\partial x} & -D_\gamma\frac{\partial}{\partial y} \\ D_\gamma\frac{\partial}{\partial x} & D_\gamma - D\frac{\partial^2}{\partial x^2} - \frac{1}{2}(1-\upsilon)D\frac{\partial^2}{\partial y^2} & -\frac{1}{2}(1+\upsilon)D\frac{\partial^2}{\partial x\partial y} \\ D_\gamma\frac{\partial}{\partial y} & -\frac{1}{2}(1+\upsilon)\frac{\partial^2}{\partial x\partial y} & D_\gamma - \frac{1}{2}(1-\upsilon)D\frac{\partial^2}{\partial x^2} - D\frac{\partial^2}{\partial y^2} \end{bmatrix} \begin{Bmatrix} w \\ \varphi_x \\ \varphi_y \end{Bmatrix} = \begin{Bmatrix} p \\ q_x \\ q_y \end{Bmatrix}$$

Messages of the Chapter 3

- We end up with three simultaneous differential equations, each of the second-order. If two degrees of freedom are eliminated a sixth-order differential equation will be found. This means that three boundary conditions must be specified at each edge, which implies, that the vertical displacement, the rotation normal to the edge face and the rotation in the edge face can be prescribed. If one or more of these displacements is unconstrained, the associated distributed edge loads in those directions can be prescribed.