Delft University of Technology
Faculty of Civil Engineering and Geosciences

Theory of Elasticity Ct 5141

Direct Methods

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J. Blaauwendraad
April 2003

Introduction to the course

When the civil engineering student chooses for the course "Theory of Elasticity", (s)he is already extensively familiarised with the mathematical description of structural behaviour by means of differential equations. With this in mind reference can be made to the courses "Elastostatics" and "Elastic Plates and Slabs". This course continues along the same line and extends on it.
During the compilation of these lecture notes, the course material has been restructured. The reason for this is that three main objectives are aimed for:

1. Historically a number of essential subjects in structural mechanics exist, which have to be dealt with. The solution of these problems is incorporated in the basic knowledge of a structural engineer entering the building practice. This part of the course is directed at "*results*". The course material contains a number of known solutions for plate problems and torsional problems for beams with solid or hollow cross-sections.
2. The engineer also should be capable of finding solutions for entirely new problems. For this aspect the course should give directions along which line a solution can be obtained. It appears that two strategies can be followed, in order to arrive at a consistent *analytical modelling*. These strategies will be discussed. In this part of the course the emphasis lies on the development of modelling skills for new problems. It also will be demonstrated that the classic problems (mentioned under objective 1) can be fitted in as applications or examples. The exercises and examples are mainly focussed on structural systems that can be regarded as one-dimensional.
3. The great merit of the Theory of Elasticity is that via an analytical approach *exact* solutions could be obtained for continuous problems of mechanics, long before numerical methods were developed and became available on a large scale as they are today. However, it should be noticed that the realised exact solutions were usually limited to a certain class of problems, sometimes with a relatively academic geometry or for a limited number of not to complex boundary conditions. After the introduction of the computer, the number of possibilities is increased enormously, which made it possible to compute solutions for continua with an arbitrary geometry. In this respect especially the Finite Element Method (FEM) is important. With this method approximate solutions are generated and the basis of the method lies in the application of energy principles. Since energy principles also play a role in the exact formulation of continua, the possibility exists to make a smooth transition from the theory of elasticity to the numerical methods. This part of the course can be considered as an introduction to the course about the Finite Element Method

In view of these three objectives the following set-up of the course is selacted. The lecture notes consists out of two parts. The first part deals with the first two objectives and "*direct methods*" will be used. In the other part attention is paid to the third objective and "*energy principles and variational methods*" will be discussed.

Chapter 1 of the first part starts with a recapitulation of the force and displacement methods for discrete bar structures. The formulation can be presented very compact with the matrix notation. Deliberately an approach is followed that makes it possible to pinpoint clearly which strategies are used.

In chapter 2 a start is made with the analysis of continuous structures. The differential equation and boundary conditions are derived for a simple one-dimensional element, i.e. the beam. For a large part this is revision of previously obtained knowledge. On basis of this well-known continuous structural element, the analogy will be demonstrated with the discrete approach.

With combinations of the different elastic cases (extension, bending, shear, torsion, elastic support) several structural systems can be modelled, such as can be found in high-rise buildings. The derived equations are always ordinary partial differential equations of the second order, fourth order or higher.

After that two-dimensional problems will be addressed. Again with the same strategy the differential equations are derived for in its plane loaded plates (chapter 3) and transversely loaded plates (chapter 4). In these chapters for a number of classic problems the solution of the partial differential equations is worked out or provided directly (objective 1).

The following two chapters deal with the theory of three-dimensional continua. Chapter 5 focuses on the basic equations. In chapter 6 a specific classic problem is formulated and worked out, namely torsion in beams with a solid or hollow cross-section (objective 1). This concludes the part “Direct Methods”.

The part “Energy principles and variational methods” will be offered as a separate set of lecture notes. In these notes the following subjects are addressed: work, energy principles, variational methods and approximate solutions. The validity of the derivations extends to general three-dimensional continua. However, the derivations itself will be worked out for one-dimensional cases. This is also the case for examples and applications.

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1 Recapitulation for discrete bar structures

1.1 Basic equations for statically determinate structures

In the lectures that preceded this course, a large amount of attention was paid to the calculation of bar structures. A number of basic ideas from that lecture material will be summarised briefly. In this summary two strategies are highlighted, which are especially suitable for the analysis of problems, namely the *displacement method* and the *force method*. Moreover, the basic ideas will be addressed in such a manner that it becomes clear that the same strategy can be used for the analysis of continuous bodies. For this purpose a simple flat truss is considered. Fig. 1.1 shows a statically determinate truss. The structure consists out of

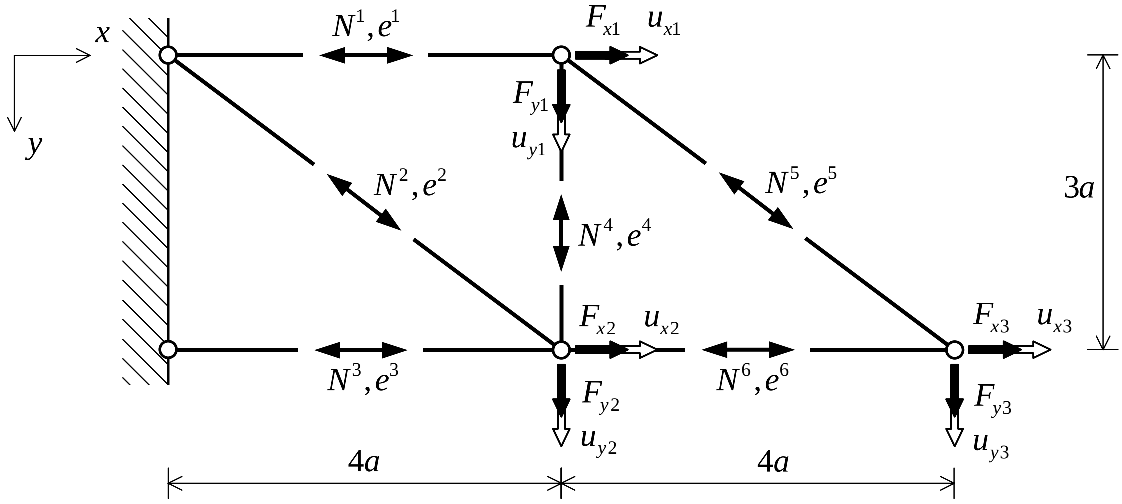


Fig. 1.1: Statically determinate truss with relevant quantities.

six members and three movable nodes, which are numbered from 1 up to 3. Quantities associated with the nodes get a subscript equal to the node number. The bars are numbered from 1 to 6. Quantities associated with the bars receive a superscript with the element number. The height and length of the truss are $3a$ and $8a$, respectively.
Each nodal point has two degrees of freedom, a displacement u_x in x-direction and a displacement u_y in y-direction. In these respective directions external forces F_x and F_y can be applied. The degrees of freedom of all nodes combined, form the vector $\boldsymbol{u}$ and all the forces form the vector $\boldsymbol{f}$.

Stresses are generated inside the structure, together with the corresponding strains. In this case the *stress resultant* N of each bar is used, together with the associated change of length (extension) e of the bar. They form the vector of generalised stresses or stress resultants $\boldsymbol{N}$ and the generalised deformations or shortly *deformations* $\boldsymbol{e}$, respectively. The sign-convention for the external quantities u_x, u_y, F_x and F_y differs from the sign-convention for the internal quantities N and e. For the external quantities a vector sign-convention is used. When they are pointing in positive x- or y-direction they are defined positive. For the internal quantity N a stress sign-convention is applicable. A positive sign is chosen for tensile forces. Likewise e is assumed positive if it concerns an elongation.

Essential in this course is the way in which the several quantities are defined. The different degrees of freedom are identified and determined. Then it is also known which external loads can be applied. Separately it is ascertained which internal (generalised) stresses will appear, after that they are identified together with the corresponding (generalised) deformations. The external vectors $\boldsymbol{u}$ and $\boldsymbol{f}$ must provide exactly the performed external work, and the internal vectors $\boldsymbol{e}$ and $\boldsymbol{N}$ determine the internal deformation work. In the coming chapters this approach will be applied to continuous structures too. In previous courses, it already has been discussed that three basic relations determine the behaviour of structures. This triplet is:

– the kinematic equations
– the constitutive equations
– the equilibrium equations

The kinematic equations provide the relation between the displacements $\boldsymbol{u}$ and the deformations $\boldsymbol{e}$. The constitutive equations relate the deformations $\boldsymbol{e}$ to the stress resultants $\boldsymbol{N}$. And the equilibrium equations prescribe how the stress resultants $\boldsymbol{N}$ are connected with

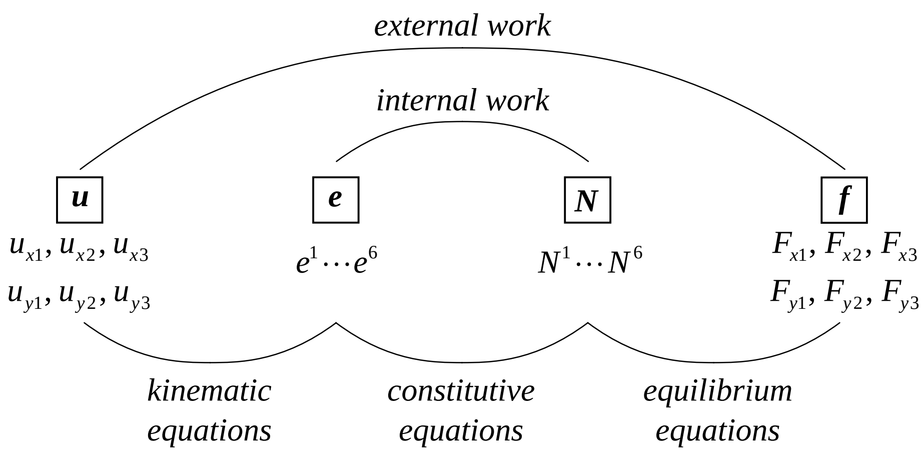


Fig. 1.2: Diagram displaying the relations between the quantities playing a role in the analysis of a truss.

the external load $\boldsymbol{f}$. The scheme in Fig.1.2 provides an overview of all these interacting relations.
Now, this triplet of equations will be worked out in detail for the example of the truss as shown in Fig. 1.1.

Kinematic relations
Considering the sign-convention for the displacements and deformations, for each of the bars the following relations can be derived (see Fig. 1.3):

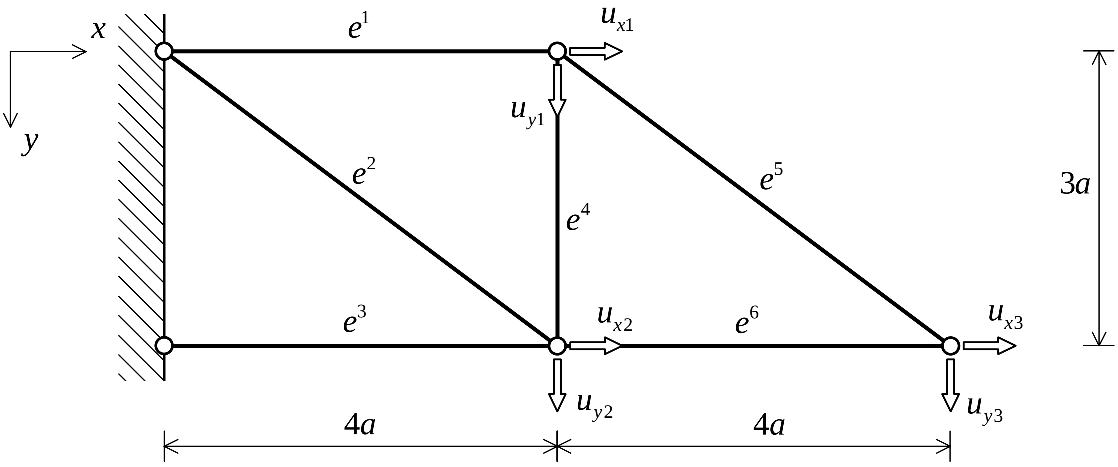


*Fig. 1.3: Relations exist between deformations **e** and displacements **u**.*

$$
\begin{aligned}
e^1 &= +\ u_{x1} \\
e^2 &= +\tfrac{4}{5}u_{x2} + \tfrac{3}{5}u_{y2} \\
e^3 &= +\ u_{x2} \\
e^4 &= -\ u_{y1} + \ u_{y2} \\
e^5 &= -\tfrac{4}{5}u_{x1} - \tfrac{3}{5}u_{y1} + \tfrac{4}{5}u_{x3} + \tfrac{3}{5}u_{y3} \\
e^6 &= -\ u_{x2} + \ u_{x3}
\end{aligned}
$$

In matrix notation this becomes:

$$
\begin{Bmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \\ e^6 \end{Bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4}{5} & \frac{3}{5} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
-\frac{4}{5} & -\frac{3}{5} & 0 & 0 & \frac{4}{5} & \frac{3}{5} \\
0 & 0 & -1 & 0 & 1 & 0
\end{bmatrix}
\begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix}
$$

This result can be rewritten briefly by the introduction of the *kinematic matrix* $\boldsymbol{B}$:

$$\boxed{\boldsymbol{e} = \boldsymbol{B}\boldsymbol{u}} \qquad (\textit{kinematic equations}) \qquad (1.1)$$

Constitutive relations

For each bar a stiffness relation exists between the normal force N and the deformation e :

$$N = \frac{EA}{l} e$$

The flexibility formulation provides the inverse form:

$$e = \frac{l}{EA} N$$

By introduction of the abbreviations:

$$D = \frac{EA}{l} \quad ; \quad C = \frac{l}{EA}$$

the formulation of the constitutive equations becomes:

$$\begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \end{Bmatrix} = \begin{bmatrix} D^1 & & & & & \\ & D^2 & & & & \\ & & D^3 & & & \\ & & & D^4 & & \\ & & & & D^5 & \\ & & & & & D^6 \end{bmatrix} \begin{Bmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \\ e^6 \end{Bmatrix}$$

or briefly:

$$\boxed{\boldsymbol{N} = \boldsymbol{D}\,\boldsymbol{e}}$$

(*constitutive relations in stiffness formulation*) (1.2)a
(*stiffness)*

and:

$$\begin{Bmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \\ e^6 \end{Bmatrix} = \begin{bmatrix} C^1 & & & & & \\ & C^2 & & & & \\ & & C^3 & & & \\ & & & C^4 & & \\ & & & & C^5 & \\ & & & & & C^6 \end{bmatrix} \begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \end{Bmatrix}$$

or briefly:

$$\boxed{\boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{N}}$$

(*constitutive relations in flexibility formulation*) (1.2)b
(*compliance*)

It will become clear that the stiffness formulation of the constitutive equations is used in the displacement method and that the flexibility formulation is used in the force method.

Equilibrium equations
The next three pairs of equilibrium equations are obtained from the equilibrium of all nodes in the direction of the respective degrees of freedom (see Fig. 1.4):

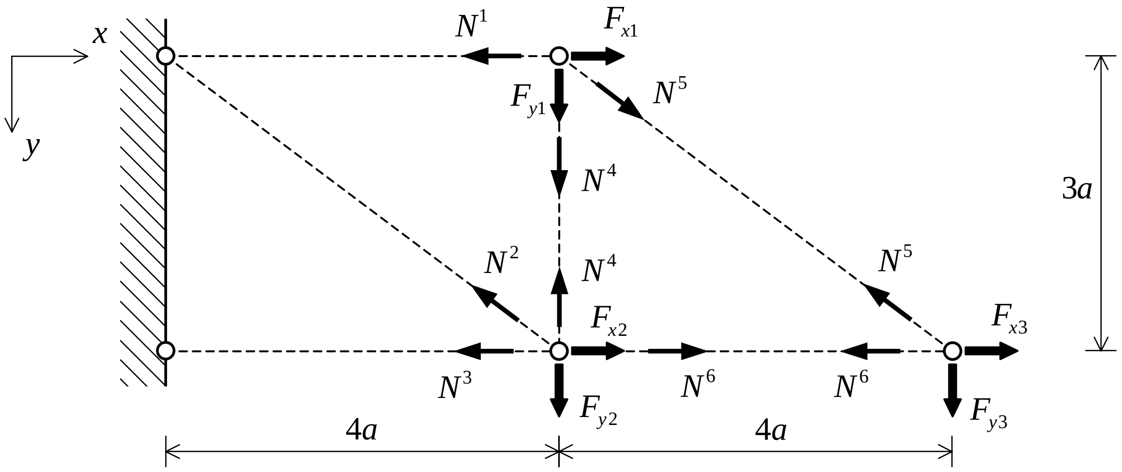


Fig. 1.4: For each node equilibrium exist between the normal forces ***N*** *and the external loads* ***f****.*

$$
\begin{array}{lllllll}
+N^1 & & & & -\frac{4}{5}N^5 & & = F_{x1} \\
 & & & -N^4 & -\frac{3}{5}N^5 & & = F_{y1} \\
 & +\frac{4}{5}N^2 & +N^3 & & & -N^6 & = F_{x2} \\
 & +\frac{3}{5}N^2 & & +N^4 & & & = F_{y2} \\
 & & & & +\frac{4}{5}N^5 & +N^6 & = F_{x3} \\
 & & & & +\frac{3}{5}N^5 & & = F_{y3}
\end{array}
$$

In matrix form this reads:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{4}{5} & 0 \\
0 & 0 & 0 & -1 & -\frac{3}{5} & 0 \\
0 & \frac{4}{5} & 1 & 0 & 0 & -1 \\
0 & \frac{3}{5} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4}{5} & 1 \\
0 & 0 & 0 & 0 & \frac{3}{5} & 0
\end{bmatrix}
\begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \end{Bmatrix}
=
\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix}
$$

Comparison of this matrix with the previously found kinematic matrix $\boldsymbol{B}$ shows that it is exactly the transposed of $\boldsymbol{B}$. Therefore it can be written:

$$\boxed{\boldsymbol{B}^T \boldsymbol{N} = \boldsymbol{f}} \qquad (\textit{equilibrium equations}) \qquad (1.3)$$

where the superscript “ T ” is the internationally accepted symbol to indicate the transposed of a matrix.

1.2 Strategies for the analysis of statically determinate structures

In the previous section the following set of basic equations have been found:

$$\boxed{\begin{array}{l} \boldsymbol{e} = \boldsymbol{B}\,\boldsymbol{u} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.4)\\ \boldsymbol{N} = \boldsymbol{D}\,\boldsymbol{e} \quad \text{or} \quad \boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{N} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.5)\\ \boldsymbol{B}^T\boldsymbol{N} = \boldsymbol{f} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.6) \end{array}}$$

Historical

The **first step** is the calculation of the stress quantities $\boldsymbol{N}$. In the case of a statically determinate truss the vectors $\boldsymbol{N}$ and $\boldsymbol{f}$ have the same number of components, which means that the matrix $\boldsymbol{B}^T$ in the equilibrium equation (1.6) is square. Therefore, the stress quantities can be determined directly by inversion of $\boldsymbol{B}^T$:

$$\boxed{\boldsymbol{N} = \boldsymbol{B}^{-T}\boldsymbol{f}} \qquad (\text{“}\textit{Cremona}\text{”}) \qquad (1.7)$$

This is the mathematical formulation of the classical graphical method involving the drawing of Cremona diagrams. The **second step** is the calculation of the deformation quantities $\boldsymbol{e}$. When the normal forces $\boldsymbol{N}$ are known, the changes in member length $\boldsymbol{e}$ directly follow from the flexibility formulation of the constitutive equations $\boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{N}$.
Then in the **third step**, the displacements can be obtained from the kinematic relations given in (1.4). In a statically determinate truss the vectors $\boldsymbol{e}$ and $\boldsymbol{u}$ again have the same number of components. So, the matrix $\boldsymbol{B}$ is square and can be inverted. The required displacements subsequently can be obtained from:

$$\boxed{\boldsymbol{u} = \boldsymbol{B}^{-1}\boldsymbol{e}} \qquad (\text{“}\textit{Williot}\text{”}) \qquad (1.8)$$

This is the mathematical description of the classical graphical method, in which the displacements are determined from the changes in bar length by construction of a Williot diagram.
The described computational method in these lecture notes contains the same consecutive phases, which also students historically have to follow during the learning process of applied mechanics. First, the force transmission and the equilibrium are thoroughly discussed. Then the concept of deformations is introduced and thirdly the displacements are calculated.
The triplet of equations as listed below is evaluated in the order indicated by the arrow:

“Historical” ↓
– equilibrium equations
– constitutive equations (flexibility formulation)
– kinematic equations

Numerical
After the introduction of the first computers, algorithms have been developed that literally followed above procedure, and basically only replaced the graphical element by a numerical technique. However, simultaneously the *displacement method* or *stiffness method* came into use, which appeared to be more suitable for computer analysis. In this method the triplet of equations is solved in the reversed order:

"Numerical"	$\uparrow$	– equilibrium equations – constitutive equations (stiffness formulation) – kinematic equations

When the kinematic equations (1.4) are substituted in the constitutive equations (1.5), followed by a substitution into the equilibrium equations (1.6), the result becomes:

$$\boxed{\boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, \boldsymbol{u} = \boldsymbol{f}} \qquad (\textit{equations}) \tag{1.9}$$

Each of these matrices is square. Since $\boldsymbol{D}$ is a symmetrical matrix and the pre-multiplication matrix $\boldsymbol{B}^T$ is the transposed of the post-multiplication matrix $\boldsymbol{B}$, the final product will be a square and symmetrical matrix. This matrix is indicated by $\boldsymbol{K}$ and is called the *stiffness matrix*, i.e.:

$$\boxed{\boldsymbol{K} = \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}} \qquad (\textit{stiffness matrix}) \tag{1.10}$$

The system of equations can now be summarised as follows:

$$\boxed{\boldsymbol{K}\, \boldsymbol{u} = \boldsymbol{f}} \tag{1.11}$$

From this system the displacements can be solved. In the standard displacement method the matrix $\boldsymbol{K}$ is assembled from the individual stiffness matrices of the several members. In this course intentionally for another derivation of $\boldsymbol{K}$ is chosen, as an introduction on the next chapters dealing with continuous structures.
In the displacement method first the displacements are calculated. After that, the deformations can be obtained from the kinematic relations and finally from the constitutive equations the stress quantities can be determined. This means that the triplet of equations is considered again in the same order.

The formulation of the equations given by (1.9) will be considered again during the discussion of continuous structures. It is quite obvious, that the product of $\boldsymbol{B}^T$, $\boldsymbol{D}$ and $\boldsymbol{B}$ has to deliver a symmetrical matrix. For linear-elastic structures this follows directly from Maxwell's law of reciprocal deflections. Conversely, it can be concluded that $\boldsymbol{B}^T$ always has to be the transposed of $\boldsymbol{B}$, irrespective of the structure considered.

1.3 Basic equations for statically indeterminate structures

The formulation of the basic equations as described in section 1.1 will be repeated for a statically indeterminate structure. The same truss is considered with three free nodes, however with an extra seventh member as shown in Fig. 1.5. This makes the structure statically indeterminate to the first degree.

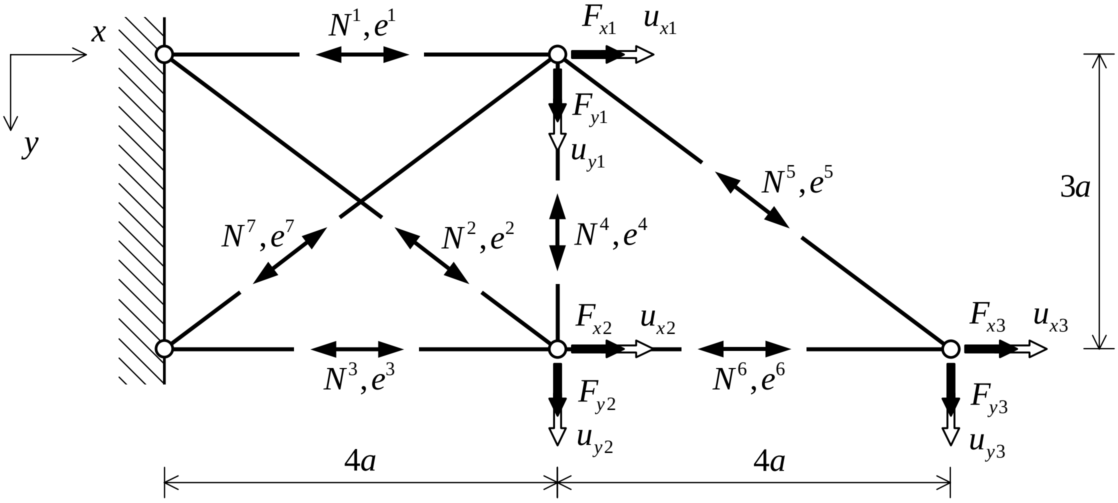


Fig. 1.5: Statically indeterminate truss with relevant quantities.

There still are two times six external quantities (degrees of freedom $\boldsymbol{u}$ and forces $\boldsymbol{f}$), but internally the number is larger. Seven normal forces in $\boldsymbol{N}$ and seven corresponding elongations in $\boldsymbol{e}$ are present.
Again the triplet of equations will be formulated. After the detailed analysis of section 1.1 this can be done briefly.

Kinematic equations

$$\begin{Bmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \\ e^6 \\ e^7 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -\frac{4}{5} & -\frac{3}{5} & 0 & 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & 0 & -1 & 0 & 1 & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

Again this can be written as:

$$\boxed{\boldsymbol{e} = \boldsymbol{B}\,\boldsymbol{u}} \qquad (1.12)$$

Now the matrix $\boldsymbol{B}$ has seven rows and six columns and therefore is not square anymore. The first six rows are identical to the matrix $\boldsymbol{B}$ of section 1.1. The seventh row is an extension due to the extra member.

Constitutive equations
Also in this case, the stiffness and flexibility formulations of the constitutive equations read:

$$\boxed{\boldsymbol{N} = \boldsymbol{D}\boldsymbol{e} \quad ; \quad \boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{N}} \tag{1.13}$$

Now $\boldsymbol{N}$ and $\boldsymbol{e}$ both contain seven components and $\boldsymbol{D}$ and $\boldsymbol{C}$ are square matrices with seven rows and seven columns.

Equilibrium equations
The six nodal equilibrium equations are now expressed in seven stress quantities:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{4}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 & -1 & -\frac{3}{5} & 0 & -\frac{3}{5} \\ 0 & \frac{4}{5} & 1 & 0 & 0 & -1 & 0 \\ 0 & \frac{3}{5} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{5} & 0 & 0 \end{bmatrix} \begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \\ N^7 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix}$$

Also in this case, the matrix is the transposed of matrix $\boldsymbol{B}$. Therefore, it briefly can be written:

$$\boxed{\boldsymbol{B}^T \boldsymbol{N} = \boldsymbol{f}} \tag{1.14}$$

Again the first six columns are equal to $\boldsymbol{B}^T$ from section 1.1, the seventh columns is an extension.

1.4 Strategies for the analysis of statically indeterminate structures

On basis of the basic equations it formally will be described how the force method and the displacement method will work out.

1.4.1 Force method

1st step: Equilibrium
Normally, the first step would have been the solution of the equilibrium equations. However, this is not possible because the number of unknowns exceeds the number of equations by one. In such a case, the old and well-tried method of making the structure statically determinate can be used, where one of the members is cut and a redundant is introduced on the cutting face. In the example of Fig. 1.6, bar 7 is cut and the redundant ϕ is introduced.

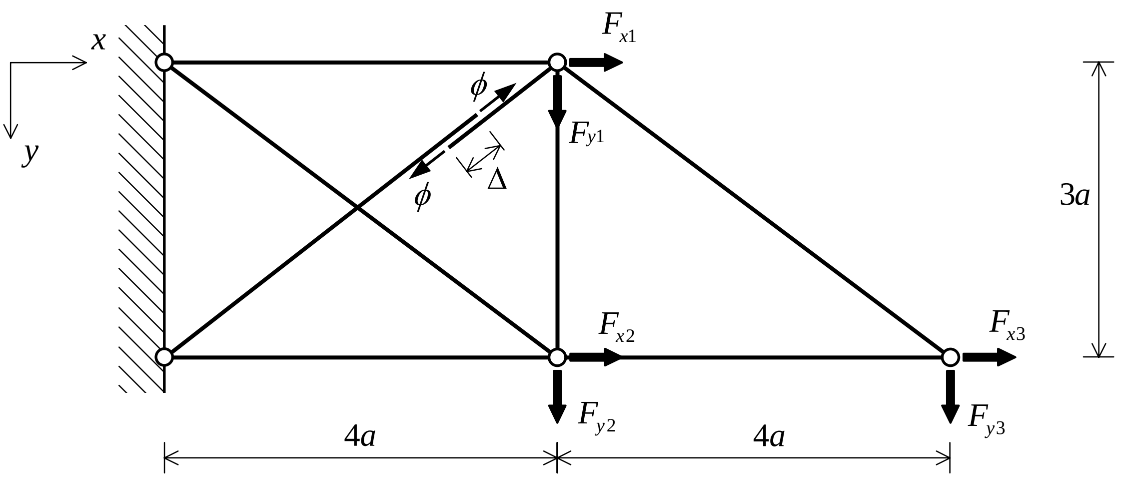


Fig. 1.6: Introduction of redundant ϕ.

Doing so a statically determinate main system is created. On top of the six external components of $\boldsymbol{f}$, also the redundant ϕ has to be considered as an external load (still unknown) on the main system. This means that this main system is subjected to two load vectors:

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix} \quad ; \quad \begin{Bmatrix} -\frac{4}{5} \\ +\frac{3}{5} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \phi$$

The second vector contains the components of the redundant ϕ acting in the three nodes. In section 1.2 it has been shown, how by (1.7) the normal forces can be calculated of the statically determinate main system.
For the first load vector this provides:

$$\begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & \frac{5}{3} & 0 & \frac{5}{3} & 0 & \frac{5}{3} \\ 0 & -\frac{4}{3} & 1 & -\frac{4}{3} & 1 & -\frac{8}{3} \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 1 & -\frac{4}{3} \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix} \tag{1.15}$$

where the matrix is the inverse of the matrix $\boldsymbol{B}^T$ from section 1.2. When in (1.15) the vector with external forces is replaced by the second vector with the components of the redundant ϕ, the normal forces in the main system resulting from ϕ can be found. The matrix-vector product then results in:

$$\begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \end{Bmatrix} = \begin{Bmatrix} -\frac{4}{5} \\ 1 \\ -\frac{4}{5} \\ -\frac{3}{5} \\ 0 \\ 0 \end{Bmatrix} \phi \tag{1.16}$$

The seventh normal force is independent from the six external forces and is directly equal to the redundant, i.e.:

$$N^7 = \phi \tag{1.17}$$

The sum of these three intermediate results delivers the normal forces for the external load together with the redundant. This sum can be written as:

$$\begin{Bmatrix} N^1 \\ N^2 \\ N^3 \\ N^4 \\ N^5 \\ N^6 \\ N^7 \end{Bmatrix} = \underbrace{\left[\begin{array}{cccccc|c} \multicolumn{6}{c|}{\boldsymbol{B}^{-T}} & \\ 1 & 0 & 0 & 0 & 0 & \frac{4}{5} & -\frac{4}{5} \\ 0 & \frac{5}{3} & 0 & \frac{5}{3} & 0 & \frac{5}{3} & 1 \\ 0 & -\frac{4}{3} & 1 & -\frac{4}{3} & 1 & -\frac{8}{3} & -\frac{4}{5} \\ 0 & -1 & 0 & 0 & 0 & -1 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{4}{3} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\boldsymbol{P}_f \;|\; \boldsymbol{P}} \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ \hline \phi \end{Bmatrix} \begin{matrix} \\ \\ \\ \boldsymbol{f} \\ \\ \\ \phi \end{matrix} \tag{1.18}$$

The columns of this matrix associated with the load vector $\boldsymbol{f}$ form the matrix $\boldsymbol{P}_f$ and the column working on the redundant ϕ is called $\boldsymbol{P}$. For cases with more than one redundant, $\boldsymbol{P}$ will contain more columns and $\boldsymbol{\phi}$ more than one component. So, matrix relation (1.18) can briefly be written as:

$$\boxed{\boldsymbol{N} = \boldsymbol{P}_f\, \boldsymbol{f} + \boldsymbol{P}\boldsymbol{\phi}} \qquad (\textit{equilibrium system}) \tag{1.19}$$

These stress resultants form an equilibrium system and therefore will satisfy the equilibrium equations (1.14) from section 1.3.

2nd step: Constitution

The force method utilises the constitutive relations in the flexibility formulation:

$$\boxed{\boldsymbol{e} = \boldsymbol{C}\, \boldsymbol{N}} \qquad (\textit{constitution}) \tag{1.20}$$

3rd step: Compatibility

The solution process continues as follows. In Fig. 1.6 it was already shown that the bar ends on the cutting face can move independently with respect to each other. The overlap Δ (or more generally the gap) is the result of both the external forces and the redundant.
For the determination of the still unknown redundant ϕ the so-called *compatibility condition* is required, which is given by:

$$\Delta = 0 \tag{1.21}$$

In words: the gap caused by external forces has to be eliminated by the gap resulting from the redundant. For the truss of the example, it will be checked how the gap is related to the deformations of the members. This has to be a purely kinematic relation, which is governed by the geometry of the structure.
For the calculation of Δ it has to be clear how much the distance is reduced between the nodes 1 and 4. This reduction will be called $\Delta_{1,4}$. Its magnitude is:

$$\Delta_{1,4} = -\tfrac{4}{5} u_{x1} + \tfrac{3}{5} u_{y1}$$

The value of u_{x1} can directly be expressed in e^1:

$$u_{x1} = e^1$$

The magnitude of u_{y1} is not directly known. However, it is known that:

$$u_{y1} = u_{y2} - e^4$$

the displacement u_{y2} of which directly can be expressed in the deformations:

$$u_{y2} = \tfrac{5}{3} e^2 - \tfrac{4}{3} e^3$$

With these results, the gap $\Delta_{1,4}$ can be written as:

$$\Delta_{1,4} = -\tfrac{4}{5} e^1 + e^2 - \tfrac{4}{5} e^3 - \tfrac{3}{5} e^4$$

The gap at the position of the redundant is equal to this result increased by the change of length of bar 7:

$$\Delta = \Delta_{1,4} + e^7$$

Thus:

$$\Delta = -\tfrac{4}{5} e^1 + e^2 - \tfrac{4}{5} e^3 - \tfrac{3}{5} e^4 + e^7$$

In matrix notation this becomes:

$$\Delta = \left\{ -\tfrac{4}{5} \quad 1 \quad -\tfrac{4}{5} \quad -\tfrac{3}{5} \quad 0 \quad 0 \quad 1 \right\} \begin{Bmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \\ e^6 \\ e^7 \end{Bmatrix}$$

Closer inspection reveals that the row-matrix is just equal to the transposed of the matrix $\boldsymbol{P}$. Therefore the gap equals:

$$\boxed{\boldsymbol{\Delta} = \boldsymbol{P}^T \boldsymbol{e}} \tag{1.22}$$

The gap is here written as a vector, because for statically indeterminate structures to higher degrees more than one gap is present. In that case, the matrix $\boldsymbol{P}^T$ will contain more rows. The *compatibility condition* can now briefly be formulated as the requirement that:

$$\boxed{\boldsymbol{P}^T \boldsymbol{e} = \boldsymbol{0}} \qquad (\textit{compatibility}) \tag{1.23}$$

In the example of the truss this is:

$$-\tfrac{4}{5} e^1 + e^2 + \tfrac{4}{5} e^3 - \tfrac{3}{5} e^4 + e^7 = 0$$

Now, formally the recipe for the compatibility condition has been derived on bases of kinematic considerations, which has been done in such a manner that a physical interpretation can be given. From a mathematical point of view, condition (1.23) can also directly be derived from the kinematic relations (1.12):

$$\boldsymbol{e} = \boldsymbol{B}\,\boldsymbol{u}$$

which contain seven equations with six unknown displacements. This means that one dependent relation between the seven deformations can be formulated. In order to find this relation, the displacements have to be eliminated. This can be done by linear combination of the rows of $\boldsymbol{B}$ in such a way that a row of zeros is created. The weight factors with which the rows have to be multiplied just form the row-matrix $\boldsymbol{P}^T$.

This formal recipe: the elimination of the displacements from the kinematic relations in order to find the compatibility condition, shall be applied again to continuous structures.

System of equations
With the three intermediate results for equilibrium (1.19), constitution (1.20) and compatibility (1.23) a system of equations is created for the calculation of the redundant(s) $\boldsymbol{\phi}$.
Substitution of (1.19) into (1.20) leads to:

$$\boxed{\boldsymbol{e} = \boldsymbol{C}\left(\boldsymbol{P}_f\, \boldsymbol{f} + \boldsymbol{P}\boldsymbol{\phi}\right)} \tag{1.24}$$

Combination of this result with the compatibility condition (1.23) yields:

$$\boxed{\boldsymbol{P}^T\boldsymbol{C}\,\boldsymbol{P}_f\,\boldsymbol{f} + \boldsymbol{P}^T\boldsymbol{C}\,\boldsymbol{P}\,\boldsymbol{\phi} = \boldsymbol{0}} \tag{1.25}$$

The first term in this relation is the gap resulting from the external load $\boldsymbol{f}$. This is a known term and will be called $\boldsymbol{\Delta}_f$. Therefore, the redundant(s) $\boldsymbol{\phi}$ can be obtained from the equations:

$$\boxed{\boldsymbol{P}^T\boldsymbol{C}\,\boldsymbol{P}\,\boldsymbol{\phi} = -\boldsymbol{\Delta}_f} \qquad (\textit{equations}) \tag{1.26}$$

This formulation will be used again for continuous structures. The product of the three matrices $\boldsymbol{P}^T$, $\boldsymbol{C}$ and $\boldsymbol{P}$ delivers a symmetrical matrix because $\boldsymbol{C}$ is symmetrical and $\boldsymbol{P}^T$ is the transposed of $\boldsymbol{P}$.
The product matrix has to be symmetrical since it has to satisfy Maxwell's law, which proves that $\boldsymbol{P}^T$ is always the transposed of $\boldsymbol{P}$. When for the product of the three matrices the total flexibility matrix $\boldsymbol{F}$ is introduced:

$$\boldsymbol{F} = \boldsymbol{P}^T\boldsymbol{C}\,\boldsymbol{P}$$

The system of equations to be solved becomes:

$$\boldsymbol{F}\,\boldsymbol{\phi} = -\boldsymbol{\Delta}_f$$

Remark 1
It already was stated that the stresses given by (1.19) satisfy the equilibrium equations (1.14).
Substitution of these stresses into the equilibrium equations provides the condition:

$$\boldsymbol{B}^T\boldsymbol{P}_f\,\boldsymbol{f} + \boldsymbol{B}^T\boldsymbol{P}\,\boldsymbol{\phi} = \boldsymbol{f}$$

Since both $\boldsymbol{f}$ and $\boldsymbol{\phi}$ have to be different from zero, from this relation it can be concluded:

$$\boxed{\begin{aligned}\boldsymbol{B}^T\boldsymbol{P}_f &= \boldsymbol{I}\\ \boldsymbol{B}^T\boldsymbol{P} &= \boldsymbol{0}\end{aligned}} \tag{1.27}$$

Here $\boldsymbol{I}$ is the unit matrix and $\boldsymbol{0}$ is the zero matrix.
In this manner the compatibility condition can be derived as well. When in the kinematic relation (1.12) both the left-hand and right-hand sides are multiplied by $\boldsymbol{P}^T$, it follows:

$$\boldsymbol{P}^T \boldsymbol{e} = \boldsymbol{P}^T \boldsymbol{B}\, \boldsymbol{u}$$

From (1.27) it can be seen that $\boldsymbol{B}^T \boldsymbol{P}$ is a zero matrix. The matrix $\boldsymbol{P}^T \boldsymbol{B}$ is its transposed and therefore a zero matrix too. This means that the right-hand side of the above equation is equal to zero, reducing it to the already obtained compatibility condition (1.23).

Remark 2
In the case of a statically determinate structure, the matrix $\boldsymbol{P}_f$ equals $\boldsymbol{B}^{-1}$ and the matrix $\boldsymbol{P}$ is not there (in that case it has zero columns).

Calculation of stress quantities
When the redundants are obtained from the equations (1.26), the stress quantities (the normal forces) can be calculated from (1.19):

$$\boldsymbol{N} = \boldsymbol{P}_f\, \boldsymbol{f} + \boldsymbol{P}\, \boldsymbol{\phi}$$

Calculation of displacements
The intermediate result $\boldsymbol{B}^T \boldsymbol{P}_f = \boldsymbol{I}$ of remark 1, can be used to calculate the displacements. From the calculated stress quantities $\boldsymbol{N}$, first the deformations are obtained from (1.20):

$$\boldsymbol{e} = \boldsymbol{C}\, \boldsymbol{N}$$

After that the displacements $\boldsymbol{u}$ can be calculated. For that purpose both right-hand and left-hand sides of the kinematic relations (1.12) are multiplied by $\boldsymbol{P}_f^T$, i.e.:

$$\boldsymbol{e} = \boldsymbol{B}\, \boldsymbol{u} \quad \rightarrow \quad \boldsymbol{P}_f^T \boldsymbol{e} = \boldsymbol{P}_f^T \boldsymbol{B}\, \boldsymbol{u}$$

Because $\boldsymbol{B}^T \boldsymbol{P}_f$ is the unit matrix, its transposed $\boldsymbol{P}_f^T \boldsymbol{B}$ is the unit matrix too. Therefore above equation can be simplified to:

$$\boxed{\boldsymbol{P}_f^T \boldsymbol{e} = \boldsymbol{u}} \qquad (\textit{calculation of the displacements}) \qquad (1.28)$$

Which demonstrates how the displacements can be obtained from the deformations.

Summary of the force method
Overseeing the strategy of the force method, it is evident that the triplet of basic equations is evaluated two times in the order as shown below. First, the system of equations is built up, the redundants of which are solved. After that, successively the stresses, deformations and displacements are determined.

"Force method" ↓
- equilibrium equations
- constitutive equations (flexibility formulation)
- kinematic equations

1.4.2 Displacement method

The displacement method for statically indeterminate structures is exactly the same as the one for statically determinate structures. The basic equations (1.12), (1.13) and (1.14) - being a bit different in this case - are evaluated two times in the order given below.

"Displacement method" $\uparrow$

– equilibrium equations
– constitutive equations (stiffness formulation)
– kinematic equations

During the first cycle again a system of equations is derived:

$$\boxed{\boldsymbol{B}^T \boldsymbol{D}\, \boldsymbol{B}\, \boldsymbol{u} = \boldsymbol{f}} \tag{1.29}$$

In this case the matrix $\boldsymbol{B}$ is not square and has more rows than columns. Naturally, $\boldsymbol{B}^T$ contains more columns than rows. The matrix multiplication results in a square symmetrical stiffness matrix $\boldsymbol{K}$ for the structure with the same number of rows as $\boldsymbol{B}^T$ and the same number of columns as $\boldsymbol{B}$, as shown in the scheme below:

$$\begin{array}{cc} & 7 \\ 6 & \begin{bmatrix} \quad \boldsymbol{B}^T \quad \end{bmatrix}\end{array} \begin{array}{cc} & 7 \\ 7 & \begin{bmatrix} \quad \boldsymbol{D} \quad \end{bmatrix}\end{array} \begin{array}{cc} & 6 \\ 7 & \begin{bmatrix} \quad \boldsymbol{B} \quad \end{bmatrix}\end{array} \begin{Bmatrix} \boldsymbol{u} \end{Bmatrix}_6 = \begin{Bmatrix} \boldsymbol{f} \end{Bmatrix}_6$$

In the second cycle, successively the deformations are calculated from the kinematic relations and the stresses are obtained from the found deformations using the constitutive equations.

1.5 Summary for discrete bar structures

In this chapter for statically determinate and statically indeterminate structures the basic equations have been derived and two strategies have been discussed. For statically determinate structures the words "historical" and "numerical" indicated these strategies and for the statically indeterminate structures the terms "force method" and "displacement method" were used. The "numerical" method for statically determinate structures is completely identical to the displacement method for statically indeterminate structures. The word "numerical" is a bit misleading. It does not mean that the other methods are not suitable to be implemented on a computer. It only indicates that the displacement method is the most appropriate one.
The "historical" method for statically determinate structures fits into the scheme of the displacement method. However, it is a special version of it, because going through two cycles is not necessary. Without compatibility conditions directly all required quantities can be determined.

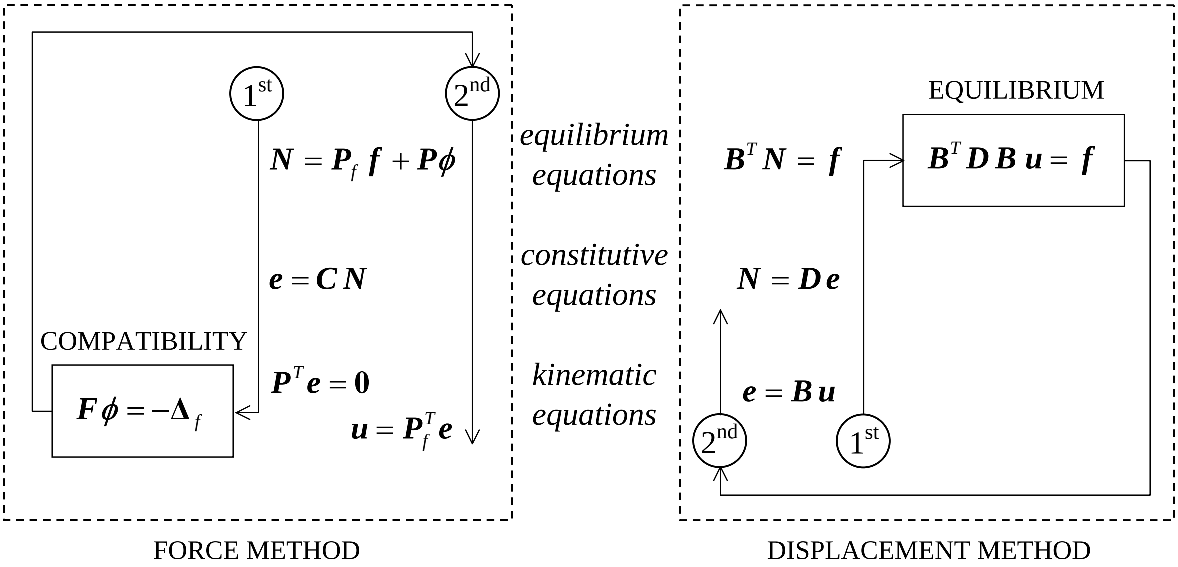


Fig. 1.7: Solution schemes for force and displacement methods.

In view of all the considerations made, it is possible to create one compact overview for all structures. In the central column of this scheme as shown in Fig. 1.7, the three basic equations are listed. In the right column it has been summarised in what form these basic equations are used for the displacement method. The two cycles are numbered by 1st and 2nd. In the left column the same has been done for the force method. However, different formulations of the kinematic equations are used in the two cycles.

Clarification

1. In the scheme of the force method, for statically determinate structures only one cycle is required. In that case the matrix $\boldsymbol{P}_f$ is equal to $\boldsymbol{B}^{-T}$ and $\boldsymbol{P}_f^T$ transforms into $\boldsymbol{B}^{-1}$.
2. In the scheme of the force method two kinematic relations are listed. For statically indeterminate structures the left relation is used in the first cycle and the right relation in the second cycle (which is the first cycle for statically determinate structures).

Two main aspects

The strategy of the displacement method does not require any clarifications. In the strategy of the force method attention is focussed on two main aspects:

1. The first one is the construction of a stress field $\boldsymbol{N}$ that satisfies the equilibrium equations and in which the (to be determined) redundants are incorporated.
2. The second one is the derivation of the compatibility equations. These are expressions in the deformations $\boldsymbol{e}$. They are found by elimination of the displacements from the kinematic equations.

Remark

The statically indeterminate truss of the example was internally statically indeterminate. For bar structures the calculation procedure remains the same for externally statically determined structures.

2 Continuous beam

The strategies considered in chapter 1 will be applied for the solution of continuous problems. In this chapter these still are beam structures. Section 2.1 will focus on statically determinate beam structures and in section 2.2 statically indeterminate structures will be highlighted. In these sections an axially loaded bar is considered, in which only a normal force is generated. This phenomenon also can be called a bar loaded in *extension*. In the sections 2.3 and 2.4 the discussion will be repeated for a beam problem with *bending*. The next section 2.5 will briefly focus on problems with *shear deformation* and *torsion*.

2.1 Statically determinate beam subjected to extension

Fig. 2.1 shows the considered structure. In this structure, the only degree of freedom is the displacement $u(x)$ in the direction of the bar axis. The displacement is defined positive if it takes place in positive x-direction. An external distributed load $f(x)$ corresponds with this degree of freedom. For this load, the same sign convention applies.

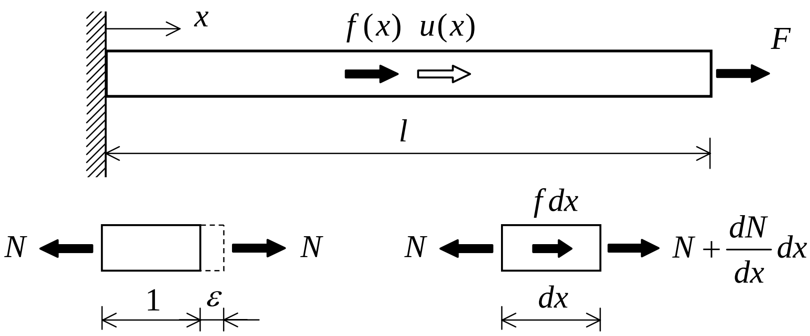


Fig. 2.1: Bar subjected to extension with relevant quantities.

Next to the two external quantities, two internal ones are present as well. They are the (generalised) stress being the normal force $N(x)$ and a specific strain $\varepsilon(x)$, which is caused by that normal force. With the choice of these two internal quantities the deformation work is uniquely determined. In the scheme of Fig. 2.2 it is depicted, which quantities exist and what relations can be established.
The three basic equations now are:

$$
\begin{aligned}
&\varepsilon = \frac{du_x}{dx} && (\textit{kinematic equation}) \\
&N = EA\,\varepsilon \quad \text{or} \quad \varepsilon = \frac{1}{EA}N && (\textit{constitutive equation}) \\
&\frac{dN}{dx} + f = 0 && (\textit{equilibrium equation})
\end{aligned}
\tag{2.1}
$$

With introduction of the two operators $\mathcal{B}$ and $\mathcal{B}'$ given by:

$$
\mathcal{B} = \frac{d}{dx} \quad ; \quad \mathcal{B}' = -\frac{d}{dx} \tag{2.2}
$$

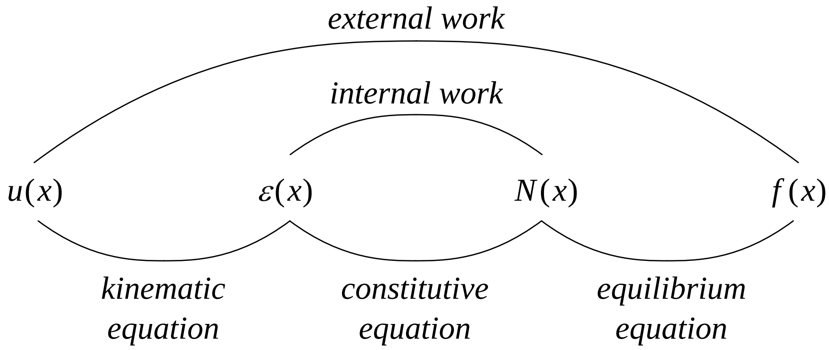


Fig. 2.2: Diagram displaying the relations between the quantities playing a role in the analysis of a bar subjected to extension.

and the stiffness D and flexibility C:

$$D = EA(x) \quad ; \quad C = \frac{1}{EA(x)} \tag{2.3}$$

the basic equations can be reformulated as:

$$\begin{array}{lll} \varepsilon = \mathcal{B}\, u & & (\textit{kinematic equation}) \\ N = D\,\varepsilon \quad \text{or} \quad \varepsilon = C\,N & & (\textit{constitutive equation}) \\ \mathcal{B}'\, N = f & & (\textit{equilibrium equation}) \end{array} \tag{2.4}$$

Comparison with the basic equations (1.1), (1.2) and (1.3) from section 1.2 shows a large analogy. For the solution of a concrete problem, the force method as well as the displacement method can be applied. Both methods will be discussed in this chapter.

Remark
That in this case a separate operator $\mathcal{B}' = -\mathcal{B}$ has been introduced has a special reason. When above relations are discretised by the Finite Element Method or Finite Difference Method these operators are replaced by matrices; then operator $\mathcal{B}$ is replaced by matrix $\boldsymbol{B}$ and operator $\mathcal{B}'$ is replaced by matrix $\boldsymbol{B}^T$. The sign difference between the operators also can be found in their matrix counterparts as shown in section 2.3.4.

2.1.1 Force method

The starting point is the equilibrium equation. This single equation contains one unknown stress quantity N, which confirms that the problem is statically determined. So, by integration the normal force $N(x)$ can directly be determined from the external load $f(x)$. In chapter 1 this boiled down to a matrix-inversion problem. Also integration (in a generalised sense) can be regarded as an inversion of differentiation. Then the constitutive equation in its flexibility formulation can be used to calculate the strains $\varepsilon(x)$. After that, integration of the kinematic equation in combination with the boundary condition directly delivers the displacement field $u(x)$.

Remark 1
The method of analysis is completely analogous to the one for statically determinate trusses.

Remark 2
The selected structure in the example of Fig. 2.1 is both internally and externally statically determinate. From the equilibrium equation it only can be established that the problem is internally statically determinate. For a conclusion about the external determinacy of the problem the boundary conditions have to be inspected.

2.1.2 Displacement method

In the displacement method the kinematic equation and the constitutive equation (in stiffness formulation) are substituted into the equilibrium equation. Doing so in (2.1), the following second order differential equation is obtained:

$$-\frac{d}{dx}\left\{ EA\left(\frac{d}{dx}u \right) \right\} = f \qquad (2.5)$$

In the case of a prismatic bar, the extensional stiffness EA is constant and the differential equation reduces to:

$$-EA\frac{d^2u}{dx^2} = f \qquad (2.6)$$

For the substitution process the operator equations (2.2) can be used too. Then the differential equation appears in the form:

$$\boxed{\mathcal{B}'D\mathcal{B}\,u = f} \qquad (2.7)$$

Now the analogy with equation (1.9) of the comparable truss problem is quite clear.

Elaboration of an example
The structure of Fig. 2.1 is considered with the values of EA and f assumed constant. In the force method successively the three basic equations are evaluated together with the two boundary conditions given by:

$$x = 0 \quad \rightarrow \quad u = 0 \quad ; \quad x = l \quad \rightarrow \quad N = 0$$

Integration of the equilibrium equation delivers:

$$N(x) = N(0) - xf$$

For $x = l$ the normal force has to be zero, so that for $N(0)$ it holds:

$$N(0) = l\,f$$

therefore:

$$N(x) = (l - x)\, f$$

Application of the constitutive equation yields:

$$\varepsilon(x) = \frac{l - x}{EA}\, f$$

Finally, from the kinematic equation it then follows:

$$u(x) = u(0) + \int_0^x \varepsilon \, dx = \frac{\left(l - \tfrac{1}{2}x\right)x}{EA}\, f$$

In the last term, the boundary condition $u(0) = 0$ already has been incorporated. The graphical representations of $N(x)$ and $u(x)$ can be found in Fig. 2.3.

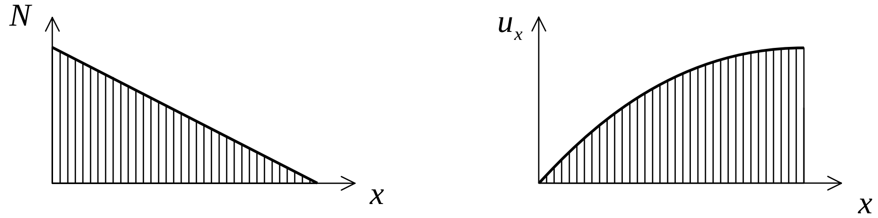


Fig. 2.3: Normal force and displacement as a function of x.

With the displacement method the same result has to be obtained by solving the differential equation:

$$-EA\frac{d^2u}{dx^2} = f$$

together with the following boundary conditions:

$$x = 0 \quad \rightarrow \quad u = 0 \quad ; \quad x = l \quad \rightarrow \quad N = 0$$

The second boundary condition can be rewritten as a condition for the displacement field:

$$N = EA\,\varepsilon = EA\frac{du}{dx} = 0$$

With these two boundary conditions it indeed is found (also see the course “Elastostatics of slender structures”):

$$u = \frac{\left(l - \tfrac{1}{2}x\right)x}{EA}\, f$$

From which for $N(x)$ it follows:

$$N(x) = EA\frac{du}{dx} = (l - x)\, f$$

Remark
The discussion in this section appears to be very trivial. This has been done on purpose to achieve two goals, first to express the analogy with the discrete approach of chapter 1 and second to highlight the correspondence with next chapters in which the analysis appears to be less obvious.

2.2 Statically indeterminate beam subjected to extension

The problem with the bar of previous section is extended with a system of distributed springs. These springs are connected to the bar at its axis and can deform only in the direction of this axis. The forces of the springs act in that direction too. Fig. 2.4 shows the set-up of this new problem. The springs are depicted as leaf springs restrained at the bottom and hinge-connected to the bar at the top.

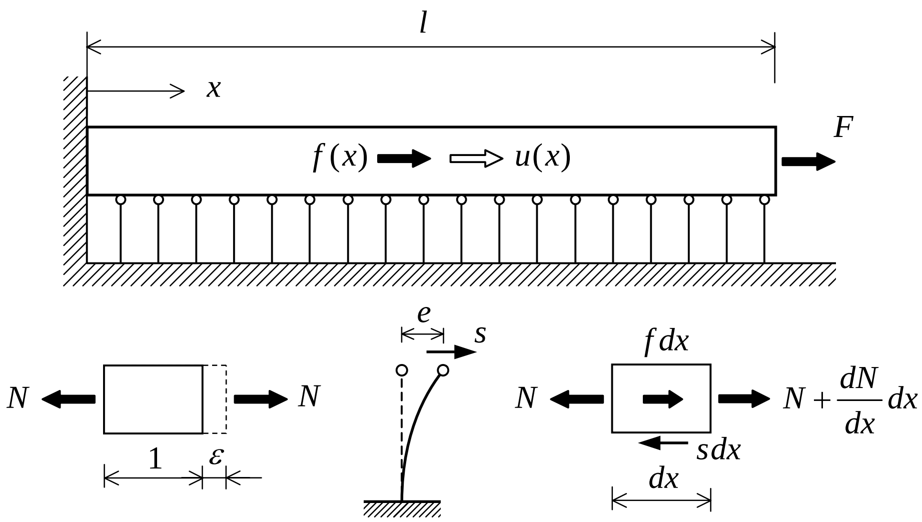


Fig. 2.4: Statically indeterminate bar with relevant quantities.

Also in this case, the displacement field is fixed with one degree of freedom $u(x)$. Therefore, there is exactly one component of external distributed load $f(x)$. With respect to the internal stress quantities and corresponding deformations the situation is different compared to the previous example. Next to the bar element there is a spring element. Deformation energy can be accumulated in both of them, such that for each of the elements separately a generalised stress and a generalised deformation occur. Therefore it makes sense to introduce separate symbols for these quantities. For the bar element these are again the normal force $N(x)$ and the specific strain $\varepsilon(x)$. In the spring element the force per unit length in x-direction is indicated by $s(x)$ and the deformation of the spring by $e(x)$. The scheme of Fig. 2.5 displays all quantities together with the governing relations.

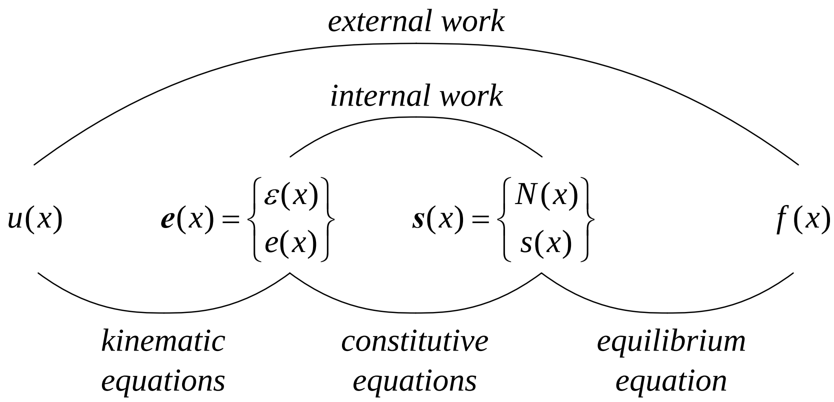


Fig. 2.5: Diagram displaying the relations between the quantities playing a role in the analysis of a spring-supported bar subjected to extension.

Governing equations

The normal force N has the dimension of a force and therefore is indicated by a capital. The load s in the springs is a force per unit of length, so it has a different dimension. For that reason a lower case letter is used. The kinematic equations for this case are:

$$\boxed{\begin{aligned}&\varepsilon=\frac{du}{dx}\\ &e=u\end{aligned}} \qquad (\textit{kinematic equations}) \qquad (2.8)$$

The constitutive equations are:

$$\boxed{\begin{aligned}&N=EA\,\varepsilon \quad \text{or} \quad \varepsilon=\frac{1}{EA}N\\ &s=k\,e \qquad\ \text{or} \quad e=\frac{1}{k}s\end{aligned}} \qquad (\textit{constitutive equations}) \qquad (2.9)$$

where EA is the extensional stiffness and k the spring modulus. For convenience sake, both parameters are taken constant. The equilibrium equation for a small section of the bar now becomes:

$$\boxed{\frac{dN}{dx}-s+f=0} \qquad (\textit{equilibrium equation}) \qquad (2.10)$$

These three sets of basic equations can be reformulated by using operators, which are defined by:

$$\boldsymbol{\mathcal{B}} = \begin{Bmatrix} \dfrac{d}{dx} \\ 1 \end{Bmatrix} \quad ; \quad \boldsymbol{\mathcal{B}}' = \begin{Bmatrix} -\dfrac{d}{dx} & 1 \end{Bmatrix} \tag{2.11}$$

$$\boldsymbol{e} = \begin{Bmatrix} \varepsilon \\ e \end{Bmatrix} \quad ; \quad \boldsymbol{s} = \begin{Bmatrix} N \\ s \end{Bmatrix} \tag{2.12}$$

$$\boldsymbol{D} = \begin{bmatrix} EA & 0 \\ 0 & k \end{bmatrix} \quad ; \quad \boldsymbol{C} = \begin{bmatrix} \dfrac{1}{EA} & 0 \\ 0 & \dfrac{1}{k} \end{bmatrix} \tag{2.13}$$

Again, the three basic equations can be written in a brief manner as discussed in chapter 1:

$$\begin{aligned} &\boldsymbol{e} = \boldsymbol{\mathcal{B}}\, u && (\textit{kinematic equations}) \\ &\boldsymbol{s} = \boldsymbol{D}\, \boldsymbol{e} && (\textit{constitutive equations}) \\ &\boldsymbol{\mathcal{B}}'\boldsymbol{s} = f && (\textit{equilibrium equation}) \end{aligned} \tag{2.14}$$

Notice that the operator $\boldsymbol{\mathcal{B}}'$ is almost the transposed of the operator $\boldsymbol{\mathcal{B}}$. During transposition the derivative changes of sign. Again, it will be discussed how these basic equations are used in the force and displacement methods.

2.2.1 Force method

In this method, the equilibrium equation (2.10) is the first equation to be evaluated. This is one equation with two unknown stress quantities N and s, which confirms that the problem is statically indeterminate. Therefore, one redundant $\phi(x)$ has to be introduced. This means that there is only one compatibility condition too. In this case a horizontal cut is made between the bar and the springs. The distributed load at both faces of the cut then becomes the redundant. In Fig. 2.6a a positive $\phi(x)$ has been drawn.

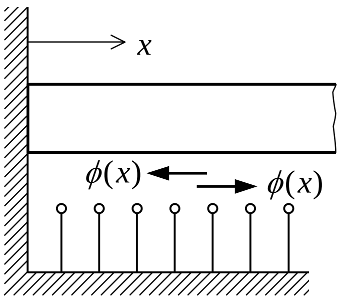


Fig. 2.6a: Selection of the redundant in a spring-supported bar subjected to extension.

It makes sense to introduce a separate symbol for the redundant. In the simple bar problem that is under investigation here, the redundant $\phi(x)$ is equal to the spring load $s(x)$, but this is not necessarily always the case. With this choice for the redundant from the equilibrium equation (2.10), the following relations for the internal stress quantities N and s can be found:

$$\boxed{\begin{aligned} &\frac{dN}{dx} = \phi - f \\ &s = \phi \end{aligned}} \qquad (\textit{equilibrium}) \tag{2.15}$$

After determination of the redundant ϕ, the force N and the load s can be obtained from the relations (2.15).
The second step in the force method is the evaluation of the constitutive equations in flexibility formulation:

$$\boxed{\begin{aligned} &\varepsilon = \frac{1}{EA} N \\ &e = \frac{1}{k}\phi \end{aligned}} \qquad (\textit{constitution}) \tag{2.16}$$

The third step is the derivation of the compatibility condition. This condition is an equation describing the relation between ε and e that has to be satisfied. The compatibility condition can be obtained from the kinematic equations (2.8) by elimination of the displacement u (notice what has been stated in the summary of chapter 1). This can be done by differentiation of the second equation with respect to x. Then both right-hand sides are equal to du/dx, which after subtraction of the two equations disappears and a relation results only containing ε and e. This is the compatibility condition and it is given by:

$$\boxed{\varepsilon - \frac{de}{dx} = 0} \qquad (\textit{compatibility}) \tag{2.17}$$

The fourth step is the derivation of the differential equation for the redundant ϕ. Analogously as done for the truss in chapter 1, first the substitution is required of the equilibrium system (2.15) into the constitutive equations (2.16). For this purpose the first equation of (2.16) is differentiated with respect to x. The two relations then become:

$$\frac{d\varepsilon}{dx} = \frac{1}{EA}\frac{dN}{dx} \quad ; \quad e = \frac{1}{k}\phi$$

Substitution of the equilibrium system (2.15) then provides:

$$\frac{d\varepsilon}{dx} = \frac{1}{EA}(\phi - f) \quad ; \quad e = \frac{1}{k}\phi \tag{2.18}$$

Next this result has to be substituted into the compatibility condition (2.17), which first is differentiated once:

$$\frac{d\varepsilon}{dx} - \frac{d^2e}{dx^2} = 0$$

Now (2.18) easily can be substituted resulting in:

$$\boxed{-\frac{1}{k}\frac{d^2\phi}{dx^2} + \frac{1}{EA}\phi = \frac{1}{EA}\, f} \tag{2.19}$$

This is a second-order differential equation with respect to the redundant ϕ. For the solution of this equation two boundary conditions have to be formulated.

Similarity with bar structures

The provided derivation of the differential equation for ϕ did not show clearly the analogy with the force method for bar structures. The recognizability is increased when the derivation is carried out a bit differently. Again, in the first step it is started with the equilibrium equation (2.10), but now an integration is carried out. The following equilibrium system is found:

$$N = \int_0^x \phi\, dx - \int_0^x f dx \quad ; \quad s = \phi \tag{2.20}$$

The integration constant is not considered, because it is not important in this case. Together with the constitutive equation in the flexibility formulation (2.9), an expression for the deformations is found:

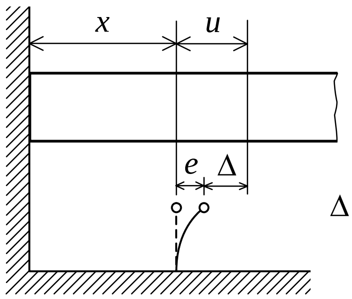
$$\varepsilon = \frac{1}{EA}\int_0^x \phi\, dx - \frac{1}{EA}\int_0^x f dx \quad ; \quad e = \frac{1}{k}\phi \tag{2.21}$$

Now, the compatibility condition is determined by elimination of the displacement from the kinematic equations (2.8). Integration of the first one followed by subtraction of the second one gives:

$$\int_0^x \varepsilon\, dx - e = 0 \tag{2.22}$$

The physical interpretation of this result is shown in Fig. 2.6b. Substitution of (2.21) into (2.22) provides the compatibility condition from which ϕ can be calculated:

$$-\frac{1}{k}\phi + \frac{1}{EA}\int_0^x\int_0^x \phi\, dx\, dx = \frac{1}{EA}\int_0^x\int_0^x f\; dx\, dx \tag{2.23}$$



$$\Delta = u - e = \int_0^x \varepsilon \, dx - e$$

Fig. 2.6b: Visual representation of the gap to be neutralised in a spring-supported bar subjected to extension.

By differentiation twice, a more suitable equation is obtained:

$$\boxed{-\frac{1}{k}\frac{d^2\phi}{dx^2} + \frac{1}{EA}\phi = \frac{1}{EA} f} \tag{2.24}$$

Now it is clear that the inclusion of integration constants in (2.20) and (2.21) makes no sense, because these constants would have disappeared anyway by the two differentiations. The differential equation (2.24) is identical to the previously found equation (2.19).

Notation with operators

Now, it will be demonstrated how above result can also be obtained with the use of operators. The similarity with chapter 1 will become clear. The internal stress quantities in (2.20) are:

$$\begin{Bmatrix} N \\ s \end{Bmatrix} = \begin{Bmatrix} -\int_0^x dx \\ 0 \end{Bmatrix} f + \begin{Bmatrix} \int_0^x dx \\ 1 \end{Bmatrix} \phi$$

Or briefly, analogously to (1.19):

$$\boldsymbol{s} = \boldsymbol{\mathcal{P}}_f \, f + \boldsymbol{\mathcal{P}} \phi$$

where:

$$\boldsymbol{\mathcal{P}}_f = \begin{Bmatrix} -\int_0^x \mathrm{d}x \\ 0 \end{Bmatrix} \quad ; \quad \boldsymbol{\mathcal{P}} = \begin{Bmatrix} \int_0^x \mathrm{d}x \\ 1 \end{Bmatrix}$$

For the strains it then follows:

$$\boldsymbol{e} = \boldsymbol{C}\left(\boldsymbol{\mathcal{P}}_f \, f + \boldsymbol{\mathcal{P}} \phi\right)$$

where $\boldsymbol{C}$ is given in (2.13). The compatibility condition (2.22) can be written as:

$$\left\{ -\int_0^x dx \quad 1 \right\} \left\{ \begin{matrix} \varepsilon \\ \\ e \end{matrix} \right\} = 0$$

Introduction of the suitable operator $\mathcal{P}'$ given by:

$$\mathcal{P}' = \left\{ -\int_0^x dx \qquad 1 \right\}$$

provides the following brief notation:

$$\mathcal{P}' \boldsymbol{e} = 0$$

In analogy with (1.23) from chapter 1, the operator $\mathcal{P}'$ is the transposed of $\mathcal{P}$ with as extra addition the sign difference in the integration term. Substitution of the matrix equation for the strains $\boldsymbol{e}$ changes this equation into:

$$\boxed{\mathcal{P}' \boldsymbol{C} \mathcal{P} \phi = -\Delta_f} \tag{2.25}$$

where Δ_f is the incompatibility resulting from the external load f, it is given by:

$$\Delta_f = \mathcal{P}' \boldsymbol{C} \mathcal{P}_f \; f \tag{2.26}$$

Written in this form the previously found differential equations (2.19) and (2.24) can be compared to the results obtained in chapter 1.

2.2.2 Displacement method

In the displacement method the constitutive equations (2.9) in stiffness formulation and the kinematic equations (2.8) are directly substituted into the equilibrium equation (2.10). This leads to:

$$\boxed{-EA\frac{d^2u}{dx^2} + k\,u = f} \tag{2.27}$$

Introduction of the operator definitions of (2.11) and the matrix $\boldsymbol{D}$ given in (2.13) provides the following matrix notation of the differential equation:

$$\boxed{\mathcal{B}' \boldsymbol{D} \mathcal{B}\, u = f} \tag{2.28}$$

In this form, the analogy with chapter 1 becomes clear again.

2.2.3 Elaboration of an example

A solution is provided for the problem of Fig. 2.7 (also see Fig. 2.4), where $f(x)$ has a constant value f and where EA and k are constants too.

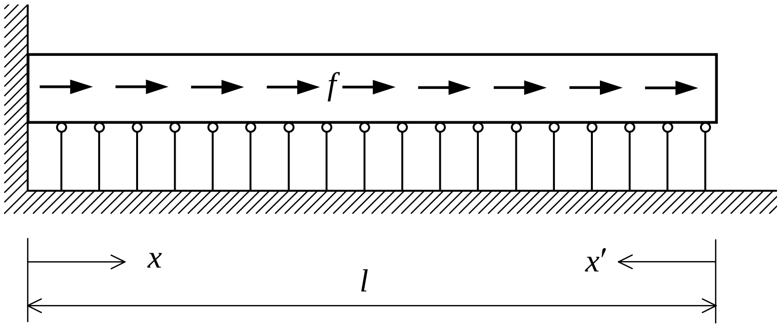


Fig. 2.7: Spring-supported bar, which is uniformly loaded in axial direction.

In the force method the differential equation is:

$$-\frac{1}{k}\frac{d^2\phi}{dx^2}+\frac{1}{EA}\phi=\frac{1}{EA}f$$

and in the displacement method the equation reads:

$$-EA\frac{d^2u}{dx^2}+k\,u=f$$

In the differential equation of the displacement method, the stiffnesses EA and k appear as constant coefficients. In the equation of the force method these are the compliances $1/k$ and $1/EA$. At the position where the stiffness k appears in one case, the compliance $1/EA$ appears in the other case and vice versa.
Both differential equations are of the second order. For both the displacement method and the force method, two boundary equations are required. They are:

$$x=0 \quad\rightarrow\quad u=0 \qquad\qquad (kinematic)$$
$$x=l \quad\rightarrow\quad N=0 \qquad\qquad (dynamic)$$

Force method
In the force method the boundary conditions are transformed into conditions expressed in ϕ. Use is made of the kinematic relation $e=u$. The kinematic boundary condition $u=0$ becomes $e=0$ and therefore $\phi=0$. The dynamic boundary condition $N=0$ becomes $EA\,du/dx=0$. Since e is equal to u, it also holds $de/dx=0$. In view of the direct relation between s and e, the condition becomes $ds/dx=0$. But s is equal to ϕ, so that for the dynamic boundary condition it has to hold $d\phi/dx=0$. Therefore, the following problem has to be solved:

$$-\frac{1}{k}\frac{d^2\phi}{dx^2}+\frac{1}{EA}\phi=\frac{1}{EA}f \qquad (\textit{differential equation})$$

$$\begin{aligned} &x=0 \quad \rightarrow \quad \phi=0 \\ &x=l \quad \rightarrow \quad \frac{d\phi}{dx}=0 \end{aligned} \qquad (\textit{boundary conditions})$$

A *particular* solution is:

$$\phi=f$$

A solution for the *homogeneous* equation (right-hand side zero) equals

$$\phi=A\,e^{rx}$$

The equation for the determination of the roots r reads:

$$\left(-\frac{r^2}{k}+\frac{1}{EA}\right)e^{rx}=0$$

By introduction of the characteristic length λ:

$$\lambda=\sqrt{\frac{EA}{k}}$$

the characteristic equation (after division by e^{rx}/EA) can be written as:

$$-r^2\lambda^2+1=0$$

From which it follows:

$$r^2=\frac{1}{\lambda^2}$$

The two roots therefore are:

$$r_1=\frac{1}{\lambda} \quad ; \quad r_2=-\frac{1}{\lambda}$$

The total solution for ϕ including the particular solution becomes:

$$\phi(x)=A_1e^{x/\lambda}+A_2e^{-x/\lambda}+f$$

This solution is often written is a somewhat different form. The first term of the right-hand side is then expressed in the coordinate x' opposite to the coordinate x and starting at the free end (see Fig. 2.7). Between x and x' the following relation holds $x=(l-x')$.
The first term then becomes $A_1e^{(1-x')/\lambda}$ or $A_1e^{l/\lambda}e^{-x'/\lambda}$. After introduction of the new constant $A_1=A_1e^{l/\lambda}$ the solution also can be written as:

$$\phi(x) = A_1 e^{-x'/\lambda} + A_2 e^{-x/\lambda} + f$$

The homogeneous part consists of a contribution that damps out from the end $x' = 0$ and of a contribution that damps out from the end $x = 0$. The constants A_1 and A_2 can be obtained from the boundary conditions. The elaboration is simplified by the assumption that the bar has such a length that both damping terms do not reach the other end. This is the case if the length l of the bar is three to four times its characteristic length λ. It then is found:

$$x = 0 \quad \rightarrow \quad \left.\begin{array}{l} e^{-x'/\lambda} = 0 \\ e^{-x/\lambda} = e^0 = 1 \end{array}\right\} \quad \rightarrow \quad \phi = A_2 + f = 0 \quad \rightarrow \quad A_2 = -f$$

$$x = l \quad \rightarrow \quad \left.\begin{array}{l} e^{-x'/\lambda} = e^0 = 1 \\ e^{-x/\lambda} = 0 \end{array}\right\} \quad \rightarrow \quad \frac{d\phi}{dx} = A_1 \frac{1}{\lambda} e^{-x'/\lambda} - A_2 \frac{1}{\lambda} e^{-x/\lambda} = A_1 \frac{1}{\lambda} = 0 \quad \rightarrow \quad A_1 = 0$$

Therefore the solution is:

$$\phi(x) = \left(1 - e^{-x/\lambda}\right) f$$

From the equilibrium equation (2.15) it directly follows:

$$N(x) = \lambda\, f\, e^{-x/\lambda}$$
$$s(x) = f\left(1 - e^{-x/\lambda}\right)$$

and the displacement becomes:

$$u(x) = \frac{f}{k}\left(1 - e^{-x/\lambda}\right)$$

Fig. 2.8 displays the normal force and displacement distributions.

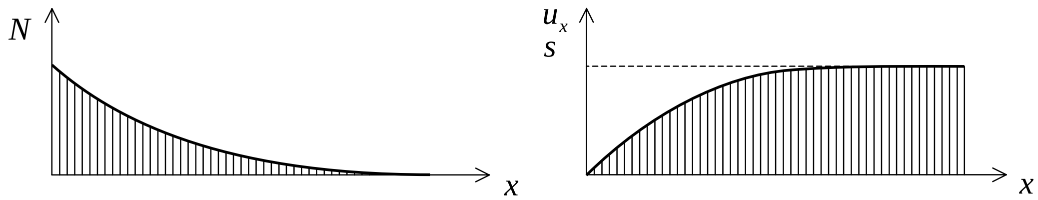


Fig. 2.8: Normal force and displacement as a function of x.

Displacement method

In the displacement method the boundary conditions have to be expressed in the displacement u. For $x = 0$ this is already the case, because at that position the kinematic boundary condition $u = 0$ applies. At the end $x = l$ the dynamic boundary condition $N = 0$ holds, which can be reformulated as $du/dx = 0$.

Therefore, the following problem has to be solved:

$$-EA\frac{d^2u}{dx^2} + k\,u = f \qquad (\textit{differential equation})$$

$$x = 0 \quad \rightarrow \quad u = 0$$

$$x = l \quad \rightarrow \quad \frac{du}{dx} = 0 \qquad (\textit{boundary conditions})$$

Now, a particular solution is:

$$u = \frac{f}{k}$$

As homogeneous solution it is selected:

$$u = A\,e^{rx}$$

This delivers the characteristic equation:

$$\left(-EA\,r^2 + k\right) e^{rx} = 0$$

After division by ke^{rx} and introduction of the characteristic length λ as previously defined, the same characteristic equation as in the force method is obtained:

$$-r^2\lambda^2 + 1 = 0$$

From this equation the same roots can be solved. In a similar way as found for ϕ in the force method, the solution for u becomes:

$$u(x) = A_1\,e^{-x'/\lambda} + A_2\,e^{-x/\lambda} + \frac{f}{k}$$

Again the restriction is introduced that $l \gg \lambda$. For the values of the constants it then can be derived:

$$x = 0 \quad \rightarrow \quad u = A_2 + \frac{f}{k} \quad \rightarrow \quad A_2 = -\frac{f}{k}$$

$$x = l \quad \rightarrow \quad \frac{du}{dx} = A_1\frac{1}{\lambda}e^{-x'/\lambda} - A_2\frac{1}{\lambda}e^{-x/\lambda} = A_1\frac{1}{\lambda} = 0 \quad \rightarrow \quad A_1 = 0$$

The final solution for u then becomes:

$$u(x) = \frac{f}{k}\left(1 - e^{-x/\lambda}\right)$$

Subsequently, from this relation the quantities N and s can be determined. Naturally, this solution is in agreement with the solution found with the force method.

2.3 Statically determinate beam subjected to bending

In this section a straight beam is considered in which bending moments are generated caused by a distributed load f perpendicular to the beam-axis. Fig. 2.9 shows the structure and the symbols to be used during the derivations.

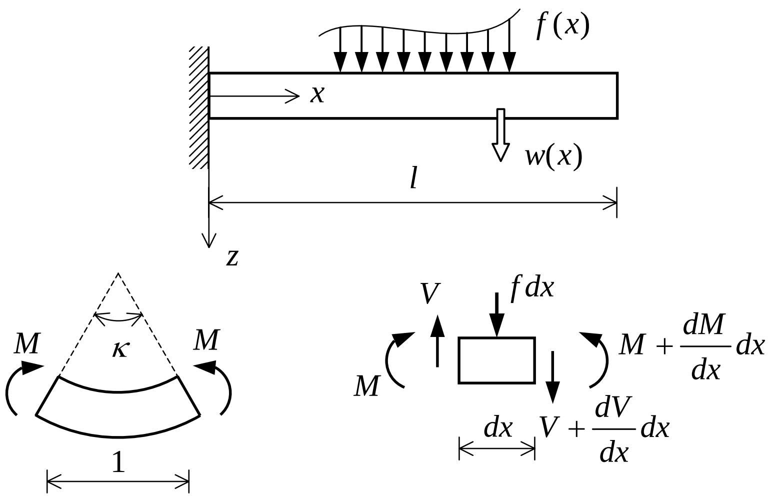


Fig. 2.9: Statically determinate beam subjected to bending with relevant quantities.

For the analysis of the problem it is assumed that the reader knows the classical beam theory. In this case it is particularly important that the deformations generated by the shear force V can be neglected. In that case the displacement field can be described by only one degree of freedom, which will be indicated by $w(x)$. Therefore, also only one external load is possible, the distributed load $f(x)$ per unit of beam length in the direction of w. Next to these two external quantities, also internal quantities play a role. They are the moment M and the curvature κ, which is caused by the moment. In Fig. 2.9 positive M and κ are depicted.

Remark 1
It is true that the shear force V is appearing as well, but not as a generalised stress that plays a role in the followed solution strategy. Generalised stresses are quantities that are coupled with deformation energy. For the shear force this is not the case because the corresponding deformation is not considered.

Remark 2
The deformation caused by the moment is indicated by κ. This parameter definition is used frequently in the engineering practice in concrete and steel. Think about the use of M-κ diagrams. This means that κ is defined positive if it is caused by a positive moment M. The quantities, which are essential for this discussion and the relations between them are indicated in the scheme of Fig. 2.10.
From the three basic equations, two can be written down immediately:

$$\boxed{\kappa = -\frac{d^2 w}{dx^2}} \qquad (\textit{kinematic equation}) \qquad (2.29)$$

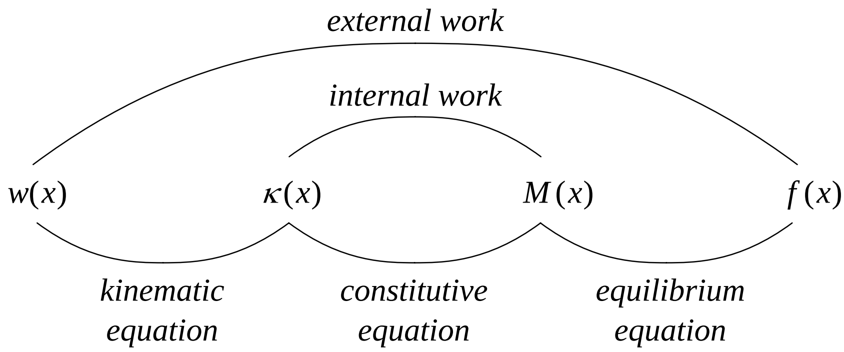


Fig. 2.10: Diagram displaying the relations between the quantities playing a role in the analysis of a beam subjected to bending.

$$\boxed{M = EI\,\kappa \quad \text{or} \quad \kappa = \frac{1}{EI}M} \qquad (\textit{constitutive equation}) \tag{2.30}$$

The equilibrium equation requires extra attention. In this formal approach, only one degree of freedom is present and consequently only one external load f acting in only one direction has to be present as well. Therefore, just one equilibrium equation can be used (acting in z-direction). However, for a small section of the beam, as drawn in the bottom-right corner of Fig. 2.9, two equilibrium equations can be formulated. Namely, one in the z-direction but also the equilibrium of moments about the y-axis:

$$\frac{dV}{dx} + f = 0 \qquad (\textit{load in z-direction})$$

$$\frac{dM}{dx} - V = 0 \qquad (\textit{moment about y-axis})$$

In the first equilibrium equation in z-direction the shear force appears, while a relation is required between the moment and the external load. For the replacement of V by an expression in M the second equilibrium equation can be applied as a help relation:

$$V = \frac{dM}{dx}$$

Substitution of this result into the equilibrium equation for the z-direction provides the required third basic equation:

$$\boxed{\frac{d^2M}{dx^2} + f = 0} \qquad (\textit{equilibrium equation}) \tag{2.31}$$

Again, the found three basic equations can be written down briefly with the use of operators. For this purpose, the following is introduced:

$$\mathcal{B} = -\frac{d^2}{dx^2} \quad ; \quad \mathcal{B}' = -\frac{d^2}{dx^2}$$
$$D = EI(x) \quad ; \quad C = \frac{1}{EI(x)} \tag{2.32}$$

The three basic equations can now briefly written down as:

$$\boxed{\begin{aligned} &\kappa = \mathcal{B}\, w \\ &M = D\kappa \quad \text{or} \quad \kappa = CM \\ &\mathcal{B}'M = f \end{aligned}} \qquad \begin{aligned} &(\textit{kinematic equation}) \\ &(\textit{constitutive equations}) \\ &(\textit{equilibrium equation}) \end{aligned} \tag{2.33}$$

Remark 1
In sections 2.1 and 2.2 it was made clear that the derivatives had to change sign when $\mathcal{B}'$ was obtained from $\mathcal{B}$. Here this is not the case. Generally it holds that a change of sign is required only for differentiations and integrations of uneven order. So, if the order is even the change of sign is not necessary. In subsection 2.3.4 more attention will be paid to this phenomenon.

Remark 2
For the selected sign convention of the moment M and the curvature κ, the curvature is equal to the second derivative of the deflection with a minus sign. Notice that the curvature κ is actually the change of curvature with respect to the unloaded state. For a straight beam, this initial curvature is zero, but for shells this is generally not the case. For shells the word curvature is introduced for the definition of the geometry too. Then the curvature is the second derivative of the function $z(x,y)$, which describes the shape of the shell surface. This is a geometrical quantity, while in this analysis κ is a deformation quantity with in addition a different sign convention.

2.3.1 Force method

Again, first the equilibrium equation is evaluated. This equation contains only one unknown, allowing direct solution by integration and substitution of the boundary conditions. Then from the constitutive equation the curvature can be determined. By integration of the kinematic relation together with the boundary conditions the displacement distribution can be found.

2.3.2 Displacement method

By successive substitution, a new equilibrium equation can be found from the three basic equations:

$$-\frac{d^2}{dx^2}\left\{EI\left(-\frac{d^2}{dx^2}w\right)\right\} = f \tag{2.34}$$

For constant EI, this equation transforms into the well-known differential equation of the fourth-order:

$$EI\frac{d^4w}{dx^4} = f \tag{2.35}$$

After introduction of the defined operators the differential equation becomes:

$$\mathcal{B}'D\mathcal{B}\,w = f \tag{2.36}$$

2.3.3 Elaboration of an example

Force method

As an example, the structure as shown in Fig. 2.9 is analysed with constant EI and f. In the force method the three basic equations are evaluated, starting with the equilibrium equation. The applied boundary conditions are:

$$x = 0 \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dx} = 0 \end{cases}$$

$$x = l \quad \rightarrow \quad \begin{cases} M = 0 \\ V = 0 \quad \rightarrow \quad \dfrac{dM}{dx} = 0 \end{cases}$$

After some elaborations the following solutions for the moment and displacement distribution can be obtained (see Fig. 2.11):

$$M = -\left(1 - \frac{2x}{l} + \frac{x^2}{l^2}\right)\frac{1}{2} f l^2 \quad ; \quad w = \left(2\frac{x^2}{l^2} - \frac{4}{3}\frac{x^3}{l^3} + \frac{1}{3}\frac{x^4}{l^4}\right)\frac{f l^4}{8EI}$$

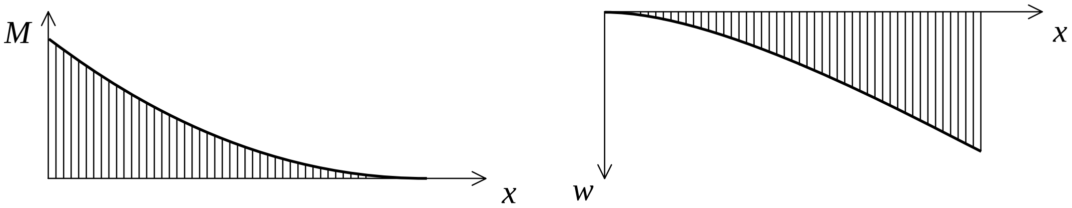


Fig. 2.11: Bending moment and displacement as a function of x.

Displacement method

With the displacement method the same solution has to be found from:

$$EI\frac{d^4w}{dx^4} = f$$

satisfying the boundary conditions given by:

$$x = 0 \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dx} = 0 \end{cases}$$

$$x = l \quad \rightarrow \quad \begin{cases} M = 0 & \rightarrow \quad \dfrac{d^2w}{dx^2} = 0 \\ V = 0 & \rightarrow \quad \dfrac{d^3w}{dx^3} = 0 \end{cases}$$

Notice that all boundary conditions are formulated in w. A kinematic condition is automatically a function of w. However, a dynamic boundary condition has to be reformulated. With this method first a solution for the deflection $w(x)$ is found. Then, from this displacement field the moment distribution $M(x)$ can be derived. Naturally, the found solution is identical to the one obtained from the force method.

2.3.4 The sign difference between $\mathcal{B}$ and $\mathcal{B}'$

In the previous sections, it was established that the operator $\mathcal{B}'$ can be obtained from operator $\mathcal{B}$ by transposition, while at the same time the sign of first derivatives are changed (see (2.11)). When the second derivative is involved the change of sign is not required (see (2.32)). Generally it holds that the sign only changes for derivatives of uneven order. So, when the order is even nothing happens. This also holds for zero-order derivatives, which are constants. This property can be clarified by making use of differential calculus*. In the discretisation process matrices replace these operators; then operator $\mathcal{B}$ is replaced by matrix $\boldsymbol{B}$ and operator $\mathcal{B}'$ is replaced by matrix $\boldsymbol{B}^T$. As will be shown, the eventual sign difference between the operators, can be found in their matrix counterparts too.

Beam subjected to extension
Again, the simple problem of Fig. 2.1 is considered in which a normal force loads a bar. In the displacement method it holds:

$$\varepsilon = \frac{du}{dx} \quad ; \quad \varepsilon = \mathcal{B}u \quad ; \quad \mathcal{B} = \frac{d}{dx}$$
$$-\frac{dN}{dx} = f \quad ; \quad \mathcal{B}'N = f \quad ; \quad \mathcal{B}' = -\frac{d}{dx}$$

The change of signs can be understood better, if the bar is discretised to a series connection of n normal-force elements. Along the bar n elements of length Δx are situated, such that $n+1$ nodes are present. For the extension e_i in element i between the nodes i and $i+1$, the following kinematic relation holds:

$$e_i = -u_i + u_{i+1}$$

Likewise for the equilibrium of node i with external load F_i it follows:

* This explanation is provided by Prof.ir. H.W. Loof, professor Numerical Mechanics at the former faculty of Civil Engineering.

$$N_{i-1} - N_i = F_i$$

The stiffness D_i, which relates e_i to N_i is given by:

$$D_i = \frac{EA}{\Delta x}$$

For all elements the kinematic relations can be written as:

$$\begin{Bmatrix} \cdot \\ \cdot \\ e_{i-1} \\ e_i \\ e_{i+1} \\ \cdot \\ \cdot \end{Bmatrix} = \begin{bmatrix} \cdot & \cdot & & & & & \text{zeros} \\ & \cdot & \cdot & & & & \\ & & -1 & 1 & & & \\ & & & -1 & 1 & & \\ & & & & -1 & 1 & \\ & \text{zeros} & & & & \cdot & \cdot \\ & & & & & & \cdot \quad \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ u_{i-1} \\ u_i \\ u_{i+1} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

or briefly:

$$\boldsymbol{e} = \boldsymbol{B}\,\boldsymbol{u}$$

The first and last elements of the vectors and matrices, which may be affected by boundary conditions, are indicated by dots. They are not essential for the demonstration of the sign difference.
Likewise all equilibrium equation form a system:

$$\begin{bmatrix} \cdot & & & & & \text{zeros} \\ \cdot & \cdot & & & & \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & 1 & -1 & \\ & & & & \cdot & \cdot \\ & \text{zeros} & & & & \cdot \quad \cdot \\ & & & & & \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ N_{i-1} \\ N_i \\ N_{i+1} \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \cdot \\ F_{i-1} \\ F_i \\ F_{i+1} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

or briefly:

$$\boldsymbol{B}^T \boldsymbol{N} = \boldsymbol{f}$$

For the total system it now holds:

$$\boldsymbol{B}^T \boldsymbol{D}\,\boldsymbol{B}\,\boldsymbol{u} = \boldsymbol{f}$$

where $\boldsymbol{D}$ is a diagonal matrix containing all element stifnesses $D_i = EA/\Delta x$. The stiffness matrix is symmetrical, because $\boldsymbol{B}^T$ is the transposed of $\boldsymbol{B}$. But at the same time it holds that $\boldsymbol{B}^T = -\boldsymbol{B}$. This sign difference can also be found between the operators $\mathcal{B}'$ and $\mathcal{B}$ in the differential equations.

Beam subjected to bending
Similarly the bending problem can be analysed. The relations are given by:

$$\kappa = -\frac{d^2 w}{dx^2} \quad ; \quad \kappa = \mathcal{B}\, w \quad ; \quad \mathcal{B} = -\frac{d^2}{dx^2}$$

$$-\frac{d^2 M}{dx^2} = f \quad ; \quad \mathcal{B}' M = f \quad ; \quad \mathcal{B}' = -\frac{d^2}{dx^2}$$

Again an equidistant node distribution along the beam is considered of $n+1$ nodes at mutual distance of Δx. As shown in Fig. 2.12 in each node i a bending moment M_i is present and a deflection w_i is generated perpendicular to the beam axis. In the course Elastic Plates it was shown that with the aid of differential calculus, the flexible bar can be replaced by a series connection of undeformable slices of length Δx, with an imaginary rotation spring between those slices. The spring constant D_i of this rotation spring, relates the moment M_i to the angle θ_i between the two adjacent rigid slices. Between θ and w the following kinematic relation holds:

$$\theta_i = \frac{1}{\Delta x}\left(-w_{i-1} + 2w_i - w_{i+1}\right)$$

This relation easily can be understood by looking at Fig. 2.12. For the angles α, β and θ_i it respectively holds:

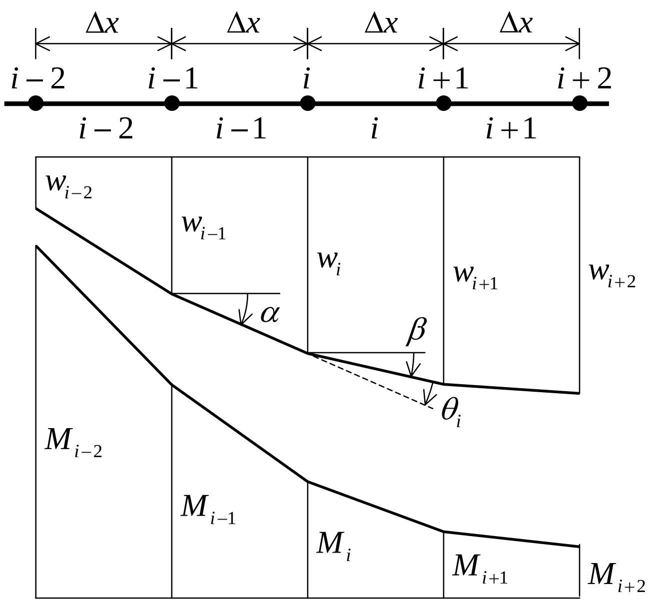


Fig. 2.12: Displacement and moment distributions.

$$\left.\begin{aligned}\alpha &= \frac{w_i - w_{i-1}}{\Delta x}\\ \beta &= \frac{w_{i+1} - w_i}{\Delta x}\end{aligned}\right\} \quad \rightarrow \quad \theta_i = \alpha - \beta$$

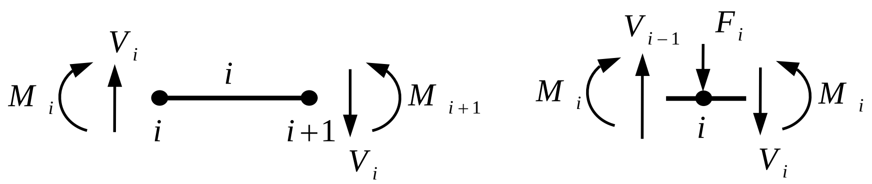
which shows that the above stated kinematic relation is correct.
For the total system of kinematic relations it then is found:

$$\begin{Bmatrix} \cdot \\ \cdot \\ \theta_{i-1} \\ \theta_i \\ \theta_{i+1} \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & & \cdot & \cdot \\ & & & & & & \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ w_{i-1} \\ w_i \\ w_{i+1} \\ \cdot \\ \cdot \end{Bmatrix}$$

or briefly:

$$\boldsymbol{\theta} = \boldsymbol{B}\,\boldsymbol{w}$$

The external distributed load $f(x)$ is discretised in such a manner that in each node i a force F_i can be applied in the direction of w_i. This means that across an element the shear force is constant. The moment equilibrium of element i then delivers (see Fig. 2.13a):



a) equilibrium of element *b) equilibrium of node*

Fig. 2.13: Equilibrium of elements and nodes.

$$V_i = \frac{M_{i+1} - M_i}{\Delta x}$$

Analogously, for element $i-1$ it follows:

$$V_{i-1} = \frac{M_i - M_{i-1}}{\Delta x}$$

For the force equilibrium of node i it holds (see Fig. 2.13b):

$$F_i = V_{i-1} - V_i$$

Substitution of the relations for the shear forces finally delivers:

$$\frac{1}{\Delta x}\left(-M_{i-1}+2M_i-M_{i+1}\right)=F_i$$

The entire system of equilibrium equations becomes:

$$\frac{1}{\Delta x}\begin{bmatrix} . & . & & & & & \\ . & . & . & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & . & . & . \\ & & & & & . & . \end{bmatrix}\begin{Bmatrix} . \\ . \\ M_{i-1} \\ M_i \\ M_{i+1} \\ . \\ . \end{Bmatrix}=\begin{Bmatrix} . \\ . \\ F_{i-1} \\ F_i \\ F_{i+1} \\ . \\ . \end{Bmatrix}$$

or briefly:

$$\boldsymbol{B}^T\boldsymbol{M}=\boldsymbol{f}$$

Substitution of the relation between $\boldsymbol{M}$ and $\boldsymbol{\theta}$ given by:

$$\boldsymbol{M}=\boldsymbol{D}\,\boldsymbol{\theta}$$

in combination with $\boldsymbol{\theta}=\boldsymbol{B}\,\boldsymbol{w}$ provides the required system of equations for the entire structure:

$$\boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\,\boldsymbol{w}=\boldsymbol{f}$$

The introduced $\boldsymbol{D}$ is a diagonal matrix containing all rotation stiffnesses $D_i = EI$.
Now $\boldsymbol{B}^T$ is not only the transposed of $\boldsymbol{B}$, but it also is completely identical to $\boldsymbol{B}$. This explains that in the differential equation the operator $\mathcal{B}'$ is completely identical to operator $\mathcal{B}$ and no change in sign is involved.

General conclusion
It has been shown that for uneven derivatives, differential calculus leads to $\boldsymbol{B}^T=-\boldsymbol{B}$ and for the even derivatives $\boldsymbol{B}^T=+\boldsymbol{B}$. This principle is generally valid.
It also can be stated differently. For uneven derivatives (of first, third, etc. order) the matrix $\boldsymbol{B}$ becomes asymmetrical with respect to the main diagonal and for even derivatives (of second fourth, etc. order) it becomes symmetrical. For that reason $\boldsymbol{B}^T$ becomes $-\boldsymbol{B}$ in the first case and $+\boldsymbol{B}$ in the other case.

2.4 Statically indeterminate beam subjected to bending

The beam problem of Fig. 2.9 is extended by an elastic foundation of uniformly distributed springs as shown in Fig. 2.14. In view of the extensive comments provided in the previous sections the derivations will be given without much explanation. Notice that in both deforming elements (beam and spring) the stress and strain quantities are introduced independently, separately from the displacement field. The distributed load $s(x)$ in the

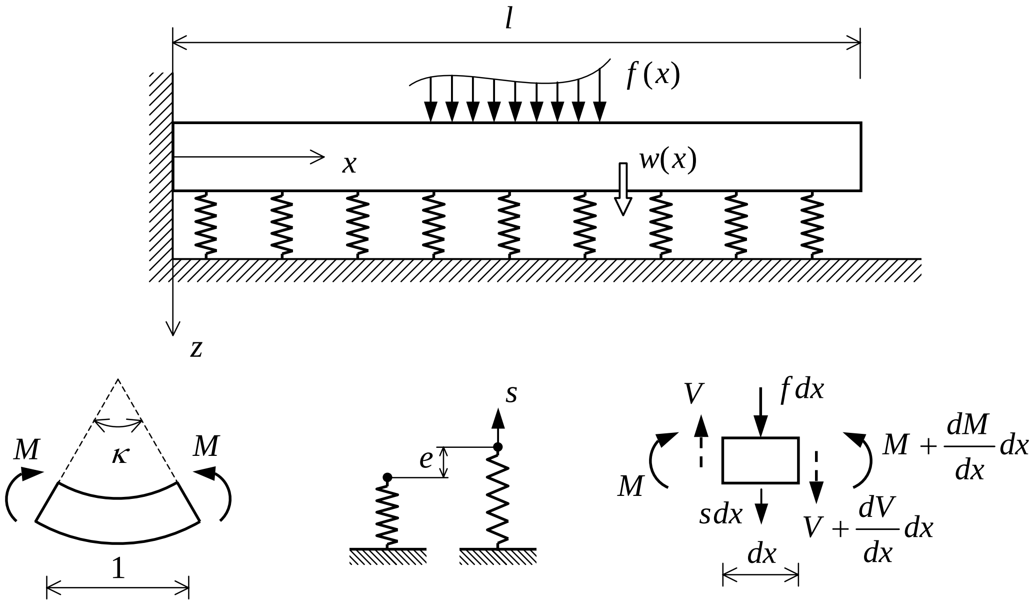


Fig. 2.14: Spring-supported statically indeterminate beam subjected to bending including relevant quantities.

springs is positive for tension. Likewise, $e(x)$ is positive for extension. Again distinction has to be made between $s(x)$ and $V(x)$. The scheme of Fig. 2.15 displays the several quantities and their mutual relations.

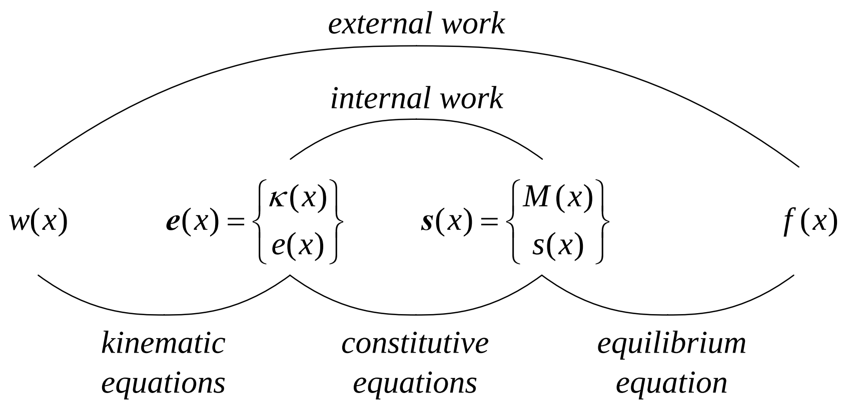


Fig. 2.15: Diagram displaying the relations between the quantities playing a role in the analysis of a spring-supported beam subjected to bending.

The kinematic relations are:

$$\boxed{\begin{aligned}&\kappa = -\frac{d^2 w}{dx^2}\\ &e = -w\end{aligned}} \qquad (\textit{kinematic equations}) \tag{2.37}$$

The constitutive equations read:

$$\boxed{\begin{array}{lll} M = EI\kappa & \text{or} & \kappa = \dfrac{1}{EI}M \\ s = ke & \text{or} & e = \dfrac{1}{k}s \end{array}} \qquad (\textit{constitutive equations}) \tag{2.38}$$

The equilibrium equation becomes:

$$\boxed{\frac{d^2M}{dx^2} + s + f = 0} \qquad (\textit{equilibrium equation}) \tag{2.39}$$

These equations can be written briefly by introduction of the following operators and matrices:

$$\begin{array}{ll} \boldsymbol{\mathcal{B}} = \begin{Bmatrix} -\dfrac{d^2}{dx^2} \\ -1 \end{Bmatrix} \quad ; & \boldsymbol{D} = \begin{bmatrix} EI & 0 \\ 0 & k \end{bmatrix} \\ \boldsymbol{\mathcal{B}}' = \begin{Bmatrix} -\dfrac{d^2}{dx^2} & -1 \end{Bmatrix} \quad ; & \boldsymbol{C} = \begin{bmatrix} \dfrac{1}{EI} & 0 \\ 0 & \dfrac{1}{k} \end{bmatrix} \end{array} \tag{2.40}$$

The result is:

$$\begin{array}{ll} \boldsymbol{e} = \boldsymbol{\mathcal{B}}\, w & (\textit{kinematic equations}) \\ \boldsymbol{s} = \boldsymbol{D}\,\boldsymbol{e} \quad \text{or} \quad \boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{s} & (\textit{constitutive equations}) \\ \boldsymbol{\mathcal{B}}'\,\boldsymbol{s} = f & (\textit{equilibrium equation}) \end{array} \tag{2.41}$$

2.4.1 Force method

The equilibrium equation cannot be solved because it contains two unknowns. Therefore, a redundant $\phi(x)$ is introduced, which is the spring load function on the cutting face between beam and springs, as shown in Fig. 2.16. This redundant sees to it that the gap Δ between beam and springs becomes zero. From the equilibrium equation (2.39) and the choice for ϕ it can be concluded that:

$$\frac{d^2M}{dx^2} = -\phi - f \quad ; \quad s = \phi$$

This result for M and s will be substituted into the constitutive equations. Before doing so, it is handy to differentiate the constitutive equations twice with respect to x. After substitution of the above relations, it then follows:

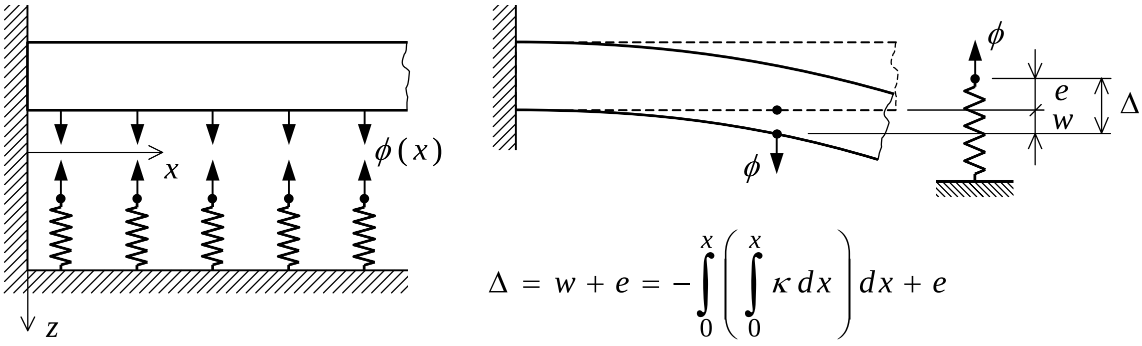


Fig. 2.16: Selection of the redundant and visual interpretation of the gap.

$$\frac{d^2\kappa}{dx^2} = \frac{1}{EI}\left(-\phi - f\right) \quad ; \quad e = \frac{1}{k}\phi$$

Further, the compatibility condition is obtained by elimination of the deflection w from the kinematic relations (2.37), which delivers:

$$\frac{d^2e}{dx^2} - \kappa = 0$$

This compatibility condition is differentiated twice:

$$\frac{d^4e}{dx^4} - \frac{d^2\kappa}{dx^2} = 0$$

Now, the previously found values for $d^2\kappa / dx^2$ and e can be substituted, leading to the following differential equation for ϕ:

$$\boxed{\frac{1}{k}\frac{d^4\phi}{dx^4} + \frac{1}{EI}\phi = -\frac{1}{EI} f} \qquad (2.42)$$

Remark

The compatibility condition has been visualised in Fig. 2.16. The total gap Δ to be eliminated is the sum of w and e:

$$w + e = 0$$

Differentiation twice transforms this condition into:

$$\frac{d^2w}{dx^2} + \frac{d^2e}{dx^2} = 0$$

The first term is exactly $-\kappa$, therefore the compatibility condition becomes:

$$-\kappa + \frac{d^2 e}{dx^2} = 0$$

This is the same equation as the one above, which was mathematically obtained by elimination of the deflection w from the kinematic equations.

2.4.2 Displacement method

In the displacement method, the kinematic and the constitutive equations are substituted into the equilibrium equation. This leads to:

$$\boxed{EI \frac{d^4 w}{dx^4} + k\, w = f} \qquad (2.43)$$

2.4.3 Elaboration of an example

Force method

The found results can be applied to the beam of Fig. 2.14 for constant EI and f. When the force method is applied, differential equation (2.42) has to be solved. The boundary conditions are partly kinematic and partly dynamic:

$$x = 0 \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dx} = 0 \end{cases} \qquad (\mathit{kinematic})$$

$$x = l \quad \rightarrow \quad \begin{cases} M = 0 \\ V = 0 \end{cases} \qquad (\mathit{dynamic})$$

The requirement that $w = 0$ for $x = 0$ means that e is zero and therefore that $\phi = 0$. Likewise, from the requirement that $dw/dx = 0$ it follows that $d\phi/dx$ is zero. This means that the kinematic boundary conditions are transformed into:

$$x = 0 \quad \rightarrow \quad \begin{cases} \phi = 0 & (1) \\ \dfrac{d\phi}{dx} = 0 & (2) \end{cases}$$

The boundary conditions for $x = l$ can also be interpreted as conditions for ϕ. To achieve this, the compatibility condition is used:

$$\frac{d^2 e}{dx^2} - \kappa = 0$$

The condition $M = 0$ is identical to $\kappa = 0$. Therefore, it follows:

$$\frac{d^2 e}{dx^2} = 0 \quad \rightarrow \quad \frac{d^2 \phi}{dx^2} = 0 \qquad\qquad (3)$$

That V is zero means:

$$\frac{dM}{dx} = 0 \quad \rightarrow \quad \frac{d\kappa}{dx} = 0$$

Then it has to hold:

$$\frac{d^3 e}{dx^3} = 0 \quad \rightarrow \quad \frac{d^3 \phi}{dx^3} = 0 \qquad\qquad (4)$$

Displacement method
When the displacement method is applied, differential equation (2.43) has to be solved. The above-mentioned kinematic boundary conditions can be used directly:

$$x = 0 \quad \rightarrow \quad \begin{cases} w = 0 & \qquad (1) \\ \dfrac{dw}{dx} = 0 & \qquad (2) \end{cases}$$

The two dynamic boundary conditions $M = 0$ and $V = 0$ for $x = l$ are reformulated into conditions for the deflection w.

$$x = l \quad \rightarrow \quad \begin{cases} \dfrac{d^2 w}{dx^2} = 0 & \qquad (3) \\ \dfrac{d^3 w}{dx^3} = 0 & \qquad (4) \end{cases}$$

Solution
For both the force method and the displacement method the homogeneous solution for a beam on an elastic foundation contains four components (so the particular solution is not considered). In the book "*Elastostatica van slanke structuren*[*]" of A.L. Bouma, section 11.4 for these components it can be found:

$$\begin{array}{lcl} e^{x/\lambda} \sin\left(x/\lambda\right) & ; & e^{-x/\lambda} \sin\left(x/\lambda\right) \\ e^{x/\lambda} \cos\left(x/\lambda\right) & ; & e^{-x/\lambda} \cos\left(x/\lambda\right) \end{array}$$

where λ is a characteristic length:

$$\lambda = \sqrt[4]{\frac{4EI}{k}}$$

[*] English translation: "*Elastostatics of slender structures*"

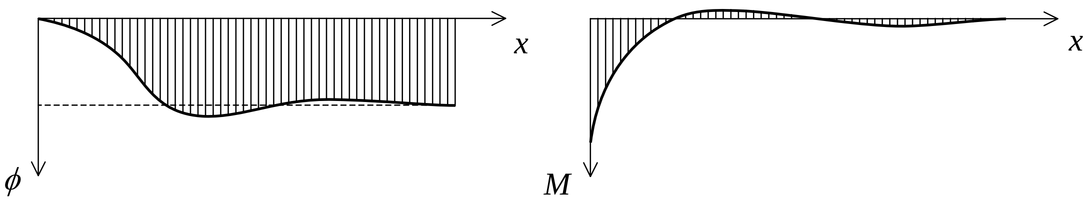


Fig. 2.17: Redundant and moment as a function of x.

When the beam length is at least four times its characteristic length λ a solution is obtained as shown in Fig. 2.17.

2.5 Beams subjected to torsion or shear

Torsion

In the case of torsion, the degree of freedom is the rotation $\varphi(x)$ about the x-axis (see Fig. 2.18). This rotation is associated with a distributed torque $t(x)$. In the cross-section of the

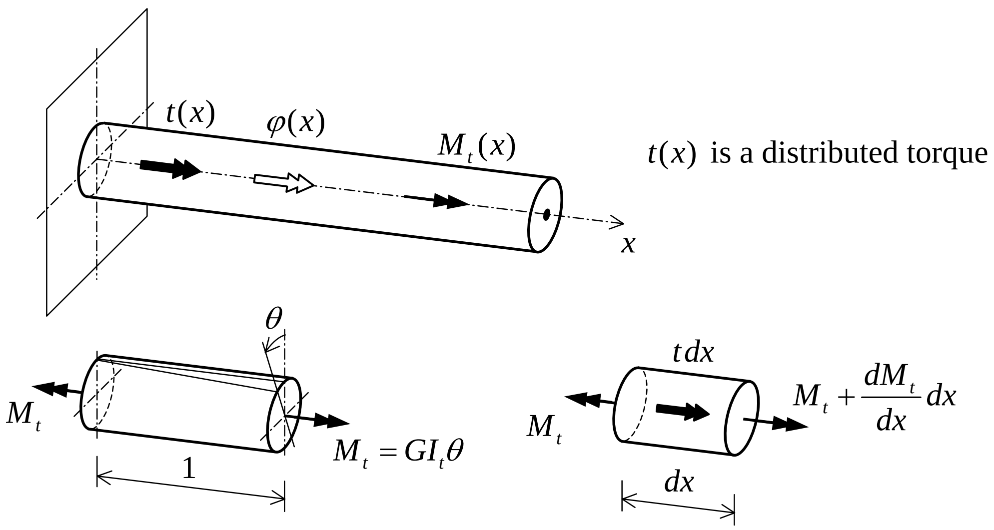


Fig. 2.18: Bar subjected to torsion with relevant quantities.

beam a torsional moment M_t is generated, which goes together with an angular deflection θ. The torsional stiffness is GI_t. The scheme of relations is displayed in Fig. 2.19.

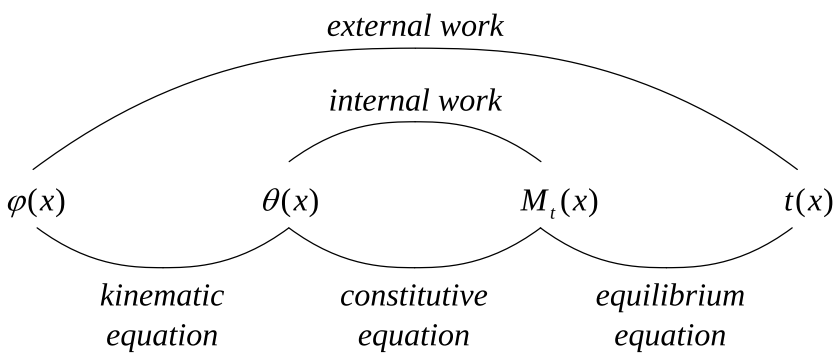


Fig. 2.19: Diagram displaying the relations between the quantities playing a role in the analysis of a beam subjected to torsion.

Exercise
Derive the basic equations for the above load case.

Shear
When in a beam only deformation occurs that is caused by a shear force, the fibres do not change in length in x-direction (see Fig. 2.20). The only deformation present is a shear

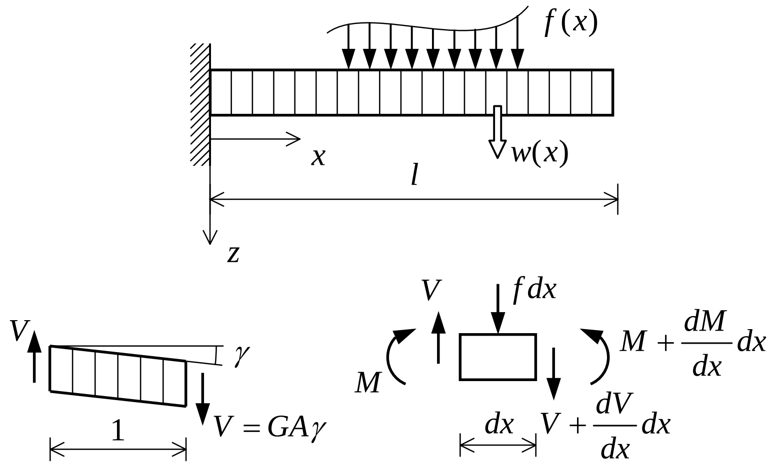


Fig. 2.20: Statically determinate beam subjected to pure shear with relevant quantities.

deformation γ caused by the shear force V. Therefore, there is only one degree of freedom w corresponding with a distributed load f. The scheme of relations can be found in Fig. 2.21.

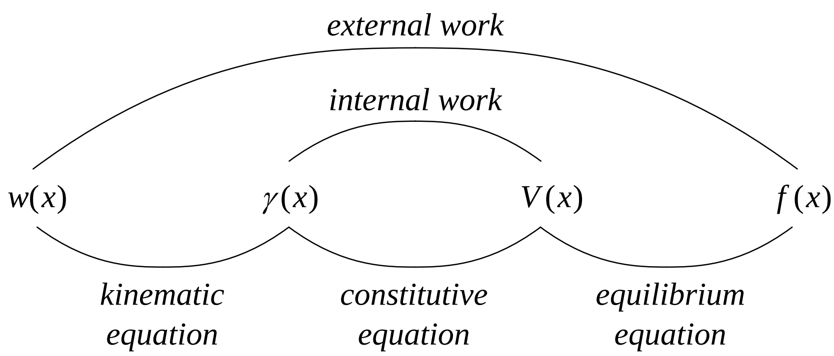


Fig. 2.21: Diagram displaying the relations between the quantities playing a role in the analysis of a beam subjected to pure shear.

Exercise
Derive the basic equations for the above load case.

2.6 Summary for load cases of continuous slender beams

In this chapter, for a number of load cases it has been shown that the basic equations can be derived in a consistent manner. For suitably selected differential operators a large analogy can be demonstrated with the formulation of discrete bar structures. A first decision concerns the choice (of the number) of independent displacements, the degrees of freedom. Every selected degree of freedom corresponds with a load. This means that the number of load components is exactly the same as the number of degrees of freedom.

Further, it is an important choice, which of the (generalised) stresses play a role in the basic equations. Only these stresses are coupled with deformations (and therefore coupled with deformation energy). It is advantageous to indicate these stresses and deformations by a separate variable, independently from the selected degrees of freedom and external loads.

The basic equations consist out of three sets:

- the kinematic equations
- the constitutive equations
- the equilibrium equations

In the force method, the constitutive equations are used in the flexibility formulation and in the displacement method the stiffness formulation is applied.
It is essential that the equilibrium equations are formulated in the direction of the degrees of freedom. The number equilibrium equations has to be equal to the number of degrees of freedom. If there are more equilibrium equations, then one or more stresses are present to which no deformation is coupled. By elimination of these equilibrium equations the number is reduced to the desired amount.

In the displacement method, the three sets of basic equations can be substituted into each other without any modifications. The substitution process starts with the kinematic equation and via the constitutive equation it ends at the equilibrium equation. The final result is a set of one or more equilibrium equations that are expressed in the degrees of freedom, the diffe .rential equation(s).

In the force method, the three basic equations are evaluated in the opposite order, first the equilibrium equations, than the constitutive equations and finally the kinematic equations. However, in this case an additional operation is required. The following three steps have to be followed:
<u>*First step*</u>: The equilibrium equations are used to express the (generalised) stresses $\boldsymbol{s}$ in one or more redundants $\boldsymbol{\phi}$. This guarantees an equilibrium system of stresses.
<u>*Second step*</u>: By substitution of this equilibrium system into the constitutive equations, the deformations $\boldsymbol{e}$ are expressed in the redundants.
<u>*Third step*</u>: From the kinematic equations the degrees of freedom are eliminated. By this operation, the equations are reduced to one or more relations between the deformations $\boldsymbol{e}$ themselves, the so-called compatibility condition(s). The results of the second and third step are combined to one or more differential equations with respect to the redundants $\boldsymbol{\phi}$.

<u>*Take caution!*</u> The kinematic relations describe the relation between deformations and the degrees of freedom. However, the compatibility condition is a requirement only containing deformations. This condition sees to it that compatibility is guaranteed, which means that no gaps or overlappings in the structure can occur.

In this chapter the following elastic elements were discussed:
– beam subjected to extension
– beam subjected to bending
– beam subjected to torsion
– beam subjected to shear
– distributed spring loads

In a number of cases, combinations of these elements were considered. Many structural problems can be reduced to a combination of these cases. When the proposed strategy is followed, a reliable and consistent differential equation will be found in all cases. A number of examples will be elaborated.

2.7 Walls coupled by springs subjected to a temperature load

Fig. 2.22 shows two high walls, which are coupled by horizontal beams of high flexural stiffness. These beams can be considered as a system of distributed springs that counteract the relative movement between the two walls. It can be assumed that the rest of the building supports the two walls in such a manner that the horizontal displacement is suppressed.
Both walls can only displace in vertical direction. Together they form the exterior wall of a building. The left wall is subjected to the outer temperature and the right wall has the interior temperature. The temperature difference between the two walls is T .

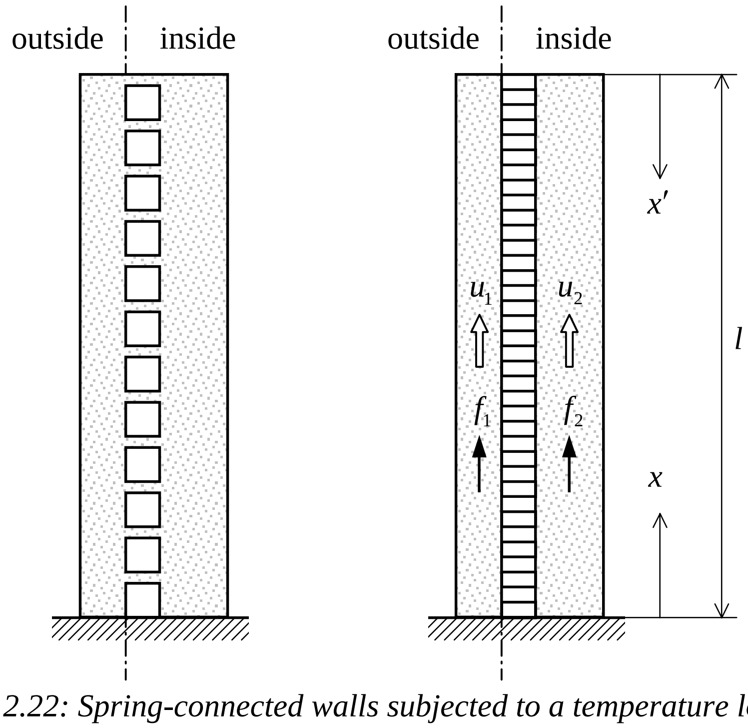


Fig. 2.22: Spring-connected walls subjected to a temperature load.

The differential equation(s) will be derived from which the shear forces can be obtained occurring in the connecting beams. The boundary conditions will be determined as well. The linear coefficient of thermal expansion is α , the axial stiffnesses of the outer and inner walls are EA_1 and EA_2 , respectively. First the force method is applied and after that the displacement method. The equations will be solved for the special case that $EA_1 = EA_2$ and

under the assumption that the walls are much taller than the characteristic length λ of the system.
The considered problem is subjected to a temperature load. In such cases the strain ε of a wall will be composed out of two contributions, en elastic strain ε_e caused by the present normal force N and a temperature strain ε_T resulting from the temperature difference T, i.e.:

$$\varepsilon_e = \frac{1}{EA} N \quad ; \quad \varepsilon_T = \alpha T$$

Then the total strain is the sum of ε_e and ε_T:

$$\varepsilon = \frac{1}{EA} N + \alpha T$$

This is the extended constitutive relation in flexibility formulation. Solving N from this equation provides the constitutive relation is stiffness formulation:

$$N = EA\left(\varepsilon - \alpha T\right)$$

In the calculation, the exterior wall is set to zero temperature and the interior wall to temperature T.
The system of the two coupled walls is described by two degrees of freedom, u_1 and u_2. Distributed loads f_1 and f_2 are possible in the direction of the degrees of freedom, however in this case they are zero. Three deformations occur, namely ε_1 and ε_2 in the walls and the displacement e in the springs. With these deformations correspond the normal forces N_1 and N_2 in the walls and a distributed shear load s in the springs. Fig. 2.23 shows the positive

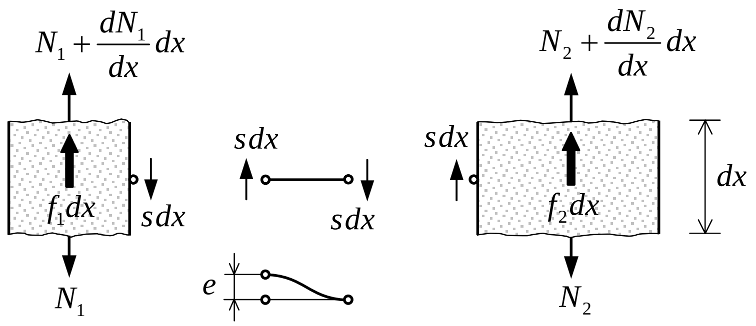


Fig. 2.23: Relevant quantities in the walls.

directions of s and e. The x-axis is pointing upwards and starts at the foundation. Fig. 2.24 displays the scheme of relations for this wall problem.
The kinematic equations are:

$$\boxed{\begin{array}{l} \varepsilon_1 = \dfrac{du_1}{dx} \quad ; \quad \varepsilon_2 = \dfrac{du_2}{dx} \\ e = u_1 - u_2 \end{array}} \tag{2.44}$$

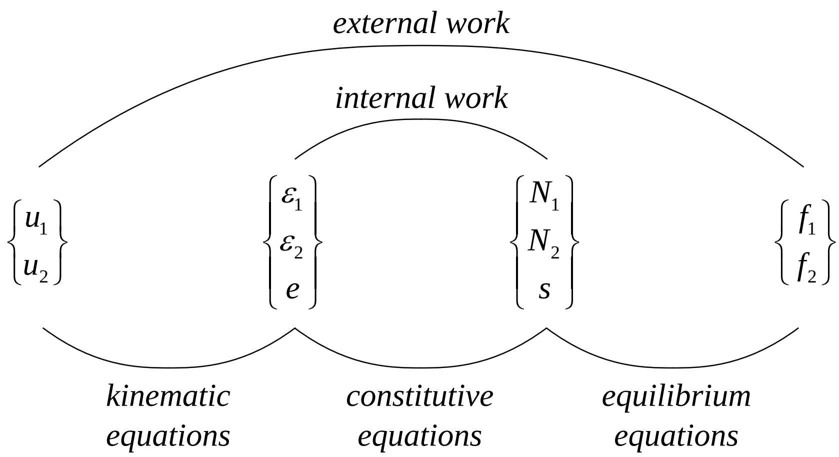


Fig. 2.24: Diagram displaying the relations between the quantities playing a role in the analysis of the spring-connected walls subjected to a temperature load.

The constitutive equations become:

$$\boxed{\begin{aligned} &N_1 = EA_1\,\varepsilon_1 \\ &N_2 = EA_2\left(\varepsilon_2 - \alpha T\right) \\ &s = k\,e \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} &\varepsilon_1 = \frac{1}{EA}N_1 \\ &\varepsilon_2 = \frac{1}{EA}N_2 + \alpha T \\ &e = \frac{1}{k}s \end{aligned}} \qquad (2.45)$$

The equilibrium equations read:

$$\boxed{-\frac{dN_1}{dx} + s = f_1 \quad ; \quad -\frac{dN_2}{dx} - s = f_2} \qquad (2.46)$$

2.7.1 Force method

Introduction of one redundant is sufficient, because two equations are present for three unknowns (N_1, N_2, s). The distributed spring load is selected as the redundant ϕ. From the equilibrium equations (first step) it then is found:

$$\frac{dN_1}{dx} = \phi \quad ; \quad \frac{dN_2}{dx} = -\phi \quad ; \quad s = \phi$$

where it has been used that $f_1 = f_2 = 0$. The three constitutive equations can now be written as (second step):

$$\frac{d\varepsilon_1}{dx} = \frac{1}{EA_1}\phi \quad ; \quad \frac{d\varepsilon_2}{dx} = -\frac{1}{EA_2}\phi \quad ; \quad e = \frac{1}{k}\phi$$

Next the compatibility condition has to be formulated (third step). For that purpose, the degrees of freedom u_1 and u_2 have to be eliminated from the three kinematic equations. This can be done easily after that the relation for e is differentiated once with respect to x. It easily can be seen that:

$$\varepsilon_1 - \varepsilon_2 - \frac{de}{dx} = 0$$

After differentiation for the second time the found results for $d\varepsilon_1/dx$, $d\varepsilon_2/dx$ and e can be substituted, leading to:

$$\boxed{-\frac{1}{k}\frac{d^2\phi}{dx^2} + \left(\frac{1}{EA_1} + \frac{1}{EA_2}\right)\phi = 0} \qquad (2.47)$$

Two boundary conditions have to be formulated.
For $x = 0$ it holds that both u_1 and u_2 are zero, this means that $e = 0$. Then it also holds $s = 0$, i.e.:

$$\boxed{\phi = 0 \quad \text{for} \quad x = 0} \qquad (2.48)$$

For $x = l$ it is known that $N_1 = 0$ and $N_2 = 0$, this means that $\varepsilon_1 = 0$ and $\varepsilon_2 = \alpha T$. Now from the compatibility condition information can be obtained about the derivative of e:

$$\frac{de}{dx} = -\alpha T$$

This means that the derivative of ϕ is known for $x = l$:

$$\boxed{\frac{d\phi}{dx} = -k\,\alpha\,T \quad \text{for} \quad x = l} \qquad (2.49)$$

The differential equation will be solved for $EA_1 = EA_2 = EA$. After introduction of the characteristic length:

$$\lambda = \sqrt{\frac{EA}{2k}}$$

the differential equation can be rewritten and solved. It is found:

$$-\lambda^2 \frac{d^2\phi}{dx^2} + \phi = 0 \quad \rightarrow \qquad \phi(x) = C_1\, e^{-x/\lambda} + C_2\, e^{-x'/\lambda}$$

where x' is defined by $x' = l - x$.

From the two boundary conditions it follows (with $l \gg \lambda$):

$$C_1 = 0 \quad ; \quad C_2 = -\lambda\, k\, \alpha\, T$$

The solution then becomes:

$$\boxed{\phi(x) = -\lambda\, k\, \alpha\, T\, e^{-x'/\lambda}} \tag{2.50}$$

Therefore, the spring load equals:

$$\boxed{s(x) = -\lambda\, k\, \alpha\, T\, e^{-x'/\lambda} = -\frac{EA}{2\lambda}\alpha\, T\, e^{-x'/\lambda}} \tag{2.51}$$

Then for the normal forces it is found:

$$\frac{dN_1}{dx} = +s \quad \rightarrow \quad N_1 = -\frac{1}{2}EA\alpha T\, e^{-x'/\lambda} + D_1$$
$$\frac{dN_2}{dx} = -s \quad \rightarrow \quad N_2 = +\frac{1}{2}EA\alpha T\, e^{-x'/\lambda} + D_2$$

Because N_1 and N_2 are zero for $x' = 0$, the integration constants become:

$$D_1 = \frac{1}{2}EA\, \alpha\, T \quad ; \quad D_2 = -\frac{1}{2}EA\, \alpha\, T$$

Substitution of these constants into the normal forces delivers:

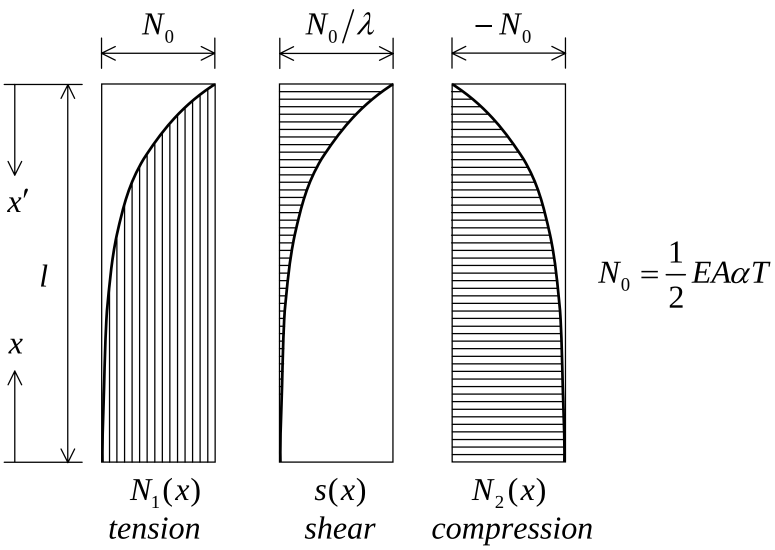


Fig. 2.25: Final results of the analysis of the spring-connected walls.

$$N_1 = \frac{1}{2} EA\alpha T\left(1 - e^{-x'/\lambda}\right) \quad ; \quad N_2 = \frac{1}{2} EA\alpha T\left(-1 + e^{-x'/\lambda}\right) \tag{2.52}$$

In Fig. 2.25, these results are depicted across he height of the structure. It shows that the sum of the normal forces N_1 and N_2 is equal to zero for every value of x. This has to be the case, since no external vertical load is present. Only the connecting beams at the upper part of the structure are subjected to a shear load. They act like dowels and see to it that the strain in both walls is identical. The wall in which the compressive force is acting becomes taller as well, because of the temperature increase in this wall.

2.7.2 Displacement method

In this case, the kinematic equations and the constitutive equations in stiffness formulation are used. Substitution of the kinematic into the constitutive equations provides:

$$N_1 = EA_1 \frac{du_1}{dx} \quad ; \quad N_2 = EA_2 \left(\frac{du_2}{dx} - \alpha\, T \right) \quad ; \quad s = k\left(u_1 - u_2\right)$$

Combination of these results with the equilibrium equations (with $f_1 = f_2 = 0$) delivers:

$$-EA_1 \frac{d^2 u_1}{dx^2} + k\, u_1 - k\, u_2 = 0 \quad ; \quad -EA_2 \frac{d^2 u_2}{dx^2} - k\, u_1 + k\, u_2 = 0 \tag{2.53}$$

In a notation with operators this set of ordinary differential equations becomes:

$$\begin{bmatrix} k - EA_1 \dfrac{d^2}{dx^2} & -k \\ -k & k - EA_2 \dfrac{d^2}{dx^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.54}$$

It easily can be seen that the matrix with even-order differentiations is symmetrical.
The boundary conditions for the bottom end $x = 0$ are:

$$x = 0 \quad \rightarrow \quad \begin{cases} u_1 = 0 \\ u_2 = 0 \end{cases} \qquad (\textit{kinematic boundary conditions}) \tag{2.55}$$

At the other end the boundary conditions read:

$$x = l \quad \rightarrow \quad \begin{cases} N_1 = 0 \\ N_2 = 0 \end{cases}$$

These dynamic boundary conditions can be rewritten as conditions for u_1 and u_2:

$$\boxed{x = l \quad \rightarrow \quad \begin{cases} \dfrac{du_1}{dx} = 0 \\ \dfrac{du_2}{dx} = \alpha T \end{cases}} \qquad \text{(\textit{dynamic boundary conditions})} \qquad (2.56)$$

For obtaining the homogeneous solution of (2.53), the following terms:

$$u_1 = A\, e^{rx} \quad ; \quad u_2 = B\, e^{rx}$$

are substituted into the differential equation. This delivers:

$$\begin{array}{llll} \left(k - EA_1 r^2\right) & A\, e^{rx} & -k & B\, e^{rx} = 0 \\ -k & A\, e^{rx} & +\left(k - EA_2 r^2\right) & B\, e^{rx} = 0 \end{array} \qquad (2.57)$$

After division by e^{rx}, the matrix formulation reads:

$$\begin{bmatrix} k - EA_1 r^2 & -k \\ -k & k - EA_2 r^2 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2.58)$$

These to equations in A and B only provide a solution for A and B if the determinant of the matrix is equal to zero, i.e.:

$$\left(k - EA_1 r^2\right)\left(k - EA_2 r^2\right) - \left(-k\right)^2 = 0$$

This is a fourth-order equation in r, which in general has four different roots r_1, r_2, r_3 and r_4. The solution will be derived for the case that EA_1 and EA_2 have the same value EA. It then holds:

$$\left(k - EA\, r^2\right)^2 - k^2 = 0 \quad \rightarrow \quad \left(EA\right)^2 r^4 - 2k\; EA\, r^2 = 0 \quad \rightarrow \quad r^2\left(EA\, r^2 - 2k\right) = 0$$

Introduction of the characteristic length:

$$\lambda = \sqrt{\frac{EA}{2k}}$$

transforms this characteristic equation into:

$$r^2\left(\lambda^2 r^2 - 1\right) = 0$$

The four roots are:

$$r_1 = 0 \quad ; \quad r_2 = 0 \quad ; \quad r_3 = \frac{1}{\lambda} \quad ; \quad r_4 = -\frac{1}{\lambda}$$

From the first equation of (2.57), for each root the ratio of A and B can be determined. With $EA_1 = EA$ and introduction of the characteristic length the equation can be rewritten as:

$$\left(k - EA\, r^2\right) A - kB = 0 \quad \rightarrow \quad \left(1 - 2\lambda^2 r^2\right) A - B = 0$$

The ratio of A and B for each of the roots then becomes:

$$\begin{array}{lllll} r_1 = 0 & \rightarrow & A_1/B_1 = 1 & ; & r_3 = \;\; 1/\lambda \quad \rightarrow \quad A_3/B_3 = -1 \\ r_2 = 0 & \rightarrow & A_2/B_2 = 1 & ; & r_4 = -1/\lambda \quad \rightarrow \quad A_4/B_4 = -1 \end{array}$$

Therefore the general solution becomes:

$$u_1 = A_1 e^{r_1 x} + A_2 e^{r_2 x} + A_3 e^{r_3 x} + A_4 e^{r_4 x} \quad ; \quad u_2 = A_1 e^{r_1 x} + A_2 e^{r_2 x} - A_3 e^{r_3 x} - A_4 e^{r_4 x}$$

Two of the four roots have the same value, namely zero. The theory of homogeneous linear differential equations with constant coefficients prescribes that for double roots r the functions e^{rx} and xe^{rx} have to be introduced. When r is zero, these functions are 1 and x. The general solution then becomes:

$$u_1 = A_1 + A_2 x + A_3 e^{x/\lambda} + A_4 e^{-x/\lambda} \quad ; \quad u_2 = A_1 + A_2 x - A_3 e^{x/\lambda} - A_4 e^{-x/\lambda}$$

By introduction of another constant A_3 for which it holds $A_3 = e^{l/\lambda} A_3$, this solution can be rewritten as (notice that $x' = l - x$):

$$u_1(x) = A_1 + A_2 x + A_3\; e^{-x'/\lambda} + A_4\; e^{-x/\lambda} \quad ; \quad u_2(x) = A_1 + A_2 x - A_3\; e^{-x'/\lambda} - A_4\; e^{-x/\lambda}$$

The first two terms form the linear part of the solution. The third term damps out from the top and the fourth term damps out from the foundation.
Now, a tall structure is considered ($l \gg \lambda$) such that the exponential terms are damped out before they reach the other end of the wall.
The kinematic conditions (2.55) then require that:

$$x = 0 \quad \rightarrow \quad \left.\begin{array}{l} A_1 + A_4 = 0 \\ A_1 - A_4 = 0 \end{array}\right\} \quad \rightarrow \quad A_1 = 0 \quad ; \quad A_4 = 0$$

This reduces the solution to:

$$u_1(x) = A_2 x + A_3 e^{-x'/\lambda} \quad ; \quad u_2(x) = A_2 x - A_3 e^{-x'/\lambda}$$

The dynamic boundary conditions (2.56) impose that:

$$x = l \quad \text{or} \quad x' = 0 \quad \rightarrow \quad \left.\begin{array}{l} A_2 + \dfrac{1}{\lambda} A_3 = 0 \\ A_2 - \dfrac{1}{\lambda} A_3 = \alpha\, T \end{array}\right\} \quad \rightarrow \quad A_2 = \frac{1}{2} \alpha\, T \quad ; \quad A_3 = -\frac{1}{2} \lambda\, \alpha\, T$$

Finally, the solution becomes:

$$\boxed{u_1(x) = \frac{1}{2} \alpha T \left(x - \lambda\, e^{-x'/\lambda} \right) \quad ; \quad u_2(x) = \frac{1}{2} \alpha T \left(x + \lambda\, e^{-x'/\lambda} \right)} \tag{2.59}$$

From these results, for the normal forces it directly follows:

$$\boxed{N_1(x) = \frac{1}{2} EA\alpha\, T \left(1 - e^{-x'/\lambda} \right) \quad ; \quad N_2(x) = \frac{1}{2} EA\alpha\, T \left(-1 + e^{-x'/\lambda} \right)} \tag{2.60}$$

and also the spring load:

$$\boxed{s(x) = -k\, \alpha\, T \lambda\, e^{-x'/\lambda} = -\frac{EA}{2\lambda} \alpha T\, e^{-x'/\lambda}} \tag{2.61}$$

The solutions (2.60) and (2.61) are identical to (2.52) and (2.51), the curves of which are displayed in Fig. 2.25.

2.7.3 Epilogue

The discussed problem of the coupled walls showed that the force method resulted in a lower number of solution steps than the displacement method. This aspect is generally valid. In the force method there is only one unknown (the redundant ϕ), contrary to two unknowns (the degree of freedoms u_1 and u_2) in the displacement method.
On the other hand, in the displacement method it is easier to determine the boundary conditions. In the force method it is often quite a hustle to reformulate the boundary conditions into conditions imposed on the redundant(s) ϕ.

2.8 Walls coupled by springs subjected to a wind load

The coupled walls of a high-rise building from section 2.7 are now subjected to a wind load. A uniformly distributed load f in horizontal direction represents the wind load. Fig. 2.26 shows the notation, variables and sign convention used. Again, the axial vertical displacements of the exterior and interior walls in positive x-direction are u_1 and u_2, respectively. In this case displacements in horizontal direction are generated as well. From the assumption that rigid floors are present, it can be concluded that the horizontal displacements of both walls are the same. The arrow w indicates this displacement and it is positive when it

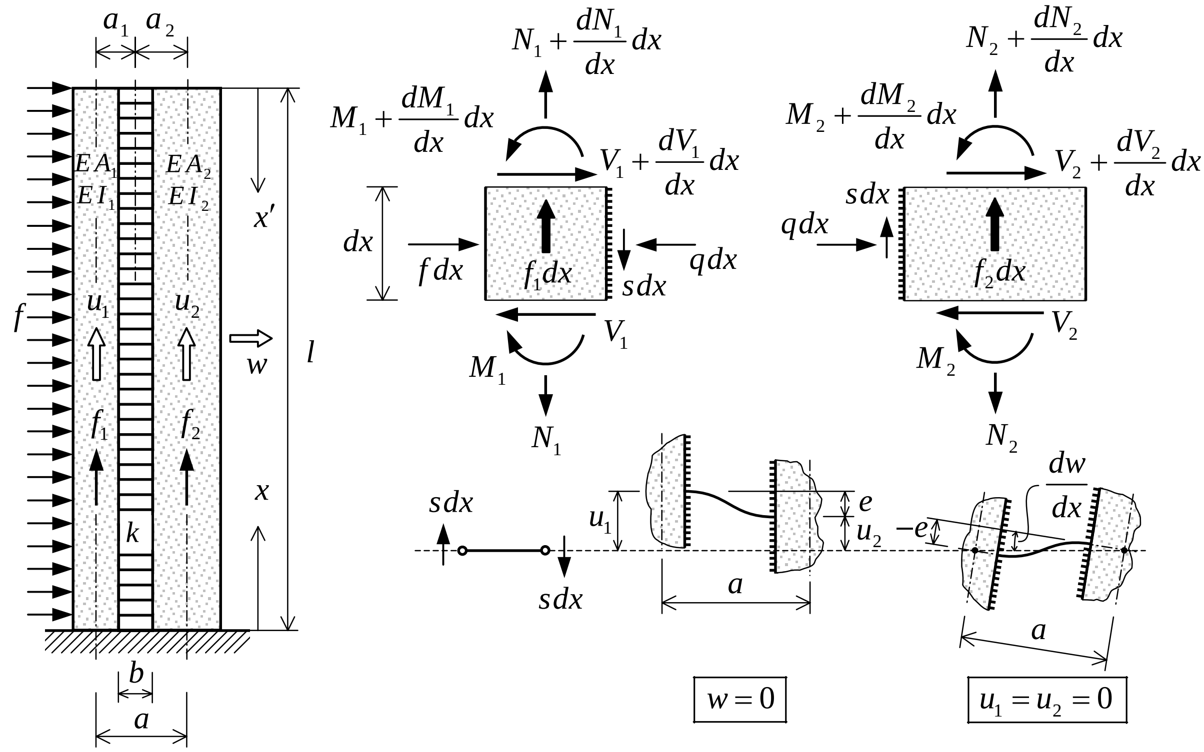


Fig. 2.26: Modelling of high wall subjected to wind load.

points to the right. Notice that in Fig. 2.26 the displacement w and the wind load f are pointing in the same direction. This also holds for the displacements u_1, u_2 and the distributed loads f_1, f_2, respectively. Again, the two load components f_1 and f_2 are equal to zero.
The centre-distance of the two walls is a and the length of the connecting beams is b. The distances of wall to beam centres are a_1 and a_2, respectively.
In the two walls, the normal forces N_1, N_2, the shear forces V_1, V_2 and the moments M_1, M_2 are generated. The distributed shear load in the beams is s. In the walls, the deformations ε and κ are considered caused by N_1, N_2 and M_1, M_2, respectively. The deformations caused by the shear forces V_1, V_2 are left out of the analysis. In the springs between the walls, the deformation e caused by the distributed shear load s is taken into account. In Fig. 2.26 a positive e has been drawn. Further, in the figure it can be observed that a deformation e is formed by both the slope dw/dx and the difference in magnitude of the two displacements u_1 and u_2. The sign conventions for ε and κ are similar to the ones in previous sections of this chapter. The stiffnesses of the two walls are EA_1, EI_1 and EA_2, EI_2, respectively. Again the spring stiffness is indicated by k.
Both walls are subjected to the same horizontal displacement w and therefore have the same curvature κ. This means that the moments M_1 and M_2 are in proportion to the stiffnesses EI_1 and EI_2. Therefore, it is advantageous to make use of the following collective quantities:

$$M = M_1 + M_2 \quad ; \quad V = V_1 + V_2 \quad ; \quad EI = EI_1 + EI_2$$

This means that the quantities will be used as indicated in the relational scheme of Fig. 2.27. The wind load f will be carried partly by wall 1 and partly by wall 2. The part of f transmitted to wall 2 is indicated by q. This introduces a compressive force in the horizontal

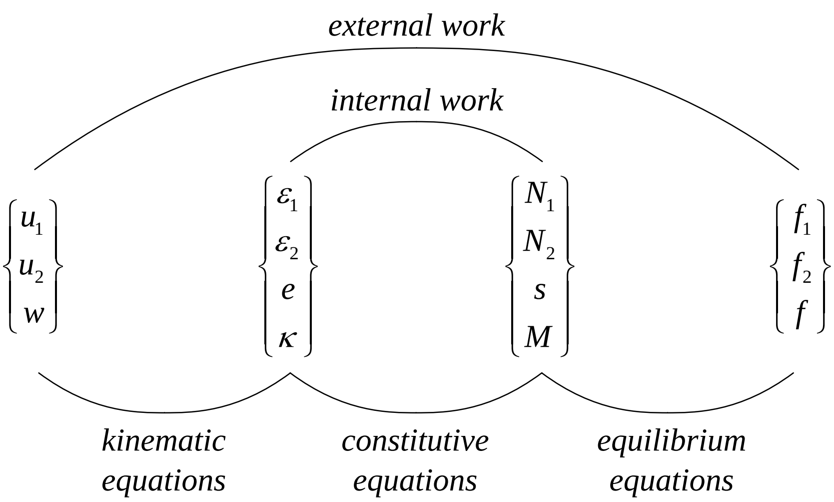


Fig. 2.27: Diagram displaying the relations between the quantities playing a role in the analysis of the spring-connected walls subjected to a wind load.

beams. The shortening of the beams is assumed negligible small. On wall 1 the load q points to the left and on wall 2 to the right.
In the displacement method three interdependent differential equations in u_1, u_2 and w can be formulated. In the force method, the utilisation of one redundant ϕ is sufficient, since three equilibrium equations exist for four unknowns. Therefore, it is obvious that the force method is applied.

Kinematic equations
The kinematic equations read:

$$\boxed{\begin{array}{lll} \varepsilon_1 = \dfrac{du_1}{dx} & ; & e = -a\dfrac{dw}{dx} + u_1 - u_2 \\ \varepsilon_2 = \dfrac{du_2}{dx} & ; & \kappa = -\dfrac{d^2 w}{dx^2} \end{array}} \tag{2.62}$$

Constitutive equations
The constitutive equations are:

$$\begin{array}{l} N_1 = EA_1\,\varepsilon_1 \\ N_2 = EA_2\,\varepsilon_2 \\ s = k\,e \\ \left.\begin{array}{l} M_1 = EI_1\,\kappa_1 \\ M_2 = EI_2\,\kappa_2 \end{array}\right\} \xrightarrow[\text{addition}]{} \; M = EI\kappa \end{array} \quad \Rightarrow \quad \boxed{\begin{array}{l} \varepsilon_1 = N_1/EA_1 \\ \varepsilon_2 = N_2/EA_2 \\ e = s/k \\ \kappa = M/EI \end{array}} \tag{2.63}$$

Equilibrium equations
The equilibrium equations read:

$$\boxed{\begin{aligned}&\frac{dN_1}{dx}-s+f_1=0\\&\frac{dN_2}{dx}+s+f_2=0\end{aligned}} \tag{2.64}$$

$$\left.\begin{aligned}\frac{dM_1}{dx}-a_1 s-V_1=0\\ \frac{dM_2}{dx}-a_2 s-V_2=0\end{aligned}\right\}\xrightarrow{\text{addition}}\ \frac{dM}{dx}-a\,s-V=0$$

$$\left.\begin{aligned}f-q+\frac{dV_1}{dx}=0\\ q+\frac{dV_2}{dx}=0\end{aligned}\right\}\xrightarrow{\text{addition}}\ f+\frac{dV}{dx}=0$$

$$\left.\begin{aligned}\frac{dM}{dx}-a\,s-V=0\\ f+\frac{dV}{dx}=0\end{aligned}\right\}\xrightarrow[\text{of } V]{\text{elimination}}\ \boxed{\frac{d^2M}{dx^2}-a\frac{ds}{dx}+f=0} \tag{2.65}$$

The distributed load s in the springs is selected as the redundant ϕ.

$$s=\phi$$

First step

From the equilibrium equations for the normal forces (2.64) it follows:

$$\frac{dN_1}{dx}=+\phi-f_1$$

$$\frac{dN_2}{dx}=-\phi-f_2$$

From the intermediate results of relation (2.65), for the moment it can be derived:

$$\left.\begin{aligned}&\left.\begin{aligned}&f+\frac{dV}{dx}=0\\&V=0\quad\text{for}\quad x=l\end{aligned}\right\}\ \rightarrow\ V=f\left(l-x\right)\\&\frac{dM}{dx}-a\,s-V=0\end{aligned}\right\}\rightarrow\ \frac{dM}{dx}=a\phi+f\left(l-x\right)$$

Second step

The constitutive equations can be rewritten as:

$$\frac{d\varepsilon_1}{dx}=\frac{1}{EA_1}\left(\phi-f_1\right)\qquad;\quad e=\frac{1}{k}\phi$$

$$\frac{d\varepsilon_2}{dx}=\frac{1}{EA_2}\left(-\phi-f_2\right)\quad;\quad\frac{d\kappa}{dx}=\frac{1}{EI}\left(a\,\phi+f\ l-f\ x\right)$$

Third step
The compatibility condition can be found by elimination of u_1 and u_2 from the kinematic equations:

$$\varepsilon_1 - \varepsilon_2 - \frac{de}{dx} + a\kappa = 0$$

Differential equation
The required differential equation is found by substitution of the constitutive relations obtained in the second step into the compatibility condition derived in the third step. But first, the compatibility condition is differentiated once:

$$\frac{1}{EA_1}(\phi - f_1) - \frac{1}{EA_2}(-\phi - f_2) - \frac{1}{k}\frac{d^2\phi}{dx^2} + \frac{1}{EI}\left(a^2\phi + fal - fax\right) = 0 \quad \rightarrow$$

$$\left(\frac{1}{EA_1} + \frac{1}{EA_2} + \frac{a^2}{EI}\right)\phi - \frac{1}{k}\frac{d^2\phi}{dx^2} = \frac{f_1}{EA_1} - \frac{f_2}{EA_2} - \frac{fa(l-x)}{EI}$$

Introduction of $f_1 = f_2 = 0$ and the factor α given by (do not confuse with the linear coefficient of thermal expansion of the previous section):

$$\boxed{\alpha^2 = k\left(\frac{1}{EA_1} + \frac{1}{EA_2} + \frac{a^2}{EI}\right)} \tag{2.66}$$

provides the following required differential equation:

$$\boxed{\alpha^2\phi - \frac{d^2\phi}{dx^2} = -\frac{k}{EI}\,fa(l-x)} \tag{2.67}$$

The reciprocal of α is the characteristic length λ of this wall system:

$$\lambda = \frac{1}{\alpha}$$

In further elaborations, it is assumed that $\lambda \ll l$.

Boundary conditions
The boundary conditions at the foundation equal:

$$x = 0 \quad \rightarrow \quad \begin{cases} u_1 = 0 \\ u_2 = 0 \\ \dfrac{dw}{dx} = 0 \end{cases}$$

With the kinematic equation for e this boundary condition transforms into $e=0$. Substitution of $e=\phi/k$ finally provides:

$$\boxed{x=0 \quad \rightarrow \quad \phi=0} \tag{2.68}$$

The boundary conditions at the top equal:

$$x=l \quad \rightarrow \quad \begin{cases} N_1=0 & \rightarrow \quad \varepsilon_1=0 \\ N_2=0 & \rightarrow \quad \varepsilon_2=0 \\ M=0 & \rightarrow \quad \kappa=0 \end{cases}$$

From the compatibility equation, it now follows that $de/dx=0$. Substitution of $e=\phi/k$ changes the boundary conditions into:

$$\boxed{x=l \quad \rightarrow \quad \frac{d\phi}{dx}=0} \tag{2.69}$$

Solution

The solution of the differential equation can now be obtained. After some elaborations it is found:

$$\boxed{\begin{aligned} &\phi(x)=C_1\, e^{-\alpha x}+C_2\, e^{-\alpha(l-x)}-\frac{k}{\alpha^2 EI}\, fa\left(l-x\right) \\ &C_1=\frac{k}{\alpha^2 EI}\, fal \quad ; \quad C_2=\frac{-k}{\alpha^3 EI}\, fa \end{aligned}} \tag{2.70}$$

With the equilibrium equations and the fact that $N_1=N_2=M=0$ for $x=l$, from this solution it directly can be derived:

$$s=-\left(-l\, e^{-\alpha x}+\frac{1}{\alpha}e^{-\alpha(l-x)}+l-x\right)\frac{kaf}{\alpha^2 EI}$$

$$N_1=\left\{-\frac{l}{\alpha}e^{-\alpha x}+\frac{1}{\alpha^2}\left(1-e^{-\alpha(l-x)}\right)+\frac{1}{2}\left(l-x\right)^2\right\}\frac{kaf}{\alpha^2 EI}$$

$$N_2=-N_1$$

$$V=f(l-x)$$

$$M=\left\{-\frac{l}{\alpha}e^{-\alpha x}+\frac{1}{\alpha^2}\left(1-e^{-\alpha(l-x)}\right)+\frac{1}{2}\left(l-x\right)^2\right\}\frac{ka^2 f}{\alpha^2 EI}-\frac{1}{2}\left(l-x\right)^2 f$$

Check
At the foundation the total moment caused by the wind load is equal to $f\,l^2/2$. This moment has to be resisted by the torque of the normal forces and the sum of the moments in the two walls. Therefore, considering the sign convention it should hold:

$$\left(a\,N_1\right)+\left(-M\right)=\frac{1}{2}\,fl^2 \quad\rightarrow\quad \left(\frac{a\,N_1}{\frac{1}{2}\,f\,l^2}\right)+\left(\frac{-M}{\frac{1}{2}\,f\,l^2}\right)=1$$

Substitution of $x=0$ into the derived formulae for N_1 and M indeed shows that this relation is satisfied.

Results
In Fig. 2.28 the dimensionless spring load s/f, the dimensionless normal force $aN_1/\frac{1}{2}\,fl^2$ and the dimensionless bending moment $-M/\frac{1}{2}\,fl^2$ are represented graphically along the wall. The formulae for these dimensionless quantities are:

$$\frac{s}{f}=-\left\{-\frac{l}{a}e^{-\alpha x}+\frac{l}{a}*\frac{1}{\alpha l}e^{-\alpha(l-x)}+\frac{l}{a}\left(1-\frac{x}{l}\right)\right\}\frac{k\,a^2}{\alpha^2 EI}$$

$$\frac{a\,N_1}{\frac{1}{2}\,fl^2}=\left\{-2\frac{1}{\alpha l}e^{-\alpha l(x/l)}+\frac{2}{\left(\alpha l\right)^2}\left(1-e^{\alpha l(1-x/l)}\right)+\left(1-\frac{x}{l}\right)^2\right\}\frac{k\,a^2}{\alpha^2 EI}$$

$$\frac{-M}{\frac{1}{2}\,fl^2}=-\left\{-2\frac{1}{\alpha l}e^{-\alpha l(x/l)}+\frac{2}{\left(\alpha l\right)^2}\left(1-e^{\alpha l(1-x/l)}\right)+\left(1-\frac{x}{l}\right)^2\right\}\frac{k\,a^2}{\alpha^2 EI}+\left(1-\frac{x}{l}\right)^2$$

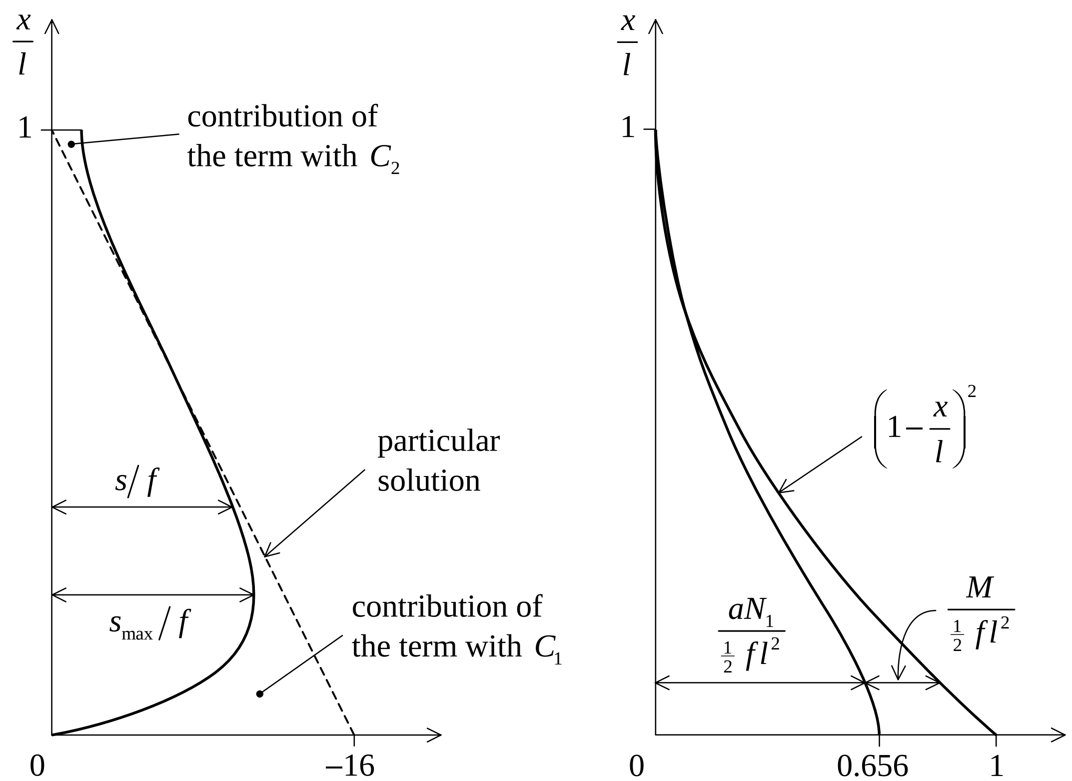


Fig. 2.28: Distributions of forces and moments in the coupled walls.

In these formulae the dimensionless parameter $ka^2/\alpha^2 EI$ is smaller than unity, because from the definition for α^2 given by (2.66) it easily can be confirmed that:

$$\frac{k\,a^2}{\alpha^2 EI}\left(\frac{I}{a^2 A_1}+\frac{I}{a^2 A_2}+1\right)=1 \quad \rightarrow \quad \frac{k\,a^2}{\alpha^2 EI}<1$$

In the graphs the following specific values have been used:

$$\alpha\, l = 10 \quad ; \quad \frac{l}{a}=20 \quad ; \quad \frac{k\,a^2}{\alpha^2 EI}=0.8$$

The graph at the left side of Fig. 2.28 displays the distributed shear load s relative to the wind load f. It has been shown how this result is obtained from the particular solution with end corrections by the homogeneous solution in order to satisfy the boundary conditions. The maximum shear load in the connecting beams appears somewhere in the lower half of the wall, but not at the foundation itself. From the formula above, the maximum value $s_{\max}$ of the shear load can be obtained. The largest shear force can be obtained by multiplication of $s_{\max}$ by the distance of the centre lines of the connecting beams. At the right side of Fig. 2.28 the curves of N_1 and M are drawn. This has been done in one graph to display clearly how the total moment caused by the wind load f is carried partly by the torque of the normal forces N_1 and N_2 and partly by the sum of the wall-moments M_1 and M_2.

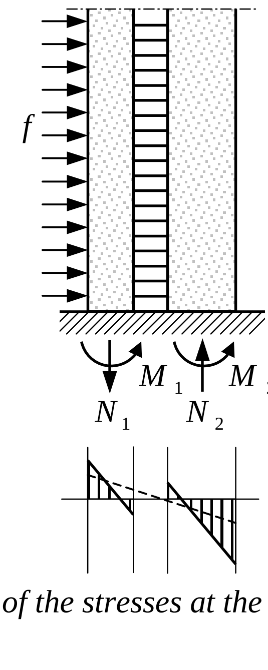


Fig. 2.29: sketch of the stresses at the bottom of both walls.

Fig. 2.29 shows how the stresses caused by N_1, M_1 and N_2, M_2 are distributed at the bottom of the walls. The dashed line indicates the stress distribution that would occur if no gaps were present between the two walls. With gaps the stresses are considerably higher.

Check on correctness of the kinematic and equilibrium equations

The matrix formulation for the kinematic equations is:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ e \\ \kappa \end{Bmatrix} = \begin{bmatrix} \dfrac{d}{dx} & 0 & 0 \\ 0 & \dfrac{d}{dx} & 0 \\ 1 & -1 & -a\dfrac{d}{dx} \\ 0 & 0 & -\dfrac{d^2}{dx^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ w \end{Bmatrix} \qquad \text{or} \qquad \boxed{\boldsymbol{e} = \boldsymbol{\mathcal{B}}\,\boldsymbol{u}}$$

The equilibrium equations can be rewritten as:

$$\begin{bmatrix} -\dfrac{d}{dx} & 0 & 1 & 0 \\ 0 & -\dfrac{d}{dx} & -1 & 0 \\ 0 & 0 & a\dfrac{d}{dx} & -\dfrac{d^2}{dx^2} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ s \\ M \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f \end{Bmatrix} \qquad \text{or} \qquad \boxed{\boldsymbol{\mathcal{B}}'\,\boldsymbol{s} = \boldsymbol{f}}$$

As expected, the operator $\boldsymbol{\mathcal{B}}'$ is the transposed of operator $\boldsymbol{\mathcal{B}}$, except for the sign of the uneven derivatives.

3 Plates loaded in-plane

In this chapter a group of problems will be addressed, which can be classified as two-dimensional. An in-plane loaded plate subjected to extension will be discussed. The plate may be in a state of *plane stress* or in a state of *plane strain*. A plane stress state occurs when a thin flat plate is loaded in-plane by a boundary load at the edge and/or a distributed load in the centre plane (see Fig. 3.1). The phenomenon of plane stress was already discussed in the course “Elastic Plates”. The plane strain state is an extension. Another new aspect in this chapter is that now both the displacement method and the force method are being considered.

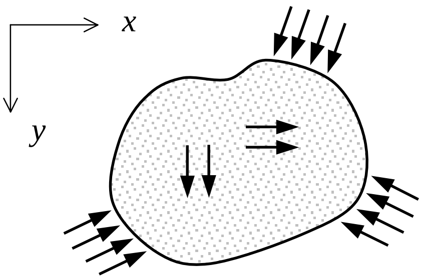


Fig. 3.1: Sketch of a plate loaded in-plane.

Plane strain occurs for example in a long cylindrical or prismatic body, the deformation of which is prevented in axial direction and the external load is independent from the axial coordinate. Moreover, the external loads (at the circumference or distributed over the volume) have to act perpendicular to the axis of the body. An example of such a load case is given in Fig. 3.2 where a straight long dam is drawn, which is subjected to its own weight and a line load.
In this case it is sufficient to analyse a slice of unit thickness, which is cut perpendicular to the cylinder-axis. This slice can be looked at as a plate with somewhat different constitutive properties as discussed hereafter.

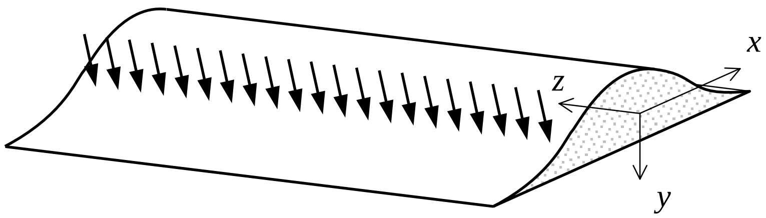


Fig. 3.2: In a straight dike or dam, the own weight or a line-load causes a state of plane strain.

Each point (x, y) of an in-plane loaded plate experiences a displacement $u_x(x, y)$ in x-direction and $u_y(x, y)$ in y-direction. So, the displacement field is completely determined by two degrees of freedom. This means that in these two directions (distributed) external loads p_x and p_y per unit of area can be applied.
Internally the plate experiences three deformations, a specific strain ε_{xx} in x-direction, a specific strain ε_{yy} in y-direction, and a specific shear deformation γ_{xy}. With those strains the stresses σ_{xx}, σ_{yy} and σ_{xy} are associated, respectively.

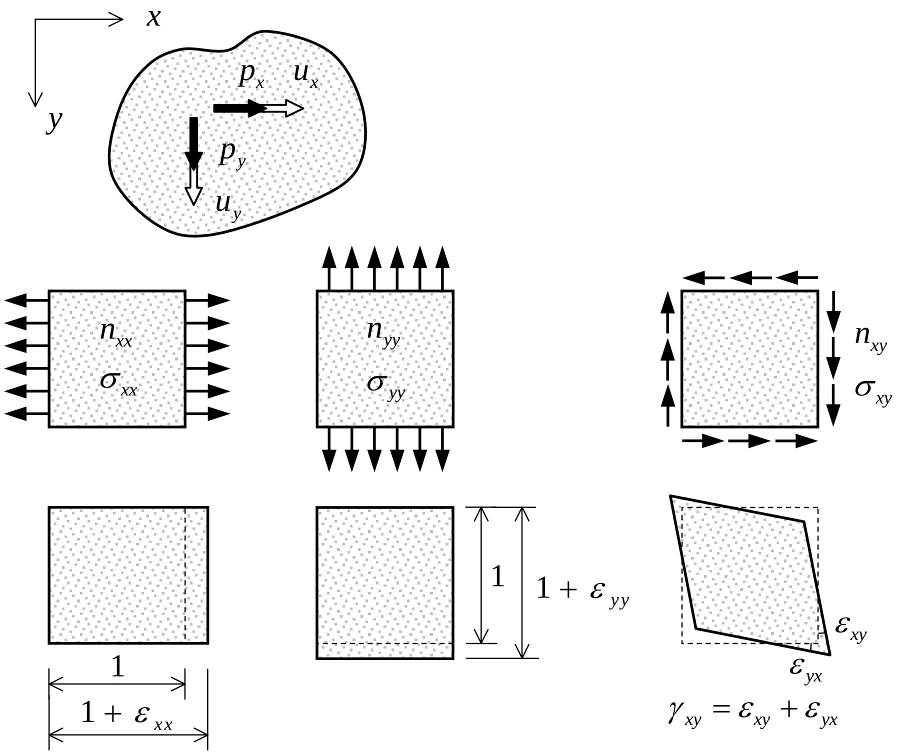


Fig. 3.3: Quantities which play a role in a plate loaded in-plane; the quantities drawn are positive.

The used sign convention reads: a stress component is *positive* when it acts in *positive* coordinate-direction on a surface with outward pointing normal in *positive* coordinate-direction, or when it acts in *negative* coordinate-direction on a surface with outward pointing normal in *negative* coordinate-direction (see Fig. 3.3).
It is customary, especially for plates loaded in-plane, to multiply the stresses by the plate thickness t. In this way, extensional forces n_{xx}, n_{yy} and n_{xy} are obtained, which are the stress resultants per unit of plate width having the dimension of force per unit length. The usual basic relations can be formulated between the previously mentioned external quantities and the internal quantities as shown in Fig. 3.4.

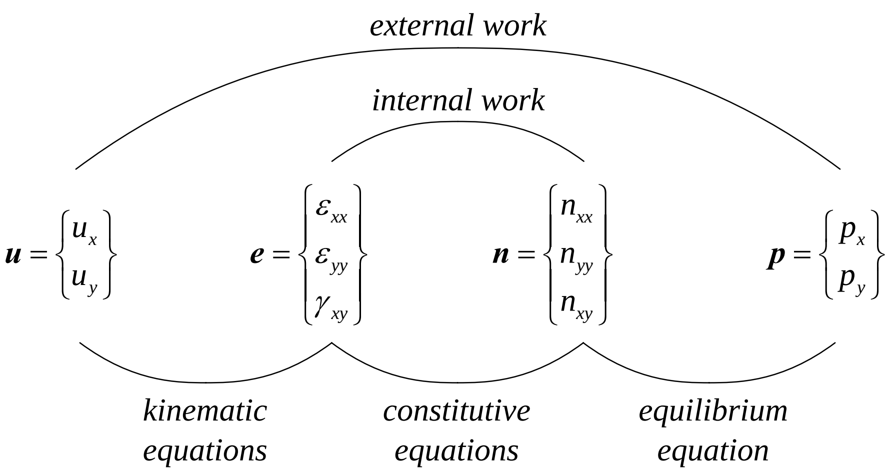


Fig. 3.4: Diagram displaying the relations between the quantities playing a role in the analysis of a plate in extension.

Since three stresses and only two load components (i.e. equilibrium equations) are present, the in-plane loaded plate is statically indeterminate. So, there is also one deformation more than displacements, which means only one compatibility equation has to be formulated.

3.1 Basic equations

The three categories of basic equations will be formulated in the order: kinematic equations, constitutive equations and equilibrium equations. The first and third categories are identical for plane stress and plane strain. However, the constitutive equations are different.

Kinematic equations
Taking the definitions of Fig. 3.3 into account, from Fig. 3.5 the following relations can be obtained:

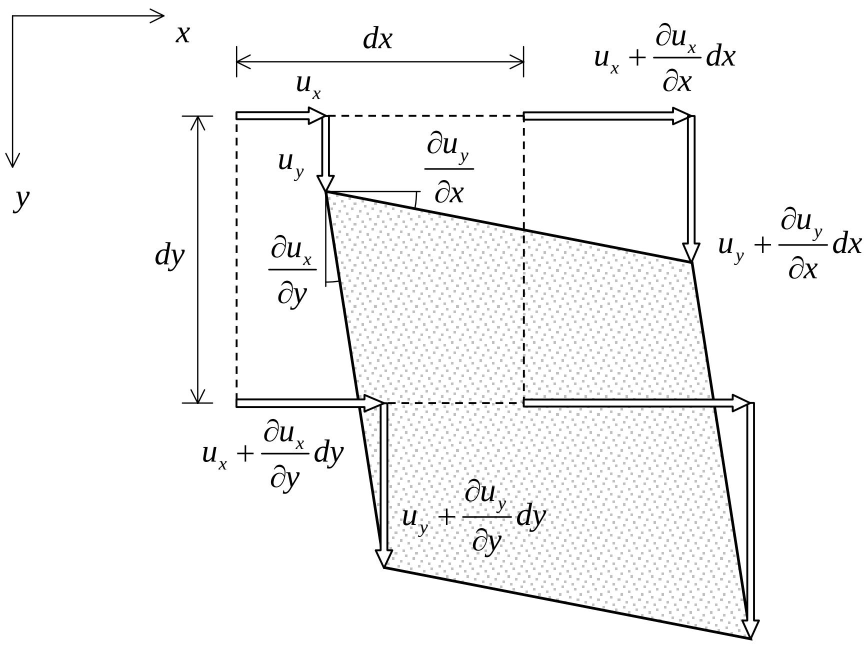


Fig. 3.5: Deformed state of an elementary plate part.

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{aligned}} \qquad \text{(\textit{kinematic equations})} \qquad (3.1)$$

Constitutive equations
For the explanation of the difference between plane stress and plane strain it is necessary to depart from Hooke's law for a linear-elastic material in its most general formulation (see Fig. 3.6).

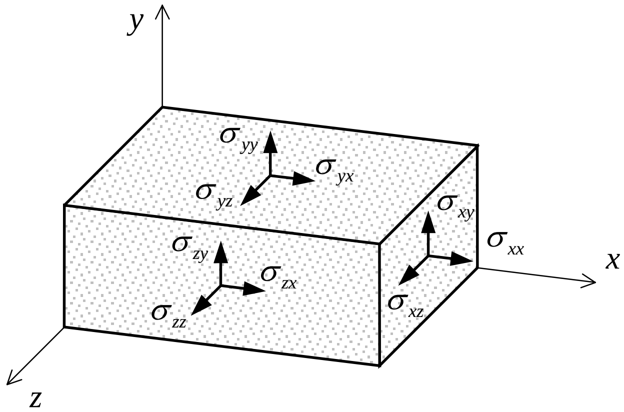


Fig. 3.6: Stresses in three dimensions.

Between the six stresses and six deformations, which may appear in the three-dimensional continuum, the following relations are valid:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\upsilon}{E}\left(\sigma_{yy} + \sigma_{zz}\right) \\
\varepsilon_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\upsilon}{E}\left(\sigma_{zz} + \sigma_{xx}\right) \\
\varepsilon_{zz} &= \frac{1}{E}\sigma_{zz} - \frac{\upsilon}{E}\left(\sigma_{xx} + \sigma_{yy}\right)
\end{aligned} \qquad (3.2)$$

and:

$$\begin{aligned}
\gamma_{xy} &= \frac{2\left(1+\upsilon\right)}{E}\sigma_{xy} \\
\gamma_{yz} &= \frac{2\left(1+\upsilon\right)}{E}\sigma_{yz} \\
\gamma_{zx} &= \frac{2\left(1+\upsilon\right)}{E}\sigma_{zx}
\end{aligned} \qquad (3.3)$$

where E is the modulus of Elasticity and υ Poisson's ratio. From the relations for the shear strains only the first one is relevant for a plate loaded in-plane:

$$\gamma_{xy} = \frac{2\left(1+\upsilon\right)}{E}\sigma_{xy} \qquad (3.4)$$

This directly follows from the definition of such a plate. From the relations for the normal strains the third one has to disappear and σ_{zz} has to be eliminated from the first two ones. This works out differently for the two stress states.

Plane stress
Considering the definition of plane stress, the stress σ_{zz} is zero and the first two equations in (3.2) become:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \upsilon\,\sigma_{yy}\right) \quad ; \quad \varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \upsilon\,\sigma_{xx}\right) \tag{3.5}$$

The third equation given by:

$$\varepsilon_{zz} = -\frac{\upsilon}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

demonstrates how the plate contracts perpendicular to its plane, which is caused by the in-plane stresses σ_{xx} and σ_{yy}. Actually, this is unnecessary information that will not be used. After introduction of the extensional forces n_{xx}, n_{yy} and n_{xy}, the equation (3.4) and (3.5) in matrix formulation read:

$$\boxed{\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E\,t}\begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix}\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix}} \quad (\textit{plane stress}) \tag{3.6}$$

This is the flexibility formulation of the constitutive relations:

$$\boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{n}$$

By inversion of (3.6) the stiffness formulation is found:

$$\boxed{\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \frac{E\,t}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}} \quad (\textit{plane stress}) \tag{3.7}$$

or briefly:

$$\boldsymbol{n} = \boldsymbol{D}\,\boldsymbol{e}$$

Plane strain

Now the corresponding constitutive laws will be given for the plane strain state. Due to the definition of such a plate the strain ε_{zz} is equal to zero. Then the third equation of (3.2) can be used to relate σ_{zz} to σ_{xx} and σ_{yy}:

$$\sigma_{zz} = \upsilon\left(\sigma_{xx} + \sigma_{yy}\right)$$

Substitution of this result into the first two equations of (3.2) provides:

$$\varepsilon_{xx} = \frac{1+\upsilon}{E}\left[(1-\upsilon)\,\sigma_{xx} - \upsilon\,\sigma_{yy}\right] \quad ; \quad \varepsilon_{yy} = \frac{1+\upsilon}{E}\left[(1-\upsilon)\,\sigma_{yy} - \upsilon\,\sigma_{xx}\right] \tag{3.8}$$

After introduction of the extensional forces, the matrix formulation of the relations (3.4) and (3.8) reads:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1+\upsilon}{E\,t} \begin{bmatrix} 1-\upsilon & -\upsilon & 0 \\ -\upsilon & 1-\upsilon & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad (\textit{plane strain}) \qquad (3.9)$$

which is the flexibility formulation of the plane strain state. In short it reads:

$$\boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{n}$$

Inversion of (3.9) again provides the stiffness formulation:

$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \frac{E\,t}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1-\upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & 0 \\ 0 & 0 & 1-2\upsilon/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad (\textit{plane strain}) \qquad (3.10)$$

or briefly:

$$\boldsymbol{n} = \boldsymbol{D}\,\boldsymbol{e}$$

Equilibrium equations
In the directions of both degrees of freedom u_x and u_y equilibrium equations can be formulated (see Fig. 3.7):

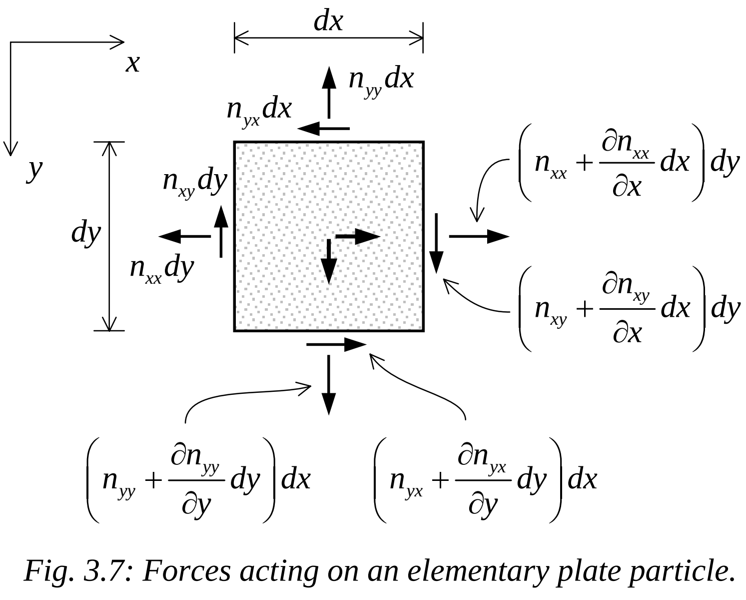


Fig. 3.7: Forces acting on an elementary plate particle.

$$\boxed{\begin{aligned}\frac{\partial n_{xx}}{\partial x}+\frac{\partial n_{yx}}{\partial y}+p_x=0\\ \frac{\partial n_{xy}}{\partial x}+\frac{\partial n_{yy}}{\partial y}+p_y=0\end{aligned}}\qquad (\textit{equilibrium equations}) \qquad (3.11)$$

All basic equation are now determined.

Set of basic equations
Similarly to the procedures in chapter 2, the kinematic equations (3.1) and the equilibrium equations (3.11) will be rephrased by introduction of the differential operators given by:

$$\mathcal{B}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix} \quad ; \quad \mathcal{B}'=\begin{bmatrix}-\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial y}\\ 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x}\end{bmatrix}$$

The basic equation can now be rewritten as:

$$\begin{aligned}&\boldsymbol{e}=\mathcal{B}\,\boldsymbol{u} && (\textit{kinematical equations})\\ &\boldsymbol{n}=\boldsymbol{D}\,\boldsymbol{e} \quad \text{or} \quad \boldsymbol{e}=\boldsymbol{C}\,\boldsymbol{n} && (\textit{constitutive equations}) \qquad (3.12)\\ &\mathcal{B}'\,\boldsymbol{n}=\boldsymbol{p} && (\textit{equilibrium equations})\end{aligned}$$

where $\boldsymbol{p}$ contains the components p_x and p_y. Just like in chapter 2, it can be seen that $\mathcal{B}'$ can be found by transposition of $\mathcal{B}$ while at the same time all differentiations of uneven order are changed of sign. In this case all non-zero terms are affected by the sign change.

Remark 1
On close inspection, the basic equations are only approximately valid for a plate subjected to plane stress. In such a plate, a strain ε_{zz} different from zero can be generated and therefore also a displacement u_z. Because n_{xx} and n_{yy} may vary across the plane of the plate, the strain ε_{zz} and displacement u_z are generally functions of x and y. Therefore, after deformation the thickness of the plate is not constant anymore. This means that the shear strains ε_{xz} and ε_{yz} are zero only at the centre plane of the plate, but may vary outside this plane. As a consequence, after deformation flat cross-sections will not be exactly flat anymore. For sufficiently small plate thicknesses all these effects can be neglected (for a more strict derivation it is referred to Sokolnikoff and Love).

Remark 2
Without explicitly mentioning, the derivations of the basic equations are valid for homogeneous isotropic plates. The theory of plates is also used for homogeneous orthotropic plates or structures that can be considered as such (see Fig. 3.8).
Then it more generally holds:

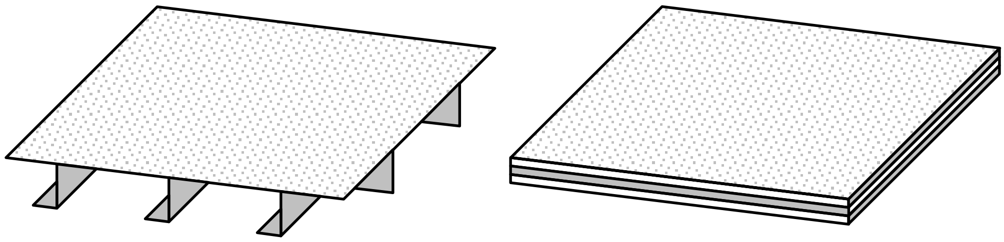


Fig. 3.8: Orthotropic plates.

$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \begin{bmatrix} D_{xx} & D_{\upsilon} & 0 \\ D_{\upsilon} & D_{yy} & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad \text{or} \quad \boldsymbol{n} = \boldsymbol{D}\,\boldsymbol{e} \tag{3.13}$$

The coefficients of the stiffness matrix have to be determined judiciously from the geometry and composition of the considered plate

3.2 Application of the force method

The force method will be elaborated for plates in a state of plane stress, which are solely subjected to loads along the perimeter. The first step in the force method is the creation of solution that satisfies the equilibrium equations, but which still contains the redundants $\boldsymbol{\phi}$. In this case only one redundant is present, because there are three unknown stress resultants n_{xx}, n_{yy} and n_{xy} in two equilibrium equations. Since the redundant is a stress quantity that is a function of the coordinates x and y, it is called a *stress function*. This function is (again) indicated by ϕ and can be determined from the compatibility condition. The compatibility condition is obtained from the three kinematic equations by elimination of the two displacements.

First step
From the equilibrium equations given by ($p_x = p_y = 0$):

$$\frac{\partial n_{xx}}{\partial x} + \frac{\partial n_{yx}}{\partial y} = 0 \quad ; \quad \frac{\partial n_{xy}}{\partial x} + \frac{\partial n_{yy}}{\partial y} = 0$$

a solution is created for the extensional forces. It turns out that the stress function has to be chosen such that it holds:

$$\boxed{n_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad ; \quad n_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad ; \quad n_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}} \tag{3.14}$$

It simply can be verified that this solution satisfies the equilibrium equations.

Second step
The solution for the extensional forces is substituted in the constitutive equations in flexibility formulation. This provides:

$$\varepsilon_{xx} = \frac{1}{E\,t}\left(\frac{\partial^2\phi}{\partial y^2} - \upsilon\frac{\partial^2\phi}{\partial x^2}\right) \quad ; \quad \varepsilon_{yy} = \frac{1}{E\,t}\left(\frac{\partial^2\phi}{\partial x^2} - \upsilon\frac{\partial^2\phi}{\partial y^2}\right) \quad ; \quad \gamma_{xy} = -\frac{1}{G\,t}\frac{\partial^2\phi}{\partial x \partial y} \tag{3.15}$$

where G is the shear modulus given by:

$$G = \frac{E}{2(1+\upsilon)}$$

Third step
In this step, the strains will be substituted in the compatibility condition. But first the compatibility condition will be derived from the kinematic equations (3.1). The first equation is differentiated twice with respect to y. The second one is differentiated twice with respect to x and the third equation once with respect to x and once with respect to y. In the third equation, the signs are changed too. Addition of all these resulting relations then leads to:

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - \gamma_{xy,xy} = 0 \qquad (\textit{compatibility condition}) \tag{3.16}$$

In the right-hand side, the displacements have disappeared, which means that (3.16) is the required compatibility condition between the three deformations. The fact that such a relation exists means that not all arbitrarily chosen combinations of strains are possible. The condition

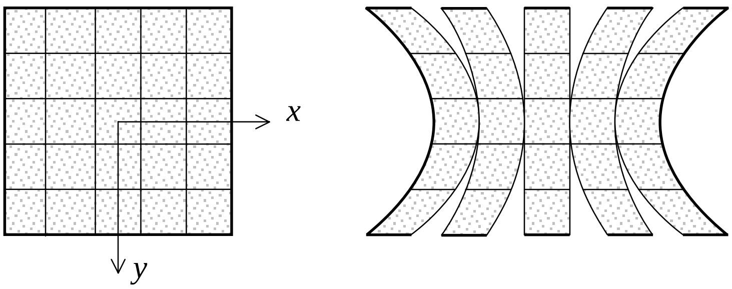


Fig. 3.9: Example of a non-compatible combination of strains.

prevents that gaps and overlappings are created. Fig. 3.9 (borrowed from Koiter) shows a combination of strains that does not satisfy the compatibility condition.

Differential equation
The substitution of the three equations (3.15) into the compatibility condition (3.16) finally leads to the required differential equation for the stress function ϕ:

$$\boxed{\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0} \tag{3.17}$$

Remark 1
It turns out that the modulus of elasticity E and Poisson's ratio υ do not appear in the differential equation.

Remark 2
In this case, the differential equation has been derived for the plane stress case. The same differential equation can be derived for a state of plane strain.

Remark 3
The differential equation can also be written as:

$$\nabla^2\nabla^2\phi = 0 \quad \text{or} \quad \nabla^4\phi = 0 \tag{3.18}$$

where ∇^2 is the Laplace-operator given by:

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

The British astronomer G.B. Airy was the first one who derived this biharmonic equation in 1862. Therefore this stress function ϕ is called the *function of Airy*.

Remark 4
Finally, the analogy will be demonstrated with the chapters 1 and 2 by presenting the solution for the stress resultants (3.14) and the compatibility equation (3.16) in operator notation. For that purpose the following operators are introduced:

$$\boldsymbol{\mathcal{P}} = \begin{Bmatrix} \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x^2} \\ -\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} \quad ; \quad \boldsymbol{\mathcal{P}}' = \begin{Bmatrix} \dfrac{\partial^2}{\partial y^2} & \dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix}$$

The relations (3.14) and (3.16) then become:

$$\boldsymbol{n} = \boldsymbol{\mathcal{P}}\phi \quad ; \quad \boldsymbol{\mathcal{P}}'\boldsymbol{e} = 0 \tag{3.19}$$

Because the differences in $\boldsymbol{\mathcal{P}}'$ are all of the second order, this operator is exactly the same as the transposed of $\boldsymbol{\mathcal{P}}$. The biharmonic equation can now be rewritten as:

$$\boldsymbol{\mathcal{P}}'\boldsymbol{C}\,\boldsymbol{\mathcal{P}} = 0$$

where $\boldsymbol{C}$ is the matrix defined in equation (3.6). This can be verified by working out the matrix multiplication. The right-hand side is zero because no distributed loads p_x and p_y are present.

Remark 5
For plane strain exactly the same differential equation is obtained. Therefore, in plane stress and plane strain the forces n_{xx}, n_{yy} and n_{xy} are the same when the stresses on the edges are the same. The deformations may be different.

Possibilities for solutions
In the nineteenth century, solutions for the function of Airy have been found for a number of characteristic problems. Typical for these problems is the condition that along the total circumference of the plate the magnitude of the stresses has to be known. When prescribed displacements are present, the use of the function of Airy is not very advisable. Names of researchers such as Boussinesq, Flamant, Biot, Filon and Lamé are attached to the solutions they found. In these lecture notes a number of not to complicated solutions will be discussed.

In the preface of these lecture notes, it already has been mentioned that nowadays the possibility exists to determine approximated computer solutions for almost every elasticity problem. The Finite Element Method (FEM) is the most elegant method for that purpose. The FEM is based on the displacement method.

3.3 Solutions in the form of polynomials

An example of a class of solutions of the biharmonic differential equation $\nabla^2\nabla^2\phi = 0$ is formed by polynomials. It simply can be confirmed that each polynomial of the third order or lower satisfies the equation. Polynomials of higher order only satisfy the equation if certain specific linear relations between the coefficients exist. Polynomials can be used for inverse solution methods. For the displacement method, this already has been applied in the course Elastic Plates. This means that one selects a solution and tries to find the corresponding problem. It will be clear that the applicability of this method is very restricted. Sometimes it is possible to find a semi-inverse solution for a problem. Then a stress function is chosen that still contains a number of undetermined coefficients. By trying to satisfy the boundary conditions, these coefficients are solved. Examples of solutions with polynomials can be obtained from literature, for example from Timoshenko & Goodier or from Biezeno & Grammel.
Two elementary examples will be discussed.

Example 1
Suppose that the stress function reads:

$$\phi = \frac{1}{2}\sigma_1 y^2 + \frac{1}{2}\sigma_2 x^2$$

Substitution of this relation into (3.17) indeed shows that this is a useful stress function. When no distributed surface load is present, the stresses in a plate of unit thickness are:

$$\sigma_{xx} = \sigma_1 \quad ; \quad \sigma_{yy} = \sigma_2 \quad ; \quad \sigma_{xy} = 0$$

This is a uniform stress distribution that is generated in a plate, which is uniformly loaded at the edges as shown in Fig. 3.10a.

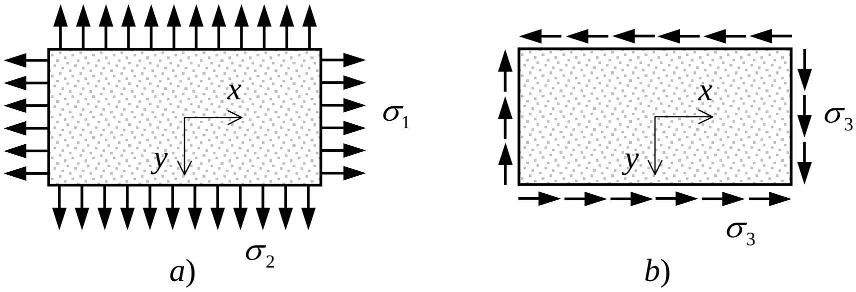


Fig. 3.10: Plate subjected to constant stresses.

A uniformly distributed shear stress $\sigma_{xy} = \sigma_3$ can be described by the following stress function (see Fig. 3.10b):

$$\phi = -\sigma_3 xy$$

Example 2
This example was previously discussed by the displacement method in the course Elastic Plates.

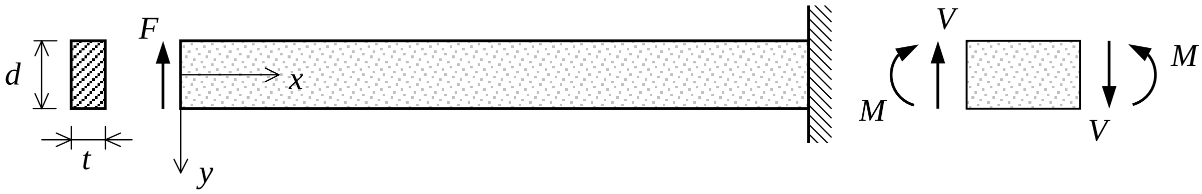


Fig. 3.11: Overhanging beam subjected to a point load.

The solution of the problem of a cantilever (height d and thickness t) loaded by a point load at the free end (see Fig. 3.11) has to be found from the following class of functions:

$$\phi = \left(C_1 xy + C_2 xy^3\right) t$$

Substitution of this relation into the differential equation shows that for all values of C_1 and C_2 the differential equation is satisfied. The corresponding stresses are:

$$\sigma_{xx} = 6C_2\, x\, y \qquad \textit{(bi-linear)}$$
$$\sigma_{yy} = 0$$
$$\sigma_{xy} = -C_1 - 3C_2\, y^2 \qquad \textit{(parabolic)}$$

The top and bottom side of the beam have to be stress free, so the shear stress σ_{xy} at those surfaces should be zero, i.e.:

$$y = \pm\frac{1}{2}d \quad \rightarrow \quad \sigma_{xy} = -C_1 - \frac{3}{4}C_2\, d^2 = 0 \quad \rightarrow \quad C_1 = -\frac{3}{4}C_2\, d^2$$

Substitution of this result into the relation for σ_{xy} provides the well-known parabolic distribution across the height of the beam:

$$\sigma_{xy} = 3C_2\left(\frac{1}{4}d^2 - y^2\right)$$

The resultant of this distribution must exactly be equal to F . Therefore:

$$F = \int_{-\frac{1}{2}d}^{+\frac{1}{2}d} \sigma_{xy}\, t\,\mathrm{d}y = \frac{1}{2}C_2\, t\, d^3 \quad \rightarrow \quad \begin{cases} C_2 = \dfrac{2F}{t\,d^3} \\ C_1 = -\dfrac{3}{2}\dfrac{F}{t\,d} \end{cases}$$

The stress function becomes:

$$\phi = -\frac{F}{d^3}\left(\frac{3}{2}xyd^2 - 2xy^3\right)$$

The corresponding stresses are:

$$\sigma_{xx} = \frac{1}{t}\frac{\partial^2\phi}{\partial y^2} = \frac{12F\,x\,y}{t\,d^3}$$

$$\sigma_{yy} = \frac{1}{t}\frac{\partial^2\phi}{\partial x^2} = 0$$

$$\sigma_{xy} = -\frac{1}{t}\frac{\partial^2\phi}{\partial x\partial y} = \frac{F}{t\,d^3}\left(\frac{3}{2}d^2 - 6y^2\right) = \frac{3}{2}\frac{F}{t\,d}\left(1 - \frac{4y^2}{d^2}\right)$$

For easy recognition, the expressions for the stresses are rewritten somewhat differently. The moment of inertia I , the cross-sectional area A and the bending moment M acting on the cross-section are introduced. They are given by:

$$I = \frac{1}{12}t\,d^3 \quad ; \quad A = t\,d \quad ; \quad M = F\,x$$

Now, the expressions for the stresses change into:

$$\sigma_{xx} = y\frac{Fx}{I} = y\frac{M}{I} \quad ; \quad \sigma_{yy} = 0 \quad ; \quad \sigma_{xy} = \frac{3}{2}\left(1 - \frac{4y^2}{d^2}\right)\frac{F}{A} \qquad (3.20)$$

Remark 1
The stress distribution (3.20) can be found by the elementary beam theory too, which is based on the hypothesis of Bernoulli stating that “plane cross-section surfaces remain plane”. The distribution of σ_{xx} is linear over the height and σ_{xy} is distributed parabolically.

Remark 2
The boundary conditions at the top and bottom side of the beam are satisfied exactly. The boundary condition at the left end is only satisfied if the point load acts through a parabolic shear stress distribution. When for example the point load acts as a concentrated force in a point of the cross-section, the stress distribution will deviate from the one provided in (3.20). However, according to the principle of De Saint-Venant the disturbance will be confined to the neighbourhood of the end of the beam.

> *The principle of De Saint-Venant reads*:
> When the forces that are acting on a small part of the surface of an elastic body are replaced by another statically equivalent system of forces, locally large changes in the stress-state are introduced; however, at a distance, which is large compared to the linear dimensions of the area that is affected by the changed forces, the influence is negligible (see Fig. 3.12).

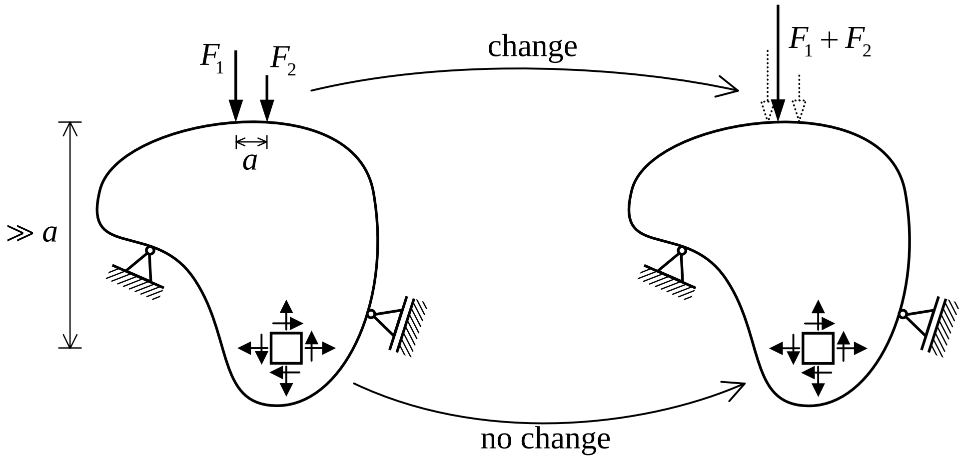


Fig. 3.12: Principle of De Saint Venant.

Remark 3
The boundary condition at the restrained end is not yet considered. It would be a large coincidence when at that position the boundary conditions were satisfied exactly. Since the clamping refers to a prevention of displacements, they first have to be determined. This will be explained below.

Continuation of example 2
The displacements can be found by integration of the expressions for the deformations. A complication is created by the fact that the deformations are found by *partial* differentiation of the displacements. By the inverse process of integration with respect to one of the variables, no integration constants are introduced but functions depending on the other variable(s).
Because $\sigma_{yy} = \sigma_{zz} = 0$ (a state of plane stress is assumed), it holds:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\sigma_{xx}}{E} = \frac{Fxy}{EI}$$

from which after integration it follows;

$$u_x = \frac{Fx^2 y}{2EI} + f(y) \tag{3.21}$$

where $f(y)$ is an undetermined function depending on y only.
An initial expression for the vertical displacement u_y is found by integration of the deformation ε_{yy}, i.e.:

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = -\upsilon \frac{\sigma_{xx}}{E} = -\upsilon \frac{Fxy}{EI} \quad \rightarrow \quad u_y = -\upsilon \frac{fxy^2}{2EI} + g(x) \tag{3.22}$$

where $g(x)$ is a still unknown function.
Now, the expressions for both displacement components u_x and u_y are found, but they still contain the to be determined functions $f(y)$ and $g(x)$. These functions can be found because the required extra data is at hand. The displacements u_x and u_y have to be in agreement with the expression for the shear stresses σ_{xy}. From (3.21) and (3.22) it follows:

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{F x^2}{2EI} - \frac{\upsilon F y^2}{2EI} + \frac{d f(y)}{dy} + \frac{dg(x)}{dx}$$

On the other hand, from Hooke's law and the last expression of (3.20) it yields:

$$\gamma_{xy} = \frac{\sigma_{xy}}{G} = \frac{3}{2}\left(1 - \frac{4y^2}{d^2}\right)\frac{F}{GA}$$

Equating the two expressions for γ_{xy} delivers:

$$\frac{Fx^2}{2EI} - \frac{\upsilon F y^2}{2EI} + \frac{d f(y)}{dy} + \frac{dg(x)}{dx} = \frac{3}{2}\left(1 - \frac{4y^2}{d^2}\right)\frac{F}{GA}$$

In this equation, terms appear that only depend on x (for example the first one and $d\,g(x)/dx$), terms that only depend on y, and a constant term (the first term of the right-hand side). The equality can be satisfied only if the sum of all terms that depend on x is a constant, just like for the terms that depend on y. The constants must have such a value that the equation is just satisfied. Therefore, it should hold:

$$\frac{df}{dy} = \frac{\upsilon F y^2}{2EI} - 6\frac{y^2}{d^2}\frac{F}{GA} + C_3 \quad ; \quad \frac{dg}{dx} = -\frac{F x^2}{2EI} + C_4$$

The constants C_3 and C_4 are interdependent, because it should hold:

$$C_3 + C_4 = \frac{3}{2}\frac{F}{GA} \quad \rightarrow \quad C_3 = \frac{3}{2}\frac{F}{GA} - C_4$$

So, the derivatives of the functions become:

$$\frac{df}{dy} = \frac{\upsilon F y^2}{2EI} + \frac{3}{2}\left(1 - \frac{4y^2}{d^2}\right)\frac{F}{GA} - C_4 \quad ; \quad \frac{dg}{dx} = -\frac{F x^2}{2EI} + C_4$$

Now, the functions $f(y)$ and $g(x)$ can be determined by ordinary integration:

$$f(y) = \frac{\upsilon F y^3}{6EI} + \frac{3}{2}\left(y - \frac{4y^3}{3d^2}\right)\frac{F}{GA} - C_4 y + C_5 \quad ; \quad g(x) = -\frac{F x^3}{6EI} + C_4 x + C_6$$

where C_5 and C_6 are real integration constants.
The found expressions for $f(y)$ and $g(x)$ are now substituted in the expressions (3.21) and (3.22) for the displacements u_x and u_y:

$$\begin{aligned} u_x &= \underbrace{\frac{F x^2 y}{2EI} + \frac{\upsilon F y^3}{6EI}}_{} \quad \underbrace{+\frac{3}{2}\left(y - \frac{4y^3}{3d^2}\right)\frac{F}{GA}}_{} \quad \underbrace{-C_4 y + C_5}_{} \\ u_y &= \underbrace{-\frac{F x^3}{6EI} - \frac{\upsilon F x y^2}{2EI}}_{\text{bending contribution}} \qquad\qquad \underbrace{\qquad\qquad}_{\text{shear contribution}} \quad \underbrace{+C_4 x + C_6}_{\text{rigid body movement}} \end{aligned} \tag{3.23}$$

These expressions clearly show which parts are contributed by the deformations caused by the bending moment and the shear force, respectively. Apart from that, there is also a contribution describing a rigid body movement, which is not associated with any development of stresses. This rigid body movement describes a pure rotation about the origin of the chosen coordinate system (C_4), and two translations (C_5 and C_6).

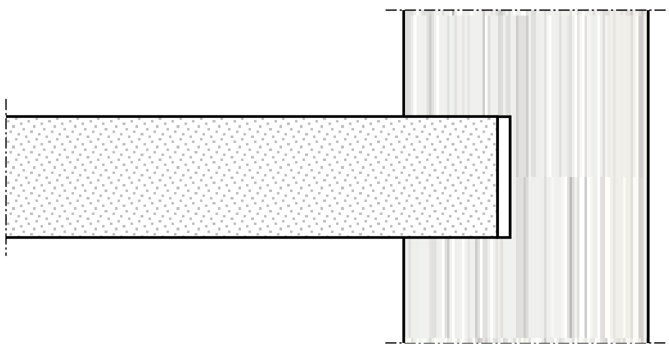


Fig. 3.13: Beam with suppressed rotation and vertical displacement at fixed end.

The value of the three constants follows from the boundary conditions at the constrained end. When the requirement is imposed that the displacements u_x and u_y of the beam-axis are zero and the beam-axis is horizontal (see Fig. 3.13), i.e. when:

$$x = l \quad \text{and} \quad y = 0 \quad \rightarrow \quad \begin{cases} u_x = 0 \\ u_y = 0 \\ \dfrac{\partial u_y}{\partial x} = 0 \end{cases}$$

for the constants it follows:

$$C_4 = \frac{Fl^2}{2EI} \quad ; \quad C_5 = 0 \quad ; \quad C_6 = \frac{Fl^3}{3EI}$$

The displacements (3.23) then become:

$$\begin{aligned} u_x &= \underbrace{\frac{F\left(x^2 - l^2\right)y}{2EI} + \upsilon \frac{F y^3}{6EI}}_{\text{bending contribution}} \quad && \underbrace{+\frac{3}{2}\left(y - \frac{4y^3}{3d^2}\right)\frac{F}{GA}}_{\text{shear contribution}} \\ u_y &= \frac{-F\left(x^3 - 3xl^2 + 2l^3\right)}{6EI} - \upsilon \frac{F x^2 y}{2EI} && \end{aligned} \qquad (3.24)$$

For the free end at $x = 0$, the vertical displacement of the beam-axis ($y = 0$) equals:

$$u_y = -\frac{Fl^3}{3EI}$$

This is exactly the same result as obtained from the elementary beam theory.
The rotation according to the elementary beam theory is found as well. For a beam without shear deformation this rotation is:

$$\frac{\partial u_x}{\partial y} = -\frac{Fl^2}{2EI}$$

This can be found from (3.24) too, with $x = 0$, $y = 0$ and GA infinitely large.
However more information can be obtained from (3.24). The elementary beam theory seems to be an approximation for the real stress-state. In both u_x and u_y a contribution is present depending on Poisson's ratio υ. When the value of υ is zero the terms disappear. When υ is different from zero, the terms have a value outside the beam-axis. They never have any influence on the values on the beam-axis, but can be interpreted as a relative displacement with respect to the beam-axis.
More important is the contribution to u_x caused by the shear deformation, because also for $\upsilon = 0$ this term has a value. For the considered restrained end as shown in Fig. 3.13 the beam deforms as depicted in Fig. 3.14.

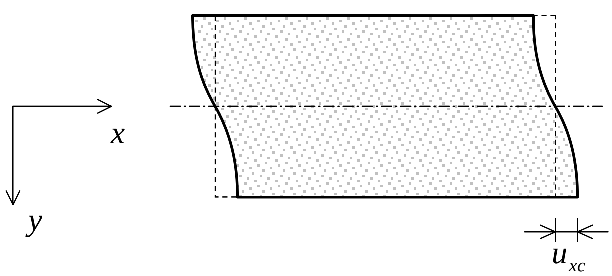


Fig. 3.14: Deformation caused by shear stresses.

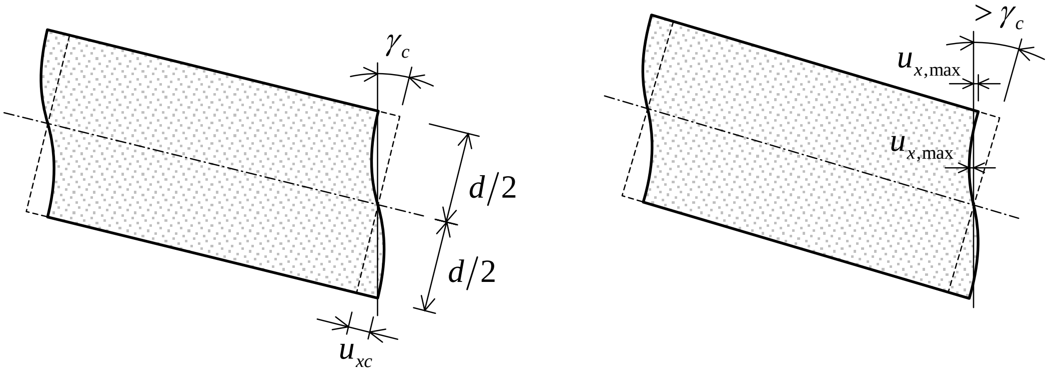
The displacement in x-direction at the restrained end equals ($x = l$, $\upsilon = 0$):

$$u_x = \frac{3}{2}\left(y - \frac{4y^3}{3d^2}\right)\frac{F}{GA}$$

The largest value u_{xc} occurs for $y = d/2$ and equals:

$$u_{xc} = \frac{F}{GA}\frac{d}{2}$$

This is just the value that also would have occurred if the shear stress distribution was *constant* over the cross-section.
When the beam would have been restrained in a different way another situation is created. For the case that the beam is glued against an undeformable block, the boundary conditions become $u_x = 0$ and $u_y = 0$ for all values of y between $-d/2$ and $+d/2$. This boundary condition cannot be satisfied for the given stress function. At best, the boundary condition can be satisfied approximately, for example by requiring that the displacement u_x is only zero at the extreme fibres ($y = -d/2$ and $y = +d/2$). This can be achieved by introducing a rotation on the previously described fixing method (see Fig. 3.15a).



a) u_x is zero at neutral line and extreme fibres *b) $u_{x,\max}$ is minimised*

Fig. 3.15: Rotation at the fixed end for incorporation of the deformation caused by the shear force.

The angle of rotation in this case is:

$$\gamma_c = \frac{u_{xc}}{\frac{1}{2}d}$$

The subscript "*c*" is used again, because this rotation would also occur for constant shear stresses across the cross-section. This directly follows from the elimination of u_{xc} from the last to equations. It then follows $\gamma_c = F/GA \rightarrow F/A = G\gamma_c$
A better approximation is obtained by introducing a rotation such that the maximum value of the horizontal displacement u_x is minimised over the full height of the cross-section. This is achieved for a rotation that is larger than γ_c (see Fig. 3.15b, $u_{x,\max}$ has been made as small as possible). This state corresponds with a shape factor η, which is larger than 1. For the

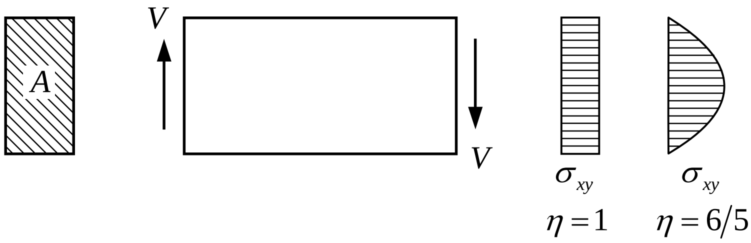


Fig. 3.16: Shape factor for a beam subjected to shear loading.

rectangular cross-section this shape factor equals $\eta = 6/5$, the meaning of which will be explained below.
Fig. 3.16 shows a rectangular beam subjected to a shear force V. With $\sigma_{xy} = G\gamma$, for a uniform shear-stress distribution, the following shear force can be calculated:

$$V = A\sigma_{xy} = GA\gamma$$

For a non-uniform shear-stress distribution, the shape factor can be introduced to obtain:

$$V = GA_s\gamma \quad ; \quad A_s = A/\eta$$

When the shear-stress distribution is parabolic, the shape factor becomes:

$$\eta = \frac{6}{5}$$

Thus for the case of minimising the horizontal displacement at the restrained end over the full height of the cross-section, the corresponding vertical displacement at the free end becomes:

$$u_y = -\frac{F\,l^3}{3EI} - \eta\,\frac{F\,l}{GA} = -\frac{F\,l^3}{3EI}\left\{1 + \frac{1}{2}\,\eta(1+\upsilon)\left(\frac{d}{l}\right)^2\right\} \tag{3.25}$$

From this expression, it easily can be seen that the deformation caused by the shear force can be neglected if $d \ll l$.

Remark
Compared to the elementary beam theory only little improvement has been obtained with respect to the most important stresses and displacements (of the neutral line). The only improvement lies in the understanding of the applicability of the elementary beam theory and its errors. Moreover, a complete consistent system of stresses and displacements is found. The horizontal displacement appears not to be a linear function of the coordinate in height direction (thus, plane cross-sections do not remain plane at all), but the stresses σ_{xx} however are just like in the elementary beam theory linearly distributed across the height.

3.4 Solution for a deep beam

The deep beam as shown in Fig. 3.17 is loaded at the bottom by a distributed sine-shaped load $f(x)$. This example was already discussed with the displacement method in the course

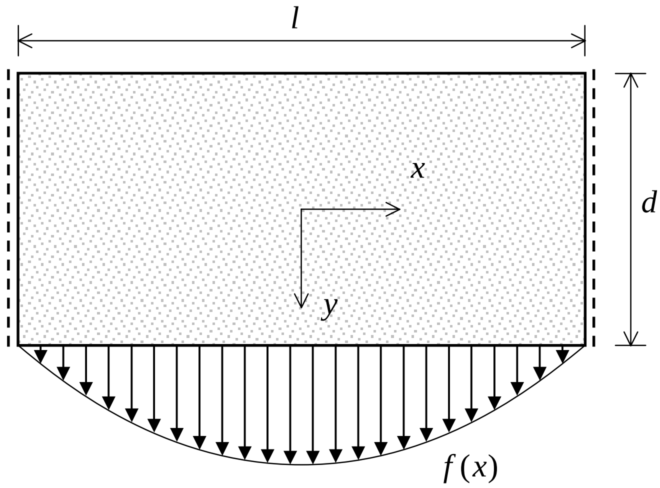


Fig. 3.17: Deep beam loaded at bottom edge.

Elastic Plates. The beam is supported in such a manner that the displacements at the left and right ends are prohibited. The aim of the exercise is to determine the stress distribution of σ_{xx} over the middle cross-section of the high beam (the cross-section $x = 0$).
For the load it can be written:

$$f(x) = f\cos(\alpha x)$$

where f is the largest value and α is equal to:

$$\alpha = \frac{\pi}{l}$$

In this case the stress function of Airy can be written as the product of two functions, of which one is only depending on x and the other one only depending on y. For the first one the same distribution as for the load is selected, namely $\cos(\alpha x)$. It can be shown that this choice leads to displacements that satisfy the boundary conditions at the beam-ends. The function in y-direction has to be selected such that the stress function satisfies the biharmonic equation of Airy and also satisfies the boundary conditions at the top and bottom of the beam. So, the stress function reads:

$$\phi(x, y) = g(y)\cos(\alpha x)$$

Substitution of this function into the biharmonic equation given by:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

delivers after division by $\cos(\alpha x)$:

$$\frac{\partial^4 g}{\partial y^4} - 2\alpha^2 \frac{\partial^2 g}{\partial y^2} + \alpha^4 g = 0$$

The advantage of this approach is that the *partial* biharmonic differential equation for $\phi(x,y)$ has been reduced to an *ordinary* fourth-order differential equation for $g(y)$.
The following trial solution is substituted:

$$g(y) = e^{ry}$$

leading to the following characteristic equation:

$$r^4 - 2\alpha^2 r^2 + \alpha^4 = 0$$

This equation can be factorised as:

$$\left(r^2 - \alpha^2\right)\left(r^2 - \alpha^2\right) = 0$$

The following four roots can be obtained:

$$r_1 = \alpha \quad ; \quad r_2 = \alpha \quad ; \quad r_3 = -\alpha \quad ; \quad r_4 = -\alpha$$

For each double root r, the terms e^{ry} and $r\,e^{ry}$ have to be included in the general solution. The general solution for $\phi(x,y)$ becomes:

$$\phi(x,y) = \underbrace{\left(C_1 e^{\alpha y} + C_2 \alpha y\, e^{\alpha y} + C_3 e^{-\alpha y} + C_4 \alpha y\, e^{-\alpha y}\right)}_{g(y)} \cos\left(\alpha x\right)$$

The four constants can be solved from the following four boundary conditions:

$$y = -\tfrac{1}{2}b \quad \rightarrow \quad \begin{cases} n_{yy} = 0 & \rightarrow \quad \phi,_{xx} = 0 \\ n_{xy} = 0 & \rightarrow \quad \phi,_{xy} = 0 \end{cases}$$

$$y = +\tfrac{1}{2}b \quad \rightarrow \quad \begin{cases} n_{yy} = f\ \cos(\alpha x) & \rightarrow \quad \phi,_{xx} = f\ \cos(\alpha x) \\ n_{xy} = 0 & \rightarrow \quad \phi,_{xy} = 0 \end{cases}$$

The straightforward calculation of the constants is left to the reader. After determination of the constants, the stress distribution can be derived from ϕ. For three special cases, the distribution of n_{xx} over the line $x = 0$ will be provided.
The expression for n_{xx} is:

$$n_{xx} = C_1 \alpha^2 e^{\alpha y} + C_2 \alpha^2 \left(2 + \alpha y\right) e^{\alpha y} + C_3 \alpha^2 e^{-\alpha y} + C_4 \alpha^2 \left(-2 + \alpha y\right) e^{-\alpha y}$$

case 1: $\boldsymbol{d/l \gg 1}$

In this case, the beam is a high wall or actually even an infinite half-space. The constants C_1 and C_2 have to be zero because the solution has to approach zero for large values of y.

case 2: $\boldsymbol{d/l \approx 1}$

In this case, it really concerns a short high beam. All four constants play a role and therefore all four terms $e^{\alpha y}$, $\alpha e^{\alpha y}$, $e^{-\alpha y}$ and $\alpha e^{-\alpha y}$ are important.

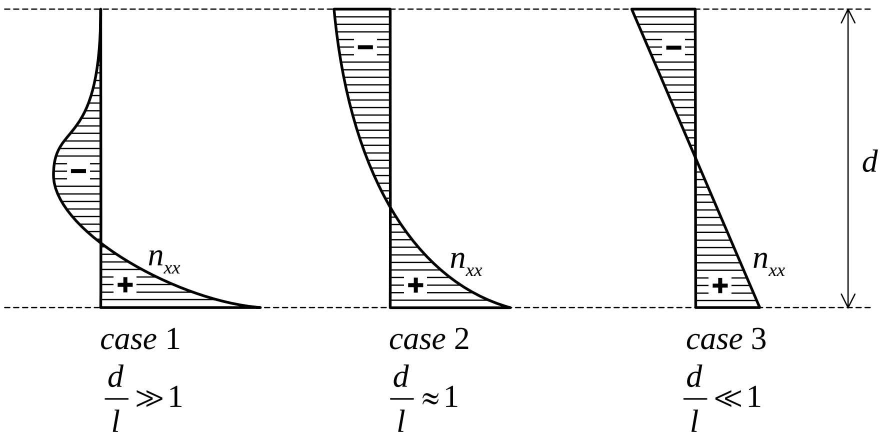


Fig. 3.18: Results for several values of the ratio of height d and span l.

***case 3:* $d/l \ll 1$**

In this case, the beam is slender and the classical beam theory applies, which will be shown below.

When $d/l \ll 1$ it also holds that $\alpha y \ll 1$. For these small values of αy the functions $e^{\alpha y}$, $\alpha e^{\alpha y}$, $e^{-\alpha y}$ and $\alpha e^{-\alpha y}$ can be approximated by a Taylor expansion around the point $y = 0$. Then the stress function $\phi(x, y)$ changes into:

$$\phi(x, y) = \left\{ \overline{C}_1 + \overline{C}_2 \alpha y + \overline{C}_3 (\alpha y)^2 + \overline{C}_4 (\alpha y)^3 + \cdots \right\} \cos(\alpha x)$$

For sufficiently small values of d/l the factor αy is so small compared to unity, that the first four terms will do. Thus, the function $g(y)$ reduces to a third-order polynomial in y. The direct consequence is that n_{xx} becomes a linear function of y (see Fig. 3.18) and n_{xy} a parabolic one. For n_{yy} the distribution is cubical in y. This is exactly the solution of the classical beam theory.

Practical application

The discussed academic case of a high wall can be used to make analytical estimations of the stress distribution in practical structures. An example is the calculation of the stress distribution in a silo wall on columns subjected to a uniformly distributed load as shown in Fig. 3.19. This load may be the dead weight or the downward directed friction forces due to silo action. In order to obtain a proper estimation of the horizontal stress between the two

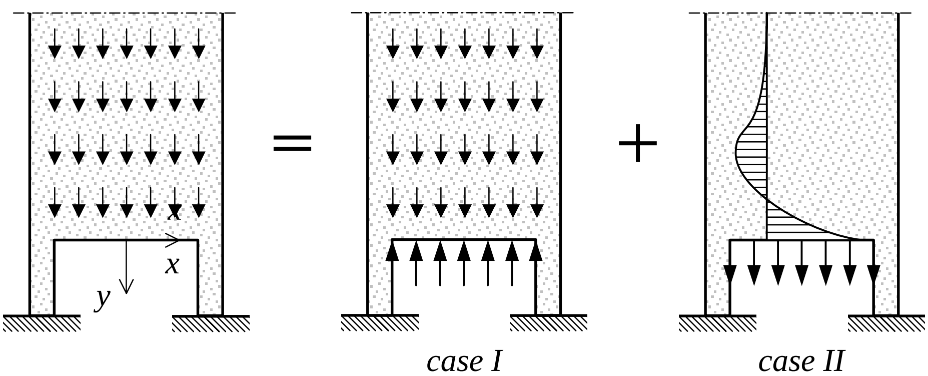


Fig. 3.19: Wall of silo on columns.

columns at the bottom of the wall the following approach is adopted. The load case is split up in a case *I* having a uniform stress distribution (without any significant horizontal stresses) and a case *II* for which the solution of the high wall can be used. The assumption for case *II* is that the distributed line load at the bottom edge of the wall can be estimated properly by the sine-shaped load.

3.5 Axisymmetry for plates subjected to extension

In this section, the special case of circular plates subjected to axisymmetric loads will be discussed. Fig. 3.20 shows such a plate with thickness t.

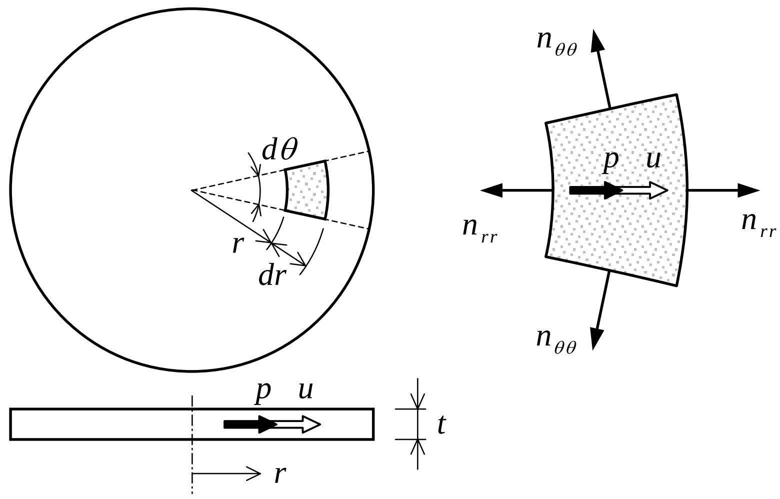


Fig. 3.20: Displacement, load and extensional forces in an axisymmetric plate problem.

For this type of problems it is handy to change to polar coordinates r and θ. Due to the symmetry condition only one degree of freedom is present, which is the displacement u that is independent from the tangential coordinate θ. The same property holds for the load p per unit of area. Only two extensional forces are present n_{rr} and $n_{\theta\theta}$. The shear stress $n_{r\theta}$ cannot occur. Therefore, only two strains exist ε_{rr} and $\varepsilon_{\theta\theta}$. The essential quantities are schematically displayed in Fig. 3.21.

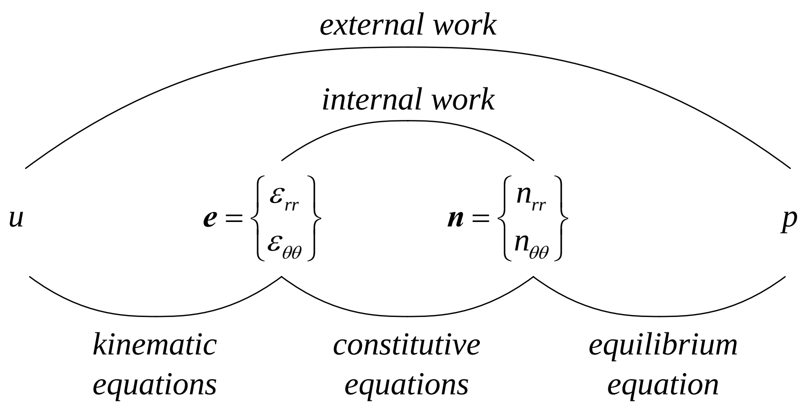


Fig. 3.21: Diagram displaying the relations between the quantities playing a role in the analysis of axisymmetric problems.

Kinematic equation

The strain ε_{rr} and the displacement u are both defined in r-direction. The well-known relation between the two reads:

$$\boxed{\varepsilon_{rr} = \frac{du}{dr}} \qquad (\textit{kinematic equation in radial direction}) \qquad (3.26)\text{a}$$

Ordinary derivatives can be used since u only depends on the coordinate r. For the derivation of the tangential strain $\varepsilon_{\theta\theta}$ a circle is considered with radius r. The circumference of this circle is $2\pi r$. After application of the axisymmetric load, each point of the circle displaces over a radial distance u. Then, the new radius of the circle is $r+u$ and the circumference $2\pi(r+u)$. The increase of the circumference is $2\pi(r+u)-2\pi r = 2\pi u$. Division of this result by the original length $2\pi r$ provides the required strain:

$$\boxed{\varepsilon_{\theta\theta} = \frac{u}{r}} \qquad (\textit{kinematic equation in tangential direction}) \qquad (3.26)\text{b}$$

Constitutive equations

The elaborations are carried out only for the plane stress case. The relations are:

$$\boxed{\begin{aligned} \varepsilon_{rr} &= \frac{1}{E\,t}\left(n_{rr} - \upsilon\, n_{\theta\theta}\right) \\ \varepsilon_{\theta\theta} &= \frac{1}{E\,t}\left(n_{\theta\theta} - \upsilon\, n_{rr}\right) \end{aligned}} \qquad (\textit{flexibility formulation}) \qquad (3.27)\text{a}$$

$$\boxed{\begin{aligned} n_{rr} &= \frac{E\,t}{1-\upsilon^2}\left(\varepsilon_{rr} + \upsilon\,\varepsilon_{\theta\theta}\right) \\ n_{\theta\theta} &= \frac{E\,t}{1-\upsilon^2}\left(\varepsilon_{\theta\theta} + \upsilon\,\varepsilon_{rr}\right) \end{aligned}} \qquad (\textit{stiffness formulation}) \qquad (3.27)\text{b}$$

Equilibrium equations

The elementary plate element of length dr and aperture angle $d\theta$ is considered as shown in Fig. 3.20. The length of the edge at the inside of the element equals $r\,d\theta$. The total force on this edge equals $n_{rr}\,r\,d\theta$ and points to the left. At the outside of the element, at a distance dr further, the force is increased to $n_{rr}\,r\,d\theta + \left\{d\left(n_{rr}\,r\,d\theta\right)/dr\right\}dr$ and points to the right. The resulting force of both circular edges is an outward-pointing force equal to $\left\{d\left(n_{rr}\,r\,d\theta\right)/dr\right\}dr$. The angle $d\theta$ is independent from r, which means that the resulting force can be written as:

$$\frac{d}{dr}\left(r\, n_{rr}\right) dr\, d\theta \tag{3.28}$$

A force $n_{\theta\theta} dr$ is acting perpendicular to each straight edge of the element. Since the angle between the two forces is $d\theta$, a force results of:

$$-n_{\theta\theta}\, dr\, d\theta \tag{3.29}$$

where the minus sign indicates the direction of the force (negative r-direction).
The distributed load p provides an outward-pointing force too. For that purpose, p has to be multiplied by the area $r\, d\theta\, dr$ of the considered plate element. The force equals:

$$p\, r\, d\theta\, dr \tag{3.30}$$

For equilibrium, the sum of the three forces (3.28), (3.29) and (3.30) has to be zero. After division by $d\theta\, dr$ the following equilibrium equation is obtained:

$$\boxed{-\frac{d}{dr}\left(r\, n_{rr}\right) + n_{\theta\theta} = r p} \qquad (\textit{equilibrium equation}) \tag{3.31}$$

In this stress problem, rigid body displacements without development of strains remain cannot occur. For each displacement u immediately a strain field develops. Further, only one combination of constant strains is possible. For the strain $\varepsilon_{\theta\theta}$ to have a constant value ε_0, from (3.26)b it follows that a displacement is required of $u = \varepsilon_0 r$. The strain ε_{rr} determined from (3.26)a then also equals ε_0. So, the only possible combination of constant strains is identical strains $\varepsilon_{\theta\theta}$ and ε_{rr}. Then from the constitutive relations (3.27) it follows that the extensional forces n_{rr} and $n_{\theta\theta}$ are equal and constant too (thus the stresses σ_{rr} and $\sigma_{\theta\theta}$). When the constant values $n_{rr} = n_0$ and $n_{\theta\theta} = n_0$ are substituted in the equilibrium equation (3.31) it appears that the distributed load p across the plate area has to be zero. The plate can only be loaded along the edge. Fig. 3.22 show two situations, a circular plate with and without hole. In both plates in each point an isotropic extensional force n_0 is present and Mohr's circle is reduced to a point.

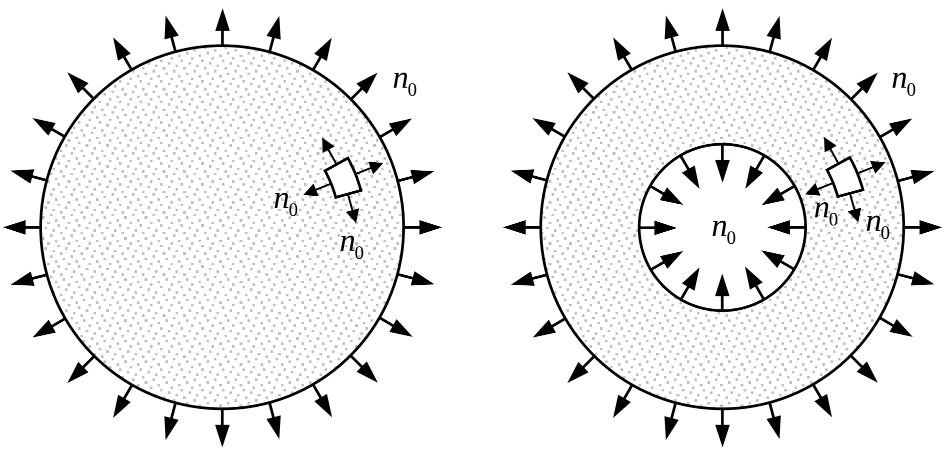


Fig. 3.22: Only one constant stress-state can occur in an axisymmetric plate; then the extensional forces n_{rr} and $n_{\theta\theta}$ are equal.

Transformation

The equilibrium equation (3.31) comprises the terms $r\,n_{rr}$ and $r\,p$. It makes sense to introduce new variables for these combinations. This will be done for $r\,n_{\theta\theta}$ too. It is defined:

$$\boxed{N_{rr} = r\,n_{rr} \quad ; \quad N_{\theta\theta} = r\,n_{\theta\theta} \quad ; \quad f = r\,p} \tag{3.32}$$

The two quantities N_{rr} and $N_{\theta\theta}$ are normal forces with the dimension of force and f is a line load with the dimension of force per unit of length.
Application of the transformations in (3.32) change the kinematic equations (3.26), the constitutive equations (3.27) and the equilibrium equation (3.31) into:

$$\boxed{\varepsilon_{rr} = \frac{du}{dr} \quad ; \quad \varepsilon_{\theta\theta} = \frac{u}{r}} \tag{3.33}$$

$$\boxed{\begin{aligned} \varepsilon_{rr} &= \frac{1}{Etr}\left(N_{rr} - \upsilon\, N_{\theta\theta}\right) \\ \varepsilon_{\theta\theta} &= \frac{1}{Etr}\left(N_{\theta\theta} - \upsilon\, N_{rr}\right) \end{aligned}} \quad \Leftrightarrow \quad \boxed{\begin{aligned} N_{rr} &= \frac{Etr}{1-\upsilon^2}\left(\varepsilon_{rr} + \upsilon\,\varepsilon_{\theta\theta}\right) \\ N_{\theta\theta} &= \frac{Etr}{1-\upsilon^2}\left(\varepsilon_{\theta\theta} + \upsilon\,\varepsilon_{rr}\right) \end{aligned}} \tag{3.34}$$

$$\boxed{-\frac{d}{dr}N_{rr} + \frac{N_{\theta\theta}}{r} = f} \tag{3.35}$$

Remark 1

This equilibrium equation can be compared with the one for a straight bar subjected to a normal force, which has been discussed in the course “Elastostatics for slender structures”:

$$-\frac{dN}{dx} = f$$

but with an extra term equal to $N_{\theta\theta}/r$. The tangential stresses help to carry the line load. When the term $-dN_{rr}/dr$ is omitted in (3.35), the equation can be used to calculate the normal force $N_{\theta\theta}$ in a ring subjected to a line load f.

Remark 2

After the transformation it can be written:

$$\boldsymbol{e} = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{Bmatrix} \quad ; \quad \boldsymbol{N} = \begin{Bmatrix} N_{rr} \\ N_{\theta\theta} \end{Bmatrix}$$

and:

$$\boldsymbol{e} = \boldsymbol{\mathcal{B}}\,u \quad ; \quad \boldsymbol{\mathcal{B}}'\,\boldsymbol{N} = f$$

where:

$$\mathcal{B} = \begin{Bmatrix} \dfrac{d}{dr} \\ \dfrac{1}{r} \end{Bmatrix} \quad ; \quad \mathcal{B}' = \begin{Bmatrix} -\dfrac{d}{dr} & \dfrac{1}{r} \end{Bmatrix}$$

Again it can be observed that in the transposed of $\mathcal{B}$ the sign of the uneven (first) derivative d/dr has to be changed, but not of the even (0^{th}) derivative $1/r$.

3.5.1 Thick-walled tube

The thick walled tube, as shown in Fig. 3.23 is subjected at the inner cylindrical surface to a uniformly distributed load q . The load is positive and points in r -direction. No distributed surface load p is present at the face of the plate, which means that f in (3.32) is zero.

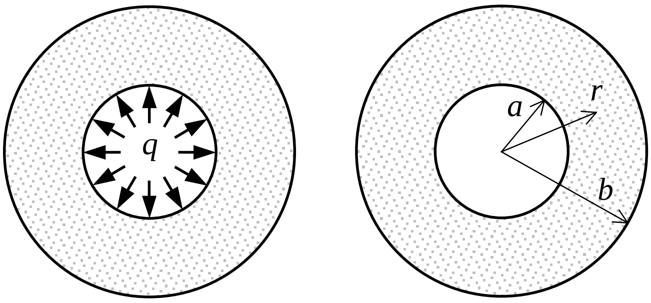


Fig. 3.23: Thick-walled pipe with load at inner wall.

In a thick-walled tube, flat sections remain flat after deformation, but the strain ε_{zz} in axial direction will not be zero. On the average σ_{zz} will be equal to zero. Therefore, the problem will be treated as a plane stress state. The analysis is carried out by both the force and displacement methods. A slice of unit thickness of the tube is considered, which is cut perpendicularly to the axial direction. This means that n_{rr} and $n_{\theta\theta}$ are equal to σ_{rr} and $\sigma_{\theta\theta}$, respectively.

Force method

The first step is to find a solution for the equilibrium equation:

$$-\frac{d}{dr} N_{rr} + \frac{N_{\theta\theta}}{r} = 0$$

This differential equation contains two unknowns. Therefore, one redundant stress function ϕ can be selected. The following choice:

$$\boxed{N_{rr} = \phi \quad ; \quad N_{\theta\theta} = r\frac{d\phi}{dr}} \tag{3.36}$$

satisfies the homogeneous differential equation.

From the constitutive relations in flexibility formulation, in the second step it follows:

$$\varepsilon_{rr} = \frac{1}{E\,t}\left(\frac{\phi}{r} - \upsilon\frac{d\phi}{dr}\right) \quad ; \quad \varepsilon_{\theta\theta} = \frac{1}{E\,t}\left(-\upsilon\frac{\phi}{r} + \frac{d\phi}{dr}\right)$$

In the third step, this result has to be substituted in the compatibility condition, which is obtained by elimination of the displacement u from the kinematic equations:

$$-\varepsilon_{rr} + \frac{d}{dr}\left(r\,\varepsilon_{\theta\theta}\right) = 0$$

Substitution of the strains leads to:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{\phi}{r} = 0 \quad \text{or} \quad r\frac{d}{dr}\left(\frac{1}{r}\frac{d\left(r\,\phi\right)}{dr}\right) = 0 \quad \text{or} \quad r\frac{d^2\phi}{dr^2} + \frac{d\phi}{dr} - \frac{\phi}{r} = 0$$

This differential equation can be rewritten as:

$$\boxed{\mathcal{L}\phi = 0} \qquad (3.37)\text{a}$$

where the operator $\mathcal{L}$ has been introduced defined by:

$$\boxed{\mathcal{L} = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r} \qquad (3.37)\text{b}$$

A second-order differential equation has been obtained, contrary to the more general case of rectangular problems for which the fourth-order biharmonic equation of Airy holds.
For the solution of differential equation (3.37) the following trial function is used:

$$\phi = C\,r^m$$

where C and m are constants. Substitution into (3.37) yields:

$$C\left(m+1\right)\left(m-1\right)r^{m-1} = 0$$

So, two roots are found:

$$m_1 = -1 \quad ; \quad m_2 = 1$$

The general solution becomes:

$$\phi(r) = C_1\frac{1}{r} + C_2\,r$$

Now, with (3.36) the force distribution can be obtained:

$$N_{rr} = \phi = C_1 \frac{1}{r} + C_2\, r \quad ; \quad N_{\theta\theta} = r\frac{d\phi}{dr} = -C_1 \frac{1}{r} + C_2 r$$

and subsequently:

$$n_{rr} = \frac{1}{r} N_{rr} = C_1 \frac{1}{r^2} + C_2 \quad ; \quad n_{\theta\theta} = \frac{1}{r} N_{\theta\theta} = -C_1 \frac{1}{r^2} + C_2 \tag{3.38}$$

This result clearly shows that only one combination of constant stresses is possible, in which case C_1 is zero and n_{rr} and $n_{\theta\theta}$ are both equal to C_2.
The two values of C_1 and C_2 are determined from the boundary conditions at the inner and outer cylindrical surface of the thick-walled tube. These conditions are:

$$r = a \quad \rightarrow \quad n_{rr} = -q \qquad ; \qquad r = b \quad \rightarrow \quad n_{rr} = 0$$

This delivers for the two constants:

$$C_1 = -\frac{a^2 b^2}{b^2 - a^2}\, q \quad ; \quad C_2 = \frac{a^2}{b^2 - a^2}\, q$$

The extensional forces become:

$$n_{rr} = \frac{a^2}{b^2 - a^2}\left(-\frac{b^2}{r^2} + 1\right) q \quad ; \quad n_{\theta\theta} = \frac{a^2}{b^2 - a^2}\left(\frac{b^2}{r^2} + 1\right) q \tag{3.39}$$

Fig. 3.24a displays this distribution of forces (stresses) across the thickness of the tube wall.

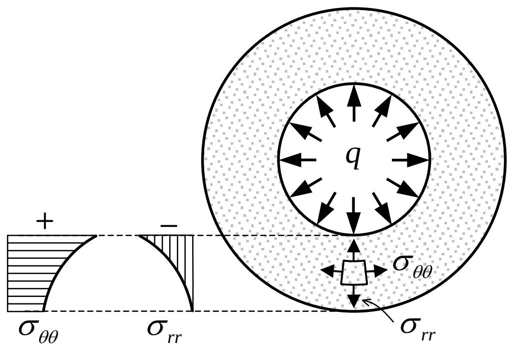


Fig. 3.24a: Stress distributions in a thick-walled pipe under internal gas pressure.

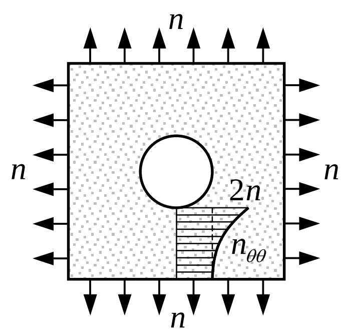


Fig. 3.24b: Stress concentration factor 2 near a hole in case of equal principal stresses.

Remark 1

When the distributed load q would have been applied on the outer wall pointing in negative r-direction, the constants would have become:

$$C_1 = \frac{a^2 b^2}{b^2 - a^2}\, q \quad ; \quad C_2 = -\frac{b^2}{b^2 - a^2}\, q$$

Addition of these values to the previously obtained ones provide the following C-values:

$$C_1 = 0 \quad ; \quad C_2 = -q$$

This is exactly the constant state of stress with equal n_{rr} and $n_{\theta\theta}$. Then both the inner and outer wall are subjected to the distributed load q.

Remark 2
When above exercise is repeated for plane strain, exactly the same differential equation will be obtained and therefore the same stresses.

Remark 3
Now the stress concentration factor can be computed in a large plate with a homogeneous stress state of equal principal stresses, in which a circular hole is made. The homogeneous membrane forces without hole are:

$$n_{rr} = n \quad ; \quad n_{\theta\theta} = n$$

In order to make the edge of the proposed hole to be stress-free a loading case must be superimposed in which the edge is loaded with an opposite load. For the boundary in this case it holds $n_{rr} = 0$. Further it is known that the stresses will vanish for large radius r. The result is that $C_1 = -n/a^2$ en $C_2 = 0$. Hence the membrane forces for this case become:

$$n_{rr} = n\left(-\frac{a^2}{r^2}\right) \quad ; \quad n_{\theta\theta} = n\left(\frac{a^2}{r^2}\right)$$

Still these membrane forces must be superimposed on the constant equal stresses for the case without hole. The final result for the large plate with hole is:

$$n_{rr} = n\left(1-\frac{a^2}{r^2}\right) \quad ; \quad n_{\theta\theta} = n\left(1+\frac{a^2}{r^2}\right)$$

Due to the hole, the maximum value of the membrane force $n_{\theta\theta}$ is twice the value n of the homogeneous stress state (stress concentration factor 2), see Fig. 3.24b.

Displacement method
In the displacement method the kinematic equations (3.33) are substituted in the constitutive equations (3.34) in stiffness formulation:

$$N_{rr} = \frac{Etr}{1-\upsilon^2}\left(\frac{du}{dr}+\upsilon\frac{u}{r}\right) \quad ; \quad N_{\theta\theta} = \frac{Etr}{1-\upsilon^2}\left(\frac{u}{r}+\upsilon\frac{du}{dr}\right)$$

Substitution of this result into the equilibrium equation (3.35), leads to the following differential equation:

$$\boxed{\frac{Et}{1-\upsilon^2}\mathcal{L}u = f} \tag{3.40}$$

where $\mathcal{L}$ is same the operator as defined in (3.37)b.
For the tube subjected an internal gas pressure q it holds $f = 0$ and therefore:

$$\mathcal{L} u = 0$$

The solution reads:

$$u = C_1 \frac{1}{r} + C_2\, r$$

The integration constants follow from the two boundary conditions:

$$\begin{aligned}
r = a \quad &\rightarrow \quad n_{rr} = -q \quad ; \quad \frac{1}{r} N_{rr} = -q \\
r = b \quad &\rightarrow \quad n_{rr} = 0 \qquad ; \quad \frac{1}{r} N_{\theta\theta} = 0
\end{aligned}$$

From the general solution it follows:

$$N_{rr} = \frac{Etr}{1-\upsilon^2} \left\{ \left(-C_1 \frac{1}{r^2} + C_2 \right) + \upsilon \left(C_1 \frac{1}{r^2} + C_2 \right) \right\}$$

Therefore:

$$n_{rr} = \frac{Et}{1-\upsilon^2} \left\{ -(1-\upsilon) C_1 \frac{1}{r^2} + (1+\upsilon) C_2 \right\}$$

Now the constants can be obtained from the two boundary conditions:

$$C_1 = (1-\upsilon) \frac{a^2 b^2}{b^2 - a^2} \frac{q}{Et} \quad ; \quad C_2 = (1-\upsilon) \frac{a^2}{b^2 - a^2} \frac{q}{Et}$$

Finally, the results for the displacement u and the stress quantities n_{rr} and $n_{\theta\theta}$ become:

$$\begin{aligned}
u(r) &= \frac{a^2}{b^2 - a^2} \frac{q}{Et} \left\{ (1+\upsilon) \frac{b^2}{r} + (1-\upsilon)\, r \right\} \\
n_{rr} &= \frac{a^2}{b^2 - a^2} \left(-\frac{b^2}{r^2} + 1 \right) q \quad ; \quad n_{\theta\theta} = \frac{a^2}{b^2 - a^2} \left(\frac{b^2}{r^2} + 1 \right) q
\end{aligned} \tag{3.41}$$

These stress quantities were previously found in (3.39)

3.5.2 Curved beam subjected to a constant moment

A curved beam is considered with a constant radius of curvature. The dashed lines in Fig. 3.25 show such a beam. The inner and outer radius are a and b, respectively. The aperture angle of the arc is φ. The beam has a rectangular cross-section with height d and width t.

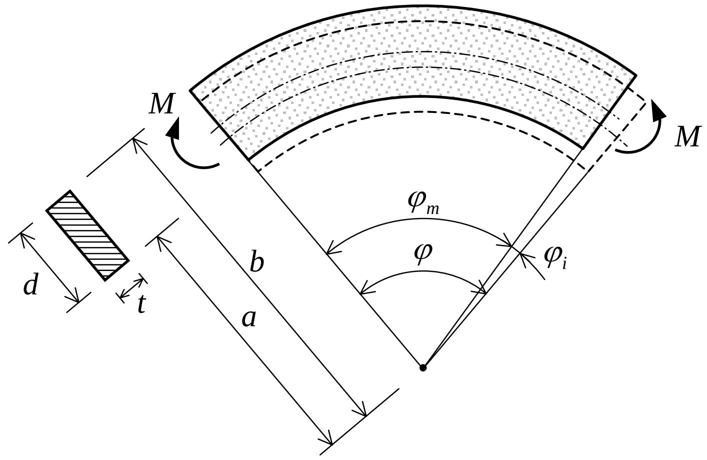


Fig. 3.25: circular bar in unloaded state (dashed lines) and in loaded state (solid lines).

So, for the height d it holds $d = b - a$. This curved bar is modelled by a thin plate with thickness t.
The radius of curvature of the beam increases, due to the application of the moment load. It is reasonable to assume that this radius is constant along the length of the curved beam. For reasons of axisymmetry, flat surfaces perpendicular to the neutral line remain flat and perpendicular to the neutral line. However the aperture angle reduces from φ to φ_m. This reduction is necessary because no normal force in the beam should be generated, which means that the beam length measured along the neutral line remains the same. But the radius of curvature increases leading to a reduction in aperture angle. This reduction is indicated by φ_i.
At first sight, this problem does not fall in the category of axisymmetric problems, because for that category the reduction φ_i does not occur. However, the problem can be included in this category by simultaneous application of the moment M and an initial strain ε_i in tangential direction. Then the magnitude of ε_i is selected such that the reduction φ_i of the angle φ is exactly compensated. It can be imagined, that the strain ε_i is caused by a temperature increase or a swelling by moisture. However, the initial strain ε_i should only be introduced in θ-direction and not in r-direction.

The initial strain ε_i is constant across the height d of the cross-section and has magnitude:

$$\varepsilon_i = \frac{\varphi_i}{\varphi}$$

Due to the simultaneous application of M and ε_i the aperture angle of the curved beam keeps its magnitude φ and no normal force develops. Each flat surface displaces exclusively in the direction of the radius r. However, the question is how this load can be applied? In axisymmetric problems, distributed surface loads on the centre plane and distributed line loads along the edges with $r = a$ and $r = b$ can be applied. In this case, these types of loading are not applicable and an alternative approach is required. The initial strain ε_i will be imposed as a “load” and then the moment M will follow from the resultant of the calculated stresses $n_{\theta\theta}$. The initial strain ε_i will be incorporated in the constitutive equation (3.34) and it will be traced how this action affects the differential equation. Then this modified differential equation will be solved with boundary conditions $n_{rr} = 0$ on the edges with $r = a$ and $r = b$.

In this way, the stress quantities n_{rr} and $n_{\theta\theta}$ are found as linear functions of the applied initial strain ε_i. The moment M is determined from the stress quantity $n_{\theta\theta}$ by integration over the height of the beam (the resulting normal force in the cross-section is zero):

$$M = \int_a^b r\, n_{\theta\theta}\, dr \tag{3.42}$$

The moment M will be a linear function of ε_i too and will be used to eliminate ε_i from the expressions for n_{rr} and $n_{\theta\theta}$, which in their turn become a function of the moment M.

Force method

The problem will only be analysed by the force method. The only difference with respect to the previous section is the formulation of the constitutive equations. They are:

$$\varepsilon_{rr} = \frac{1}{Etr}\left(N_{rr} - \upsilon N_{\theta\theta}\right) \quad ; \quad \varepsilon_{\theta\theta} = \frac{1}{Etr}\left(N_{\theta\theta} - \upsilon N_{rr}\right) + \varepsilon_i \quad \text{with} \quad \varepsilon_i > 0 \tag{3.43}$$

The differential equation (3.37)a is extended to:

$$\frac{1}{Et}\mathcal{L}\phi = -\varepsilon_i \tag{3.44}$$

Next to the in section 3.5.1 found homogeneous solution given by:

$$\phi(r) = C_1 \frac{1}{r} + C_2\, r$$

a particular solution exists corresponding with the right-hand side $-\varepsilon_i$:

$$\phi(r) = -\frac{1}{2} E\, t\, \varepsilon_i\, r \ln r$$

The correctness of this solution can be checked by substitution into (3.44). The total solution becomes:

$$\phi(r) = C_1 \frac{1}{r} + C_2\, r - \frac{1}{2} E\, t\, \varepsilon_i\, r \ln r \tag{3.45}$$

From the boundary condition $n_{rr} = 0$ on the two edges with $r = a$ and $r = b$ the two integration constants can be determined:

$$C_1 = -\frac{1}{2} E\, t\, \varepsilon_i \frac{a^2 b^2}{b^2 - a^2} \ln\left(\frac{b}{a}\right) \quad ; \quad C_2 = \frac{1}{2} E\, t\, \varepsilon_i \frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2} \tag{3.46}$$

This results in the stress quantities:

$$n_{rr} = \frac{1}{2} E\, t\, \varepsilon_i\, f_{rr}(r) \quad ; \quad n_{\theta\theta} = \frac{1}{2} E\, t\, \varepsilon_i\, f_{\theta\theta}(r) \tag{3.47}$$

where:

$$f_{rr}(r) = -\frac{a\,b}{b^2-a^2}\ln\left(\frac{b}{a}\right)\frac{a\,b}{r^2} + \frac{a^2}{b^2-a^2}\ln\left(\frac{r}{a}\right) - \frac{b^2}{b^2-a^2}\ln\left(\frac{r}{b}\right)$$
$$f_{\theta\theta}(r) = -1 + \frac{a\,b}{b^2-a^2}\ln\left(\frac{b}{a}\right)\frac{a\,b}{r^2} + \frac{a^2}{b^2-a^2}\ln\left(\frac{r}{a}\right) - \frac{b^2}{b^2-a^2}\ln\left(\frac{r}{b}\right) \qquad (3.48)$$

The moment according to (3.42) becomes:

$$M = \frac{1}{2} E\, t\, \varepsilon_i\, C \qquad (3.49)$$

where:

$$\boxed{C = \frac{1}{4}\left(b^2 - a^2\right) - \frac{a^2 b^2}{b^2-a^2}\left(\ln\frac{b}{a}\right)^2} \qquad (3.50)$$

The normal force N becomes:

$$N = \int_a^b n_{\theta\theta}\, dr = 0$$

Elimination of ε_i from (3.47) and (3.49) finally provides the required relation between the moment M and the stress quantities n_{rr} and $n_{\theta\theta}$:

$$\boxed{n_{rr} = \frac{M}{C} f_{rr}(r) \quad ; \quad n_{\theta\theta} = \frac{M}{C} f_{\theta\theta}(r)} \qquad (3.51)$$

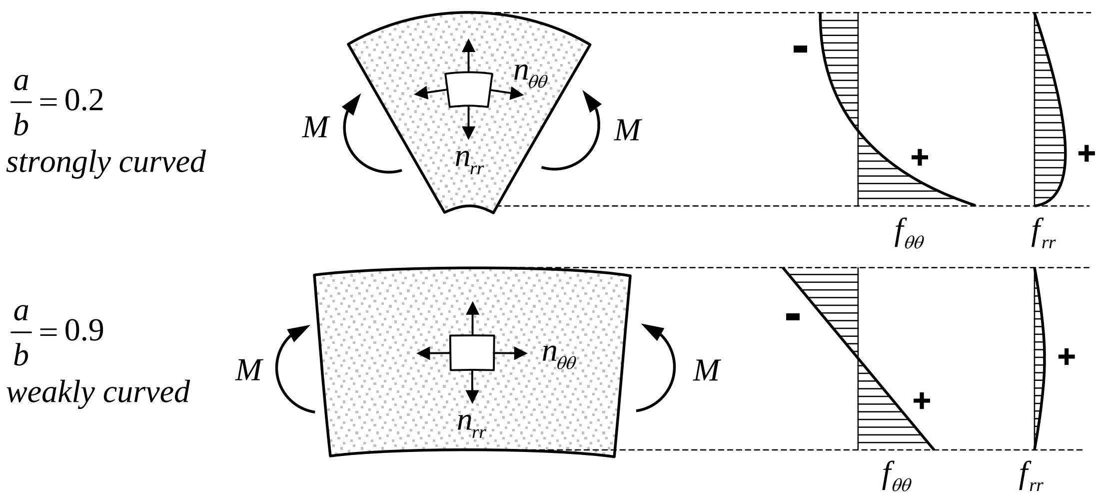


Fig. 3.26: Distribution functions for the stresses n_{rr} and $n_{\theta\theta}$ for a small and a large value of a/b.

In these expressions, C only depends on a and b, i.e. the geometry. The functions $f_{rr}(r)$ and $f_{\theta\theta}(r)$ are dimensionless and provide the distribution of stresses over the height d of the cross-section. In Fig. 3.26 this distribution is displayed for two different values of the ratio a/b, a value that is small compared to unity (strong curvature) and a value close to unity (weak curvature). For a strong curvature, the bending stress distribution deviates severely from a linear distribution, irrespective of the fact that flat cross-sections remain flat.

Remark 1
In these notes a differential equation of the second-order has been used. Timoshenko and Goodier analysed the same problem in their book “Theory of Elasticity”. They started with the fourth-order Airy equation and found the same distribution of stresses.

Remark 2
That in the case of pure bending also stresses n_{rr} are generated can be made clear by considering the part of the beam inside the neutral line. The tensile stresses $n_{\theta\theta}$ deliver a tensile force. The two tensile forces acting on the ends of the beam-part have different directions. Equilibrium is possible only if over the full length of the bar a radial outward-pointing n_{rr} is present. Therefore, it can be concluded that n_{rr} is a tensile stress. The same conclusion holds for the part of the beam outside the neutral line, where compressive stresses $n_{\theta\theta}$ are present.

3.6 Description in polar coordinates of plates subjected to extension

The discussion is extended by the analysis in polar coordinates of plates subjected to extension for which no axisymmetry holds. The starting point is the treatment of the rectangular plates in section 3.2. In that section the stress function $\phi(x,y)$ was introduced, for which it was derived:

$$n_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad ; \quad n_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad ; \quad n_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

For this stress function the biharmonic equation of Airy holds:

$$\nabla^2 \nabla^2 \phi = 0$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

This operator is a representation of the sum of the two normal forces. Fig. 3.27 shows which stresses are involved for the description in polar coordinates. Now, two degrees of freedom u_r and u_θ are present and in both directions loads can be applied.
In this case two equilibrium equations have to be formulated in both u_r and u_θ directions. From Fig. 3.28 it follows:

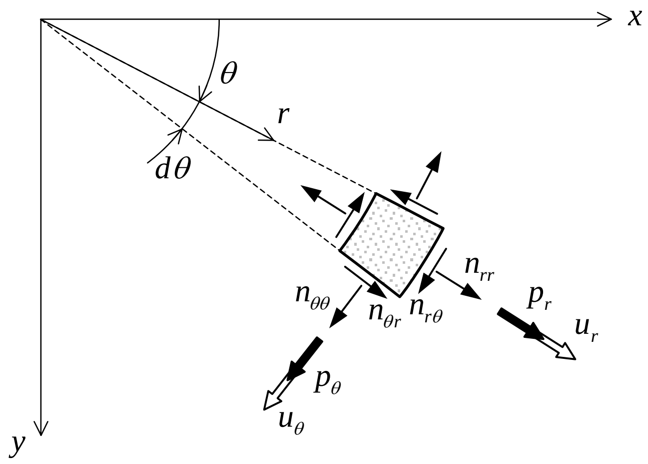


Fig. 3.27: Relevant quantities for a description in polar coordinates.

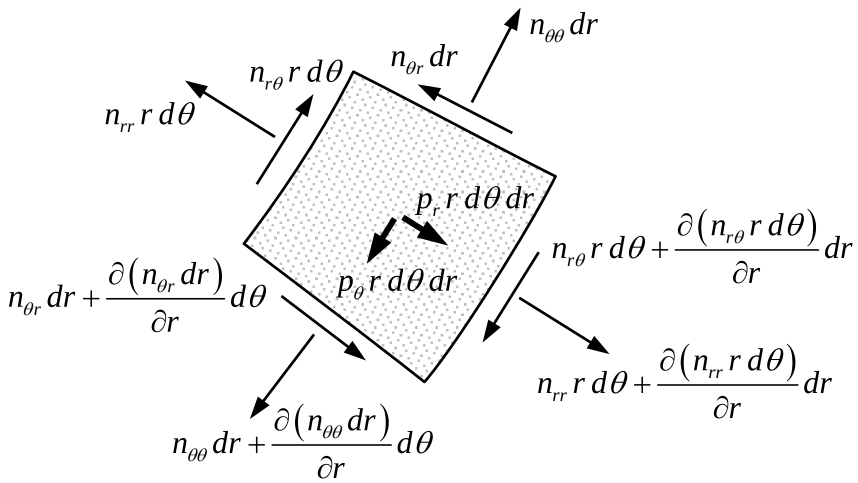


Fig. 3.28: Forces participating in the equilibrium.

$$-\frac{\partial(n_{rr}\, r)}{\partial r} + n_{\theta\theta} - \frac{\partial n_{\theta r}}{\partial \theta} = p_r\, r \quad ; \quad -\frac{\partial n_{\theta\theta}}{\partial \theta} - \frac{\partial(n_{r\theta}\, r)}{\partial r} - n_{\theta r} = p_\theta\, r \tag{3.52}$$

where the components $n_{\theta r}$ and $n_{r\theta}$ are identical. For the case $p_r = p_\theta = 0$, the equations are satisfied by the following solution:

$$\boxed{n_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \quad ; \quad n_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \quad ; \quad n_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)} \tag{3.53}$$

Now, directly the operator ∇^2 can be determined from the sum of n_{rr} and $n_{\theta\theta}$. The equation of Airy then becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0 \tag{3.54}$$

3.6.1 Point load on a half-plane

A wedge is considered with a distributed and point load as displayed in Fig. 3.29. The wedge and the load are symmetrical with respect to the axis $\theta = 0$.

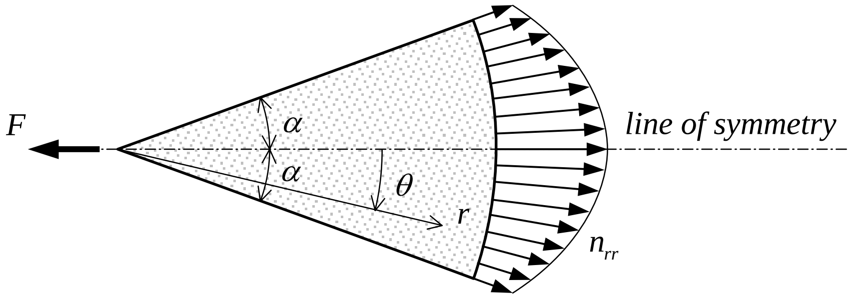


Fig. 3.29: wedge that is subjected to loads that are in equilibrium.

It can be shown that the stress function:

$$\phi = C r \theta \sin\theta$$

defines the stress-state in the wedge. The reader can verify that this stress function satisfies the biharmonic equation of Airy. Using (3.53), the following stresses can be determined:

$$n_{rr} = 2C\frac{1}{r}\cos\theta \quad ; \quad n_{\theta\theta} = 0 \quad ; \quad n_{r\theta} = 0$$

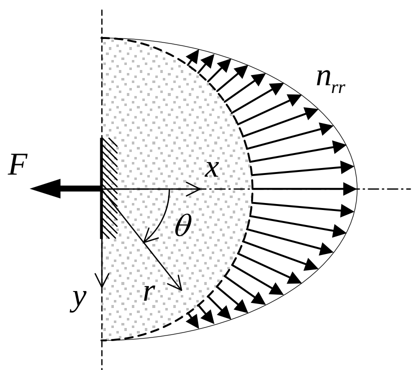
The value of C can be calculated as follows. The horizontal resultant of the boundary load n_{rr} on the circular edge has to be equal to F:

$$F = \int_{-\alpha}^{\alpha} n_{rr}\cos\theta \; r d\theta = C\int_{-\alpha}^{\alpha} 2\cos^2\theta \; d\theta = C\left(2\alpha + \sin 2\alpha\right) \quad \rightarrow \quad C = \frac{F}{2\alpha + \sin 2\alpha}$$

Consequently, for n_{rr} it can be written:

$$n_{rr} = \frac{F}{\alpha + \frac{1}{2}\sin 2\alpha}\frac{\cos\theta}{r}$$

A special case is introduced when the angle α becomes equal to $\pi/2$, which is presented in Fig. 3.30. In this way the stress distribution is obtained for a point load on an infinite 2D half-



$$n_{rr} = \frac{2F}{\pi}\frac{\cos\theta}{r}$$

$$n_{\theta\theta} = 0$$

$$n_{r\theta} = 0$$

Fig. 3.30: Solution in polar coordinates for a point-load on a half-plane.

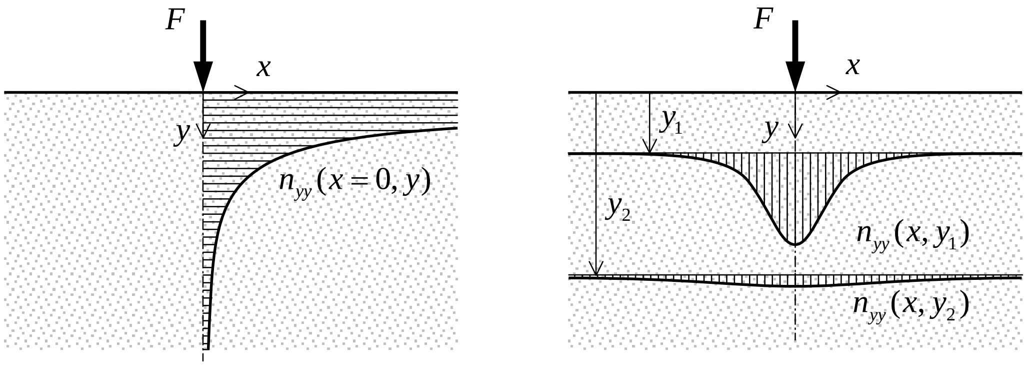
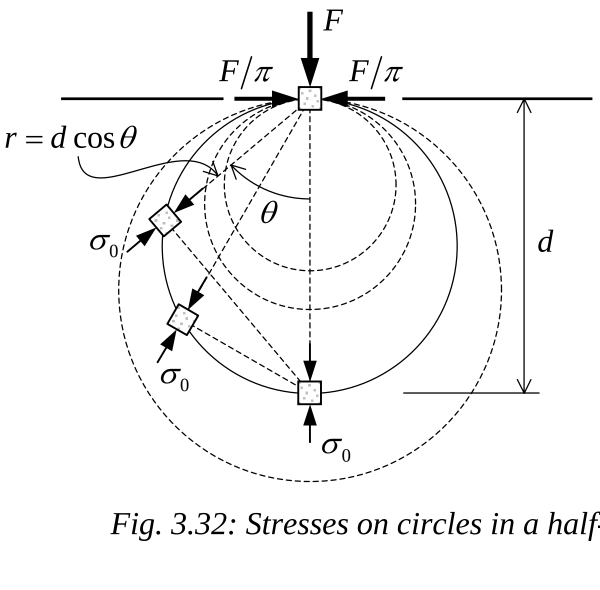


Fig. 3.31: Vertical stresses in half-space due to point load on surface.

plane, which is identical to a line-load on an infinite 3D half-space. Boussinesq even found such a type of solution for a compressive point-load F on an infinite 3D half-space, from which Flamant obtained the above-derived solution. Therefore, the solution for a point-load on a half-plane is also called the solution of Boussinesq. In each point (r,θ) a transformation can be made from the stresses n_{rr}, $n_{\theta\theta}$ and $n_{r\theta}$ to the stresses n_{xx}, n_{yy} and n_{xy}. All three the stresses are different from zero. Fig. 3.31 shows the vertical stress distribution n_{yy}.

The solution for a compressive force F on a half-plane becomes very simple when it is presented by eccentric circles (Fig. 3.32). For all points on a circle it holds $r = d\cos\theta$. Then in each circle the stress σ_{rr} is constant while the other stress components are zero. The constant value is $-\sigma_0$, in which $\sigma_0 = 2F/\pi d$ is positive. In the vertical line of symmetry, the horizontal stress is zero. Just below the point load, a horizontal force F/π is present. This can be derived from the horizontal equilibrium of a part of the half-plane. In the next section this will be demonstrated.



$$\sigma_{rr} = -\frac{2F}{\pi}\frac{\cos\theta}{r} = -\frac{2F}{\pi d} = -\sigma_0$$

$$\sigma_{\theta\theta} = 0$$

$$\sigma_{r\theta} = 0$$

Fig. 3.32: Stresses on circles in a half-plane due to a point load.

3.6.2 Brazilian splitting test

Every material laboratory has a compression test rig. Therefore, it is easy to do splitting tests on concrete cylinders (Fig. 3.33). This test is often called a Brazilian splitting test. Direct tensile tests on concrete are much more difficult to perform, because a special tensile test set-

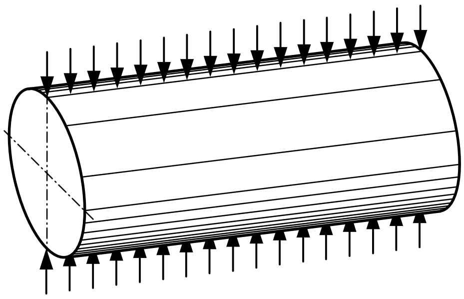


Fig. 3.33: Splitting of cylinder due to double line load along generating lines.

up needs to be available. Fortunately, there is a simple relation between the vertical splitting forces and the tensile strength in a Brazilian splitting test. For this reason, the tensile strength of concrete is often obtained from splitting tests. In this section it will be shown that a horizontal tensile stress occurs, which is constant over the height of the cylinder.
It is assumed that the stresses do not vary along the axial direction of the cylinder so that a slice of unit thickness can be considered. In Fig. 3.34 the solution for a point load on a half-plane is displayed again. The material outside the circle has been removed and replaced by the edge loading σ_0. In the same figure the mirror image of the solution is presented. When both solutions are superimposed a circular disk is obtained that is loaded by two concentrated forces F and edge stresses $\sigma_{rr} = \sigma_{\theta\theta} = -\sigma_0$. The edge stresses combined deliver a radial edge stress $-\sigma_0$. Note that no horizontal stresses are present in the vertical line of symmetry.

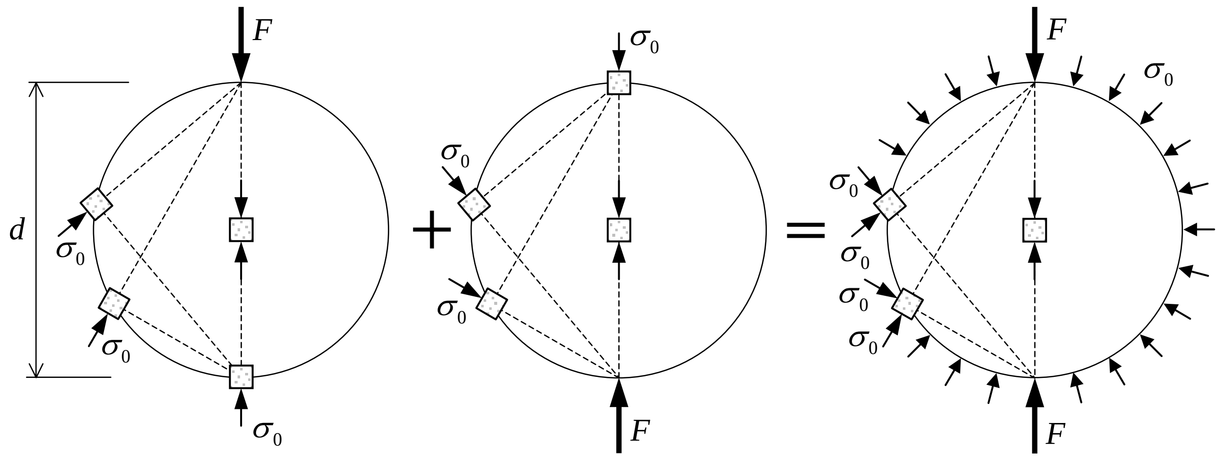


Fig. 3.34: Summation of the original and reflected solution.

The final step in the derivation is to remove the edge stress by adding the axisymmetric solution of a disk with a constant tensile stress σ_0 on the edge (see Fig. 3.35). In this load case a hydrostatic stress-state occurs with a tensile stress σ_0 acting in all directions, also in horizontal direction on the vertical line of symmetry. The result of the superposition is a circular disk subjected to two diametrically opposite point-loads F with a free unloaded edge. On the vertical line of symmetry a constant tensile stress σ_0 occurs, i.e.:

$$\sigma_{xx} = \frac{2F}{\pi d}$$

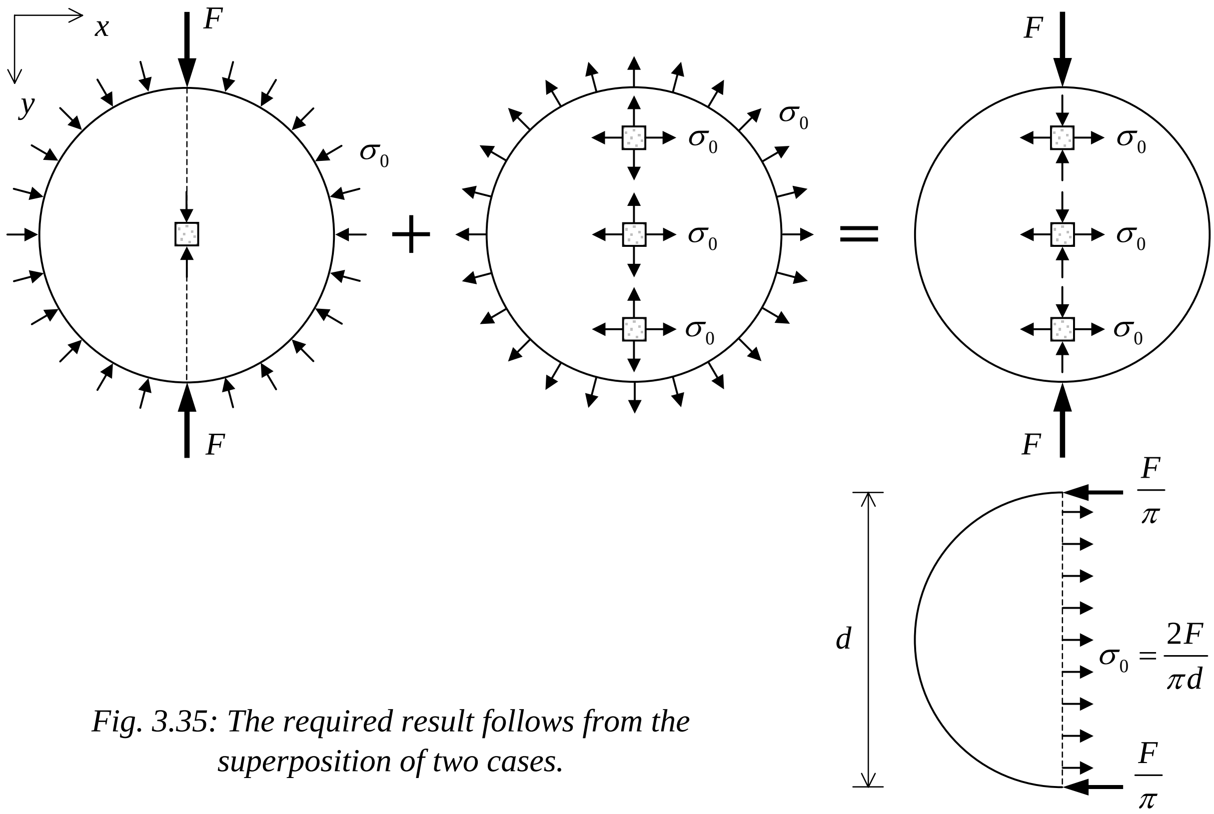


Fig. 3.35: The required result follows from the superposition of two cases.

The total horizontal force on the line of symmetry has to be zero. Therefore, the concentrated horizontal compressive force at the point of action of each force F equals $\frac{1}{2}\sigma_0 d$, which is equal to F/π.
The final result is a constant tensile stress σ_0 on the splitting face. In a test, the tensile strength σ_0 can be determined by measurement of the force F per unit length. Since in this linear elastic solution the compressive stress becomes infinite at the surface (see Fig. 3.31), the reality will be somewhat different. The compressive force F/π will be spread out over some distance and locally the material behaviour will be non-linear.

3.6.3 Hole in large plate under uniaxial stress

A plate of large size with a hole is considered, in which a uniaxial stress state is present. This stress state can be split into two parts, one with equal principal stresses of equal sign and one with equal principal stresses of opposite sign (see Fig. 3.36). The first one has already been

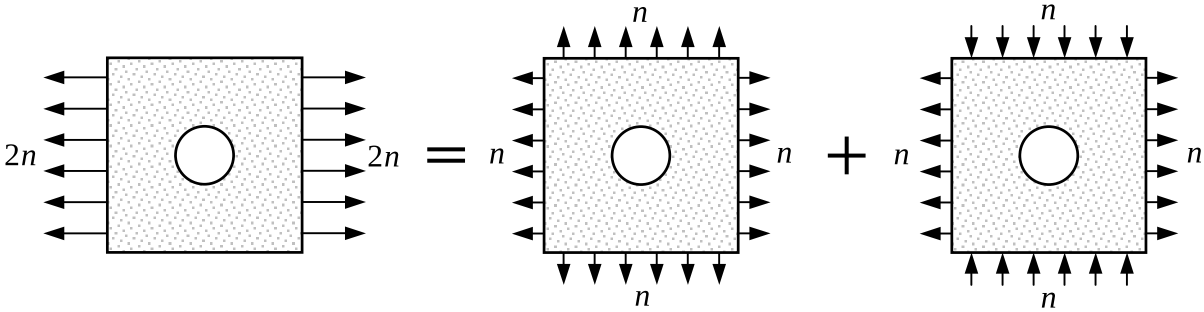


Fig. 3.36: A state of uniaxial stress can be split in two other states, which can be analysed easily.

solved in section 3.5.1. This case is similar to the axisymmetric plate with a hole. Here the second part of the problem will be dealt with. The case of equal principal stresses with opposite sign is in fact the case of a homogeneous shear stress. A hole in this plate causes a disturbance of the homogeneous state in the vicinity of the hole, which results in tensile stresses and compressive stresses at the edge of the hole, which are much higher than the homogenous stresses. These high stresses will be determined. The plate under consideration is shown in Fig. 3.37. The value of the principal membrane forces is n. It easily can be verified from Mohr`s circle or the transformation rules that the homogeneous stresses in absence of the hole would be:

$$\begin{aligned} n_{rr} &= n \cos 2\theta \\ n_{\theta\theta} &= -n \cos 2\theta \\ n_{r\theta} &= -n \sin 2\theta \end{aligned}$$

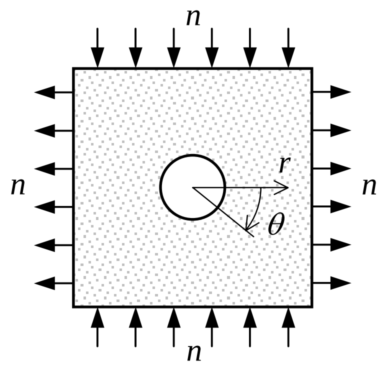


Fig. 3.37: Stress state with equal principal stresses of opposite sign.

If a hole is created, the membrane forces n_{rr} and $n_{r\theta}$ have to be made zero on the edge of the hole. This means that an edge loading has to be superimposed, which causes the same membrane forces but with an opposite sign:

$$\begin{aligned} n_{rr} &= -n \cos 2\theta \\ n_{r\theta} &= n \sin 2\theta \end{aligned}$$

The general homogeneous differential equation (3.54) has to be solved. This can be done by choosing a solution for ϕ of the form:

$$\phi(r,\theta) = g(r)\cos 2\theta$$

This means that the variables r and θ are separated. Substitution in the differential equation (3.54) yields an ordinary fourth-order differential equation for $g(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right) g = 0$$

The solution of the shape $g(r) = C\,r^m$ is tried, which results in the following equation for m:

$$\left(m-4\right)\left(m-2\right)m\left(m+2\right)=0$$

The four roots are:

$$m_1 = 4 \quad ; \quad m_2 = 2 \quad ; \quad m_3 = 0 \quad ; \quad m_4 = -2$$

The general solution for $\phi(r,\theta)$ now becomes:

$$\phi(r,\theta) = \left(C_1 r^4 + C_2 r^2 + C_3 + C_4 \frac{1}{r^2} \right) \cos 2\theta$$

From (3.53) it can be derived:

$$\begin{aligned}
n_{rr} &= -\left(2C_2 + \frac{4}{r^2} C_3 + \frac{6}{r^4} C_4 \right) \cos 2\theta \\
n_{\theta\theta} &= \left(12r^2 C_1 + 2C_2 + \frac{6}{r^4} C_4 \right) \cos 2\theta \\
n_{r\theta} &= \left(6r^2 C_1 + 2C_2 - \frac{2}{r^2} C_3 - \frac{6}{r^4} C_4 \right) \sin 2\theta
\end{aligned}$$

For infinite large r all membrane forces must vanish. Hence $C_1 = 0$ and $C_2 = 0$. At the edge $r = a$ of the hole, the membrane forces are:

$$\begin{aligned}
n_{rr} &= -n \cos 2\theta \\
n_{r\theta} &= n \sin 2\theta
\end{aligned}$$

From this information C_3 and C_4 can be derived:

$$C_3 = a^2 n \quad ; \quad C_4 = -\frac{1}{2} a^4 n$$

So the result for the membrane forces is:

$$\begin{aligned}
n_{rr} &= n \left(-4 \frac{a^2}{r^2} + 3 \frac{a^4}{r^4} \right) \cos 2\theta \\
n_{\theta\theta} &= n \left(-3 \frac{a^4}{r^4} \right) \cos 2\theta \\
n_{r\theta} &= n \left(-2 \frac{a^2}{r^2} + 3 \frac{a^4}{r^4} \right) \sin 2\theta
\end{aligned}$$

This solution still has to be superimposed on the homogenous stresses for the situation without hole. The final result is:

$$n_{rr} = n\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta$$

$$n_{\theta\theta} = n\left(-1 - 3\frac{a^4}{r^4}\right)\cos 2\theta$$

$$n_{r\theta} = n\left(-2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\sin 2\theta$$

The maximum tensile stress $n_{\theta\theta}$ at the edge appears for $\theta = \pm\pi$ and is equal to $4n$. This value is four times the applied principal membrane stresses (stress concentration factor is 4), see Fig. 3.38.

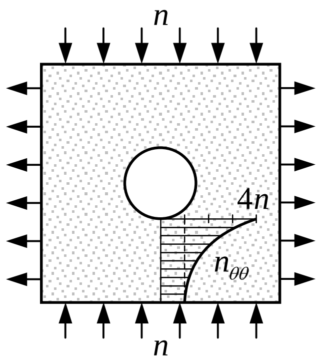


Fig. 3.38: Stress concentration factor of 4 near a hole in a constant shear field.

The uniaxial stress state is found from the superposition of the here achieved solution and the solution for the axisymmetric case in section 3.5.1, divided by 2 in order to relate it to an applied stress of the magnitude n:

$$n_{rr} = \frac{n}{2}\left\{\left(1 - \frac{a^2}{r^2}\right) + \left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta\right\}$$

$$n_{\theta\theta} = \frac{n}{2}\left\{\left(1 + \frac{a^2}{r^2}\right) - \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right\}$$

$$n_{r\theta} = n\left\{-1 - 2\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right\}\sin 2\theta$$

Now the maximum tensile stress $n_{\theta\theta}$ is three times the value of the uniaxial membrane force (stress concentration factor is 3). The stress distribution is shown in Fig. 3.39.

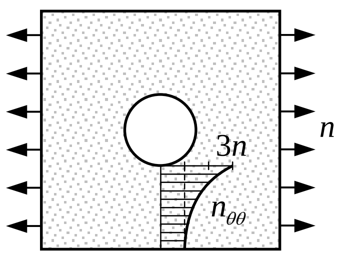


Fig. 3.39: Stress concentration factor of 3 near a hole in a uniaxial stress field.

4 Plates in bending

4.1 Rectangular plates

When a shear load is applied on a thin plate, it is subjected to bending. This problem was already discussed in the previous course “Elastic Plates”. The theory of plates, which is used to analyse this elasticity problem is based on the same assumptions as the bending theory for beams. It will be shown that in a homogeneous isotropic plate bending occurs in two directions and also torsion is present. It is assumed that the centre plane of the plate does not deform. Therefore, all stresses in this plane are zero, similarly as on the neutral line of beams. Analogously to a beam where flat cross-sections perpendicular to the neutral line remain flat and perpendicular after deformation, for a plate a similar hypothesis is introduced. A straight line perpendicular to the centre plane remains straight and perpendicular to the centre plane after the load has been applied. This is a proper assumption as long as the deformations caused by the shear loads can be neglected with respect to the deformations caused by bending and torsion. For a beam this assumption applies when the beam height t is small compared to the span l (i.e. $t \ll l$); for plates this holds too. Further, in the plate theory it is assumed that the displacement perpendicular to the centre plane is small compared to the plate thickness ($w \ll t$).
When this is not the case, non-linear terms will be introduced in the kinematic and equilibrium equations and also extensional forces will play a role (membrane action).
The *plate moments* m_{xx}, m_{yy}, m_{xy} and m_{yx} have the dimension of force (moment per unit of length). The sign convention is as follows: they are positive when the stress is positive for positive values of the z-coordinate. The torsional moments m_{xy} and m_{yx} are identical for isotropic plates. The shear forces v_x and v_y have the dimension of force per unit of length and are positive for positive stresses. Fig. 4.1 shows all the stress resultants. They are constantly indicated in positive direction. The figure also shows the only possible external load p together with the corresponding degree of freedom w.

The shear deformation caused by the shear forces v_x and v_y is neglected. Therefore, these stress resultants do not appear in the constitutive and equilibrium equations, which will be formulated in z-direction. However, the deformation due to the plate moments m_{xx}, m_{yy} and

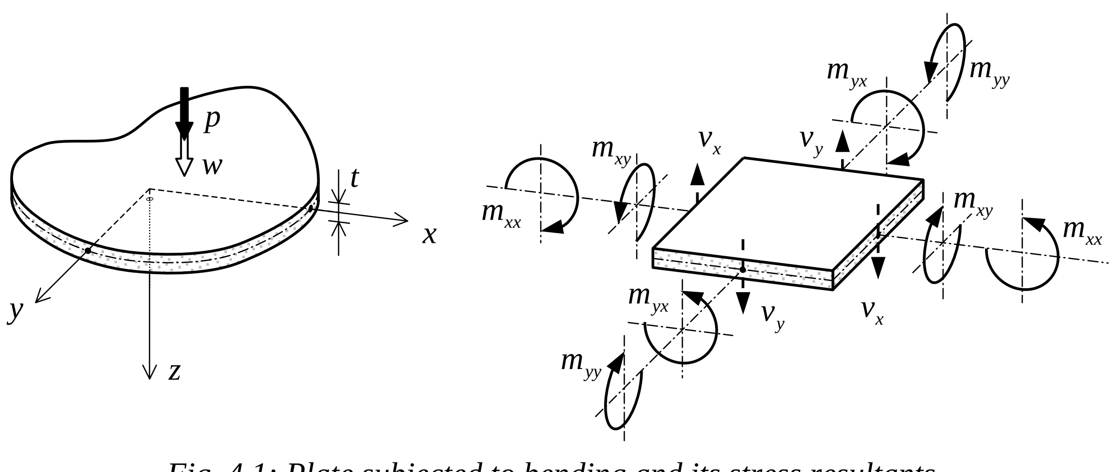


Fig. 4.1: Plate subjected to bending and its stress resultants.

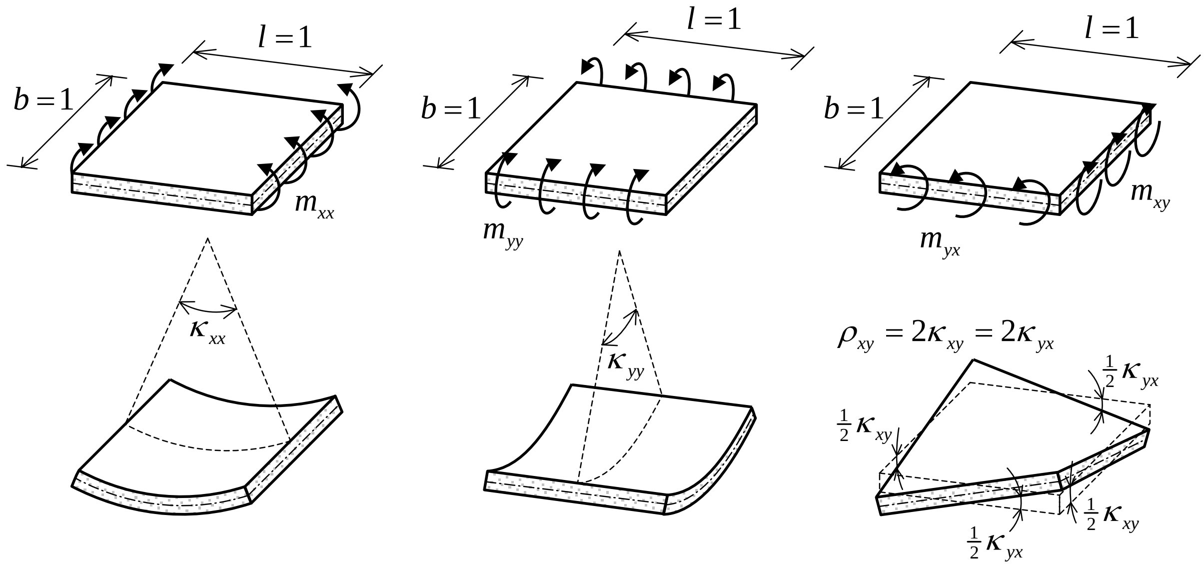


Fig. 4.2: Stress resultants and deformations in a plate subjected to bending.

m_{xy} will be considered, which means that these quantities determine the internal deformation energy. The deformations corresponding with these plate moments listed in the same order are κ_{xx}, κ_{yy} and ρ_{xy}. In literature, the deformation ρ_{xy} is often split into two equal parts κ_{xy} and κ_{yx}, corresponding with the torsional moments m_{xy} and m_{yx}, respectively. The deformations are drawn in Fig. 4.2 and the scheme of relations is given in Fig. 4.3.

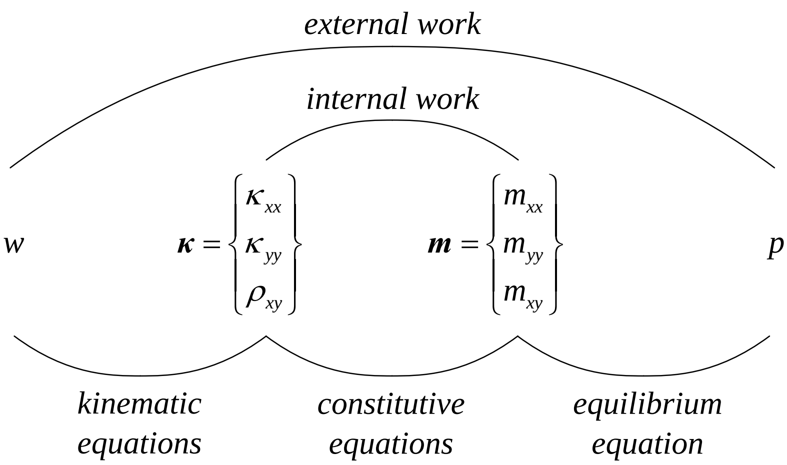


Fig. 4.3: Diagram displaying the relations between the quantities playing a role in the analysis of rectangular plates subjected to bending.

4.1.1 The three basic equations in an orthogonal coordinate system

As mentioned earlier, this type of plates was analysed already with respect to an orthogonal coordinate system in the course “Elastic Plates”. The following three sets of basic equations were found:

Kinematic equations

$$\boxed{\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}} \quad ; \quad \boxed{\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}} \quad ; \quad \boxed{\rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}} \tag{4.1}$$

Constitutive equations

$$\boxed{m_{xx} = D\left(\kappa_{xx} + \upsilon\,\kappa_{yy}\right)} \quad ; \quad \boxed{m_{yy} = D\left(\upsilon\,\kappa_{xx} + \kappa_{yy}\right)} \quad ; \quad \boxed{m_{xy} = D\left(\frac{1-\upsilon}{2}\right)\rho_{xy}} \tag{4.2}$$

where D is the plate stiffness given by:

$$D = \frac{E\,t^3}{12\left(1-\upsilon^2\right)}$$

Equilibrium equation

$$\boxed{-\left(\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2}\right) = p} \tag{4.3}$$

Remark 1

For verification purposes, the kinematic, constitutive and equilibrium equations are reformulated in operator form. With the introduction of:

$$\boldsymbol{\mathcal{B}} = \begin{Bmatrix} -\dfrac{\partial^2}{\partial x^2} \\ -\dfrac{\partial^2}{\partial y^2} \\ -2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} \quad ; \quad \boldsymbol{\mathcal{B}}' = \begin{Bmatrix} -\dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial^2}{\partial y^2} & -2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix}$$

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} \quad ; \quad \boldsymbol{m} = \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix} \tag{4.4}$$

$$\boldsymbol{D} = D\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1}{2}\left(1-\upsilon\right) \end{bmatrix} \quad ; \quad \boldsymbol{C} = \frac{1}{D\left(1-\upsilon^2\right)}\begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2\left(1+\upsilon\right) \end{bmatrix}$$

The relations (4.1), (4.2) and (4.3) can briefly be rewritten as:

$$\begin{aligned}
&\boldsymbol{\kappa} = \boldsymbol{\mathcal{B}}\, w \\
&\boldsymbol{\mathcal{B}}'\, \boldsymbol{m} = p \\
&\boldsymbol{m} = \boldsymbol{D}\,\boldsymbol{\kappa} \quad \Leftrightarrow \quad \boldsymbol{\kappa} = \boldsymbol{C}\,\boldsymbol{m}
\end{aligned} \tag{4.5}$$

Remark 2

The constitutive equations (4.2) are valid for a homogeneous isotropic plate of constant thickness, for which the plate stiffness D holds. The plate stiffness can also be formulated as:

$$\begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} D_{xx} & D_{\upsilon} & 0 \\ D_{\upsilon} & D_{yy} & 0 \\ 0 & 0 & \tfrac{1}{2} D_{xy} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} \tag{4.6}$$

where:

$$D_{xx} = D_{yy} = D \quad ; \quad D_{\upsilon} = \upsilon D \quad ; \quad D_{xy} = (1-\upsilon) D$$

The terms D_{xx} and D_{yy} represent the bending stiffness of a plate-strip of unit width and modulus of elasticity equal to $E/(1-\upsilon^2)$.
The third constitutive equation can also be written as:

$$m_{xy} = \frac{1}{2} D_{xy}\, \rho_{xy} \quad \rightarrow \quad m_{xy} = D_{xy}\, \kappa_{xy}$$

The term D_{xy} is the torsional stiffness GI_{xy} in the x-direction of a plate-strip with unit width and shear modulus $G = E/2(1+\upsilon)$. In chapter 6 it will be derived that the torsional moment of inertia of a cross-section with thickness t and unit width is equal to $\tfrac{1}{3}t^3$. Half of it is caused by the horizontal stresses in the cross-section, the other half results from the vertical stresses. In this case, in the plate only horizontal stresses occur, therefore the torsional moment of inertia becomes $I_{xy} = \tfrac{1}{6}t^3$.
The quantity κ_{xy} is the specific torsional deformation of the plate-strip. (In section 6.1 this quantity will be indicated by θ). For a plate-strip of unit width a corresponding torsional stiffness D_{yx} in the y-direction holds. The stiffnesses D_{xy} and D_{yx} are equal because m_{xy} and m_{yx} have the same value.

Remark 3

In the derivation it was assumed that the plate is homogeneous, isotropic and of constant thickness. The three basic equations are also used for the analysis of orthotropic plates or plates, which can be schematised as such (see Fig. 4.4).
In such a plate it still holds that $\kappa_{xy} = \kappa_{yx}$, but m_{xy} is not equal to m_{yx} anymore. Therefore it makes sense to define two different torsional stiffnesses D_{xy} and D_{yx}:

$$m_{xy} = \frac{1}{2} D_{xy}\, \rho_{xy} \quad ; \quad m_{yx} = \frac{1}{2} D_{yx}\, \rho_{yx}$$

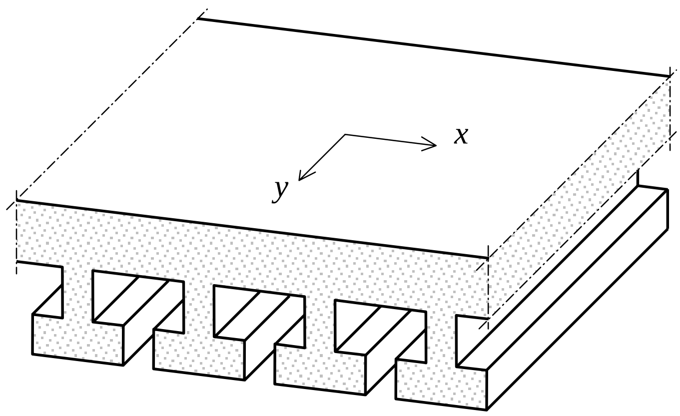


Fig. 4.4: The theory of thin plates can be applied to orthotropic plates too.

Contrary to a homogeneous plate, the quantities m_{xy} and m_{yx} are not identical and for that reason D_{xy} and D_{yx} differ too. Instead of (4.6) is can be written:

$$\begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_t \end{Bmatrix} = \begin{bmatrix} D_{xx} & D_\upsilon & 0 \\ D_\upsilon & D_{yy} & 0 \\ 0 & 0 & \frac{1}{2}D_t \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} \tag{4.7}$$

The terms D_{xx} and D_{yy} are generally not equal to each other. They are averaged values for a plate of unit width. The coupling term D_υ caused by lateral contraction has to be determined separately. The quantity m_t is the average of m_{xy} and m_{yx}, and D_t is the average of D_{xy} and D_{yx}. The designer has to determine the stiffnesses judiciously based on the considered structure.

4.1.2 Force method

In the force method a solution will be formulated for the stress resultants (in this case the moments), which satisfy the equilibrium equation. For a plate subjected to bending one equilibrium equation exists containing three moments. Therefore, two redundant stress functions ϕ_1 and ϕ_2 are present. The solution to be obtained consists of a homogeneous part depending on ϕ_1 and ϕ_2 and a particular part depending on the load p. The following type of solution complies wit the equation:

$$\begin{aligned} m_{xx} &= -\phi_{2,y} + \textit{particular part} \\ m_{yy} &= -\phi_{1,x} + \textit{particular part} \\ m_{xy} &= \frac{1}{2}\left(\phi_{1,y} + \phi_{2,x}\right) + \textit{particular part} \end{aligned} \tag{4.8}$$

The correctness of the homogeneous part can be checked by substitution of this solution into the equilibrium equation (4.3) with $p = 0$ (bear in mind that $m_{xy} = m_{yx}$). Two compatibility conditions can be formulated for the determination of the two stress functions, because the kinematic equations (4.1) contain three deformations and only one displacement. This displacement can be eliminated in two manners, namely from the first and second equation and from the second and third equation. The result equals:

$$\kappa_{yy,x} - \frac{1}{2}\rho_{xy,y} = 0 \quad ; \quad \kappa_{xx,y} - \frac{1}{2}\rho_{xy,x} = 0 \tag{4.9}$$

With the aid of the constitutive equations these two conditions can be transformed into conditions for the moments:

$$\frac{1}{D_0}\left\{m_{yy,x} - \upsilon m_{xx,x} - (1+\upsilon)m_{xy,y}\right\} = 0 \quad ; \quad \frac{1}{D_0}\left\{m_{xx,y} - \upsilon m_{yy,y} - (1+\upsilon)m_{xy,x}\right\} = 0 \tag{4.10}$$

where:

$$D_0 = \frac{E\,t^3}{12}$$

Substitution of solution (4.8) for the moments into (4.10) provides:

$$\begin{aligned}
&\frac{1}{D_0}\left(-\phi_{1,xx} - \frac{1+\upsilon}{2}\phi_{1,yy} - \frac{1-\upsilon}{2}\phi_{2,xy}\right) = -\Delta_1 \\
&\frac{1}{D_0}\left(-\phi_{2,yy} - \frac{1+\upsilon}{2}\phi_{2,xx} - \frac{1-\upsilon}{2}\phi_{1,xy}\right) = -\Delta_2
\end{aligned} \tag{4.11}$$

In this system of partial differential equations Δ_1 and Δ_2 are the gaps resulting from the particular solution. In operator form (4.11) can be written as:

$$\frac{1}{D_0}\begin{bmatrix} -\dfrac{\partial^2}{\partial x^2} - \dfrac{1+\upsilon}{2}\dfrac{\partial^2}{\partial y^2} & -\dfrac{1-\upsilon}{2}\dfrac{\partial^2}{\partial x \partial y} \\ -\dfrac{1-\upsilon}{2}\dfrac{\partial^2}{\partial x \partial y} & -\dfrac{\partial^2}{\partial y^2} - \dfrac{1+\upsilon}{2}\dfrac{\partial^2}{\partial x^2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\Delta_1 \\ -\Delta_2 \end{Bmatrix} \tag{4.12}$$

Remark 1
In practice, the force method is never applied for the derivation of analytical solutions. However, in the Finite Element Method special applications along this path can be put in practice.

Remark 2
The system (4.12) can also be obtained by introduction of the operators $\mathcal{P}$ and $\mathcal{P}'$ given by:

$$\mathcal{P} = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial y} \\ -\dfrac{\partial}{\partial x} & 0 \\ \dfrac{1}{2}\dfrac{\partial}{\partial y} & \dfrac{1}{2}\dfrac{\partial}{\partial x} \end{bmatrix} \quad ; \quad \mathcal{P}' = \begin{bmatrix} 0 & \dfrac{\partial}{\partial x} & -\dfrac{1}{2}\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & 0 & -\dfrac{1}{2}\dfrac{\partial}{\partial x} \end{bmatrix}$$

The relations (4.8) and (4.9) then become:

$$\boldsymbol{m} = \mathcal{P}\boldsymbol{\phi} + \textit{particular part} \quad ; \quad \mathcal{P}'\boldsymbol{\kappa} = \mathbf{0}$$

where $\boldsymbol{\phi}$ is the vector containing the two components ϕ_1 and ϕ_2 .With:

$$\boldsymbol{\kappa} = \boldsymbol{C}\,\boldsymbol{m}$$

the matrix system (4.12) can be written as:

$$\mathcal{P}'\,\boldsymbol{C}\,\mathcal{P}\,\boldsymbol{\phi} = -\boldsymbol{\Delta}$$

where $\boldsymbol{\Delta}$ is the vector containing the two components Δ_1 and Δ_2.

Remark 3
The force method for plates subjected to bending and the displacement method for plates subjected to extension demonstrate a large similarity. The role of ϕ_1 and ϕ_2 corresponds with the role of u_x and u_y . Also m_{xx} , m_{yy} , m_{xy} correspond with ε_{xx} , ε_{yy} , ε_{xy} , respectively (however with a sign difference). This analogy (dualism) has been used in the past to apply solutions in one field for problems in another field.

4.1.3 Displacement method

In the displacement method the kinematic and constitutive equations are substituted in the equilibrium equation. This approach is adopted in the course “Elastic Plates”. The result is the following differential equation:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = p \tag{4.13}$$

This is the biharmonic equation:

$$D\,\nabla^2\nabla^2 w = p \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4.14}$$

In literature the derivation of Lagrange (dated 1811) can be found.
No applications will be discussed, because this has been done already in the course “Elastic Plates”.

Remark 1
Relation (4.14) can be reformulated with operators. With the relations (4.5):

$$\boldsymbol{\kappa} = \mathcal{B}\,w \quad ; \quad \mathcal{B}'\,\boldsymbol{m} = p \quad ; \quad \boldsymbol{m} = \boldsymbol{D}\,\boldsymbol{\kappa}$$

equation (4.14) can be rewritten as:

$$\mathcal{B}'\boldsymbol{D}\mathcal{B}\,w = p$$

Remark 2

Again the analogy is demonstrated between the plate subjected to extension and the plate subjected to bending. For the first one the biharmonic equation holds for Airy's function and for second one the biharmonic equation for the deflection. The extensional forces n_{xx}, n_{yy}, n_{xy} correspond with the curvatures κ_{xx}, κ_{yy}, κ_{xy} (with a sign difference).

4.2 Axisymmetry for plates subjected to bending

4.2.1 Basic equations for axisymmetry

A special elaboration of the theory of plates can be made for circular plates resisting a load, which is axisymmetric with respect to the centre point of the plate. The vertical deflection w of the centre plane will solely depend on the radius r, the distance to the centre point. The moments and shear forces are defined in the direction of the polar coordinates r and θ. The moments to be distinguished are the radial moments m_{rr} and the tangential moments $m_{\theta\theta}$. Due to the axisymmetry no torsional moments $m_{r\theta}$ occur.

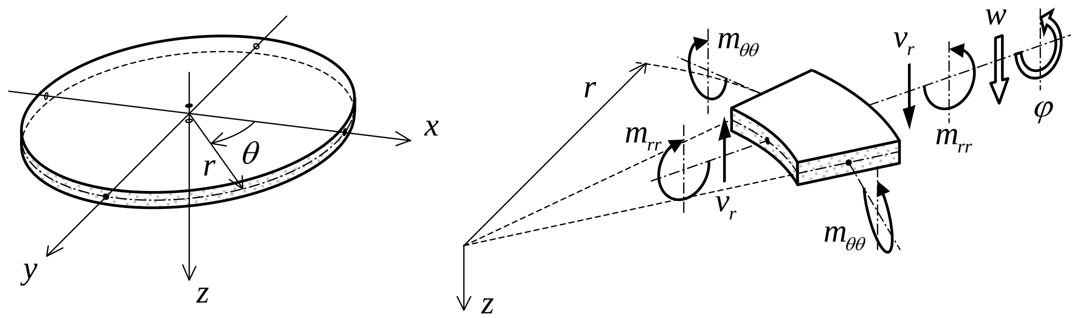


Fig. 4.5: Relevant quantities for axisymmetric plates subjected to bending.

Only one shear force component is present, namely the radial shear force v_r acting on areas of constant r. For axisymmetric reasons, on areas of constant θ no shear force v_θ develops. The relevant quantities are indicated in Fig. 4.5.
Two independent degrees of freedom w and φ are defined, analogously to the defined

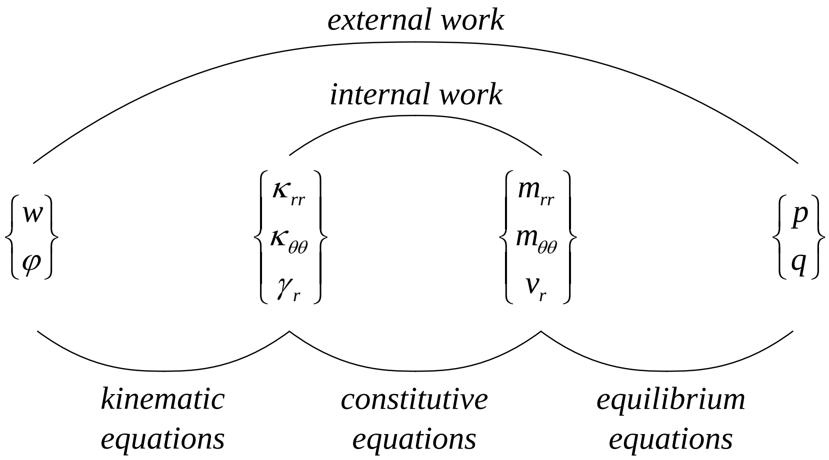


Fig. 4.6: Diagram displaying the relations between the quantities playing a role in the analysis of axisymmetric plates subjected to bending.

orthogonal coordinates in the course "Elastic Plates". So, initially deformation by shear forces is not excluded. The deflection w is positive if it is pointing in positive z-direction. The rotation φ is positive, when at the positive z-side of the centre plane a radial displacement u occurs in positive r-direction (see Fig. 4.5). Fig. 4.6 shows the scheme of relations, where p is a distributed load in w-direction and q a distributed moment in φ-direction.

Kinematic relations

The relation between the curvature κ_{rr} and the rotation φ is simple:

$$\kappa_{rr} = \frac{d\varphi}{dr}$$

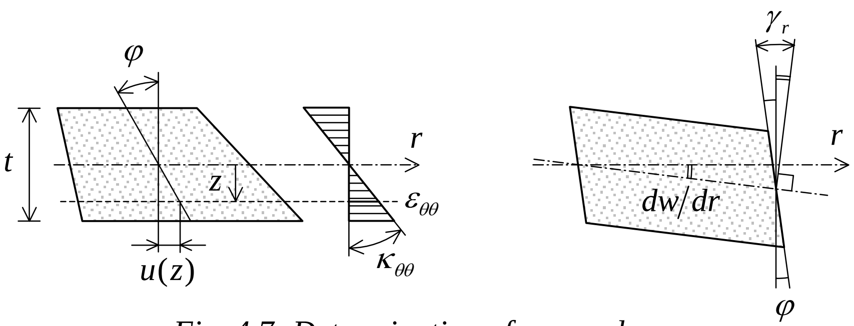


Fig. 4.7: Determination of $\kappa_{\theta\theta}$ and γ_r.

The calculation of $\kappa_{\theta\theta}$ is more complicated, because no rotation in θ-direction exists. It should be kept in mind that the curvature $\kappa_{\theta\theta}$ is equal to the gradient of the strains $\varepsilon_{\theta\theta}$ across the plate thickness, see Fig. 4.7. For axisymmetric plates subjected to extension it already was derived that:

$$\varepsilon_{\theta\theta} = \frac{u}{r}$$

At distance z from the centre plane the displacement equals $u(z) = z\varphi$, the strain at that spot equals:

$$\varepsilon_{\theta\theta}(z) = z\,\frac{\varphi}{r}$$

It also holds $\varepsilon_{\theta\theta} = z\,\kappa_{\theta\theta}$ (see Fig. 4.7), so that it can be concluded:

$$\kappa_{\theta\theta} = \frac{\varphi}{r}$$

For the shear strain it holds:

$$\gamma_r = \frac{dw}{dr} + \varphi$$

Summarising it can be stated that:

$$\kappa_{rr} = \frac{d\varphi}{dr} \quad ; \quad \kappa_{\theta\theta} = \frac{\varphi}{r} \quad ; \quad \gamma_r = \frac{dw}{dr} + \varphi \tag{4.15}$$

Constitutive relations
The constitutive equations speak for themselves:

$$m_{rr} = D\left(\kappa_{rr} + \upsilon\,\kappa_{\theta\theta}\right) \quad ; \quad m_{\theta\theta} = D\left(\upsilon\,\kappa_{rr} + \kappa_{\theta\theta}\right) \quad ; \quad v_r = D_\gamma\,\gamma_r \tag{4.16}$$

with:

$$D = \frac{E\,t^3}{12\left(1-\upsilon^2\right)} \quad ; \quad D_\gamma = \frac{E\,t}{2\left(1+\upsilon\right)}\frac{1}{\eta}$$

where η is a shape-factor (for a rectangular cross-section the value equals $6/5$).

Equilibrium relations
The equilibrium is considered of an elementary plate particle with apex angle $d\theta$ and length dr in r-direction, see Fig. 4.8. The equilibrium equations in w- and φ-directions become:

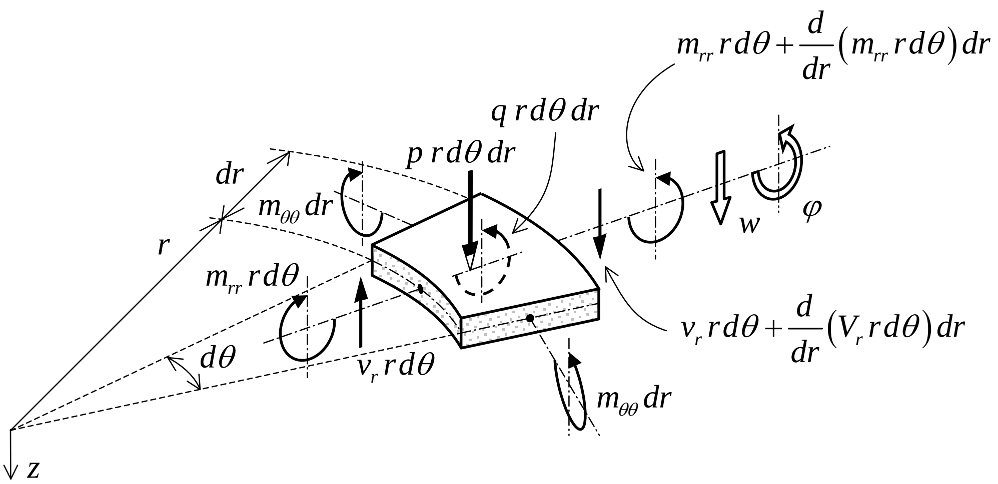


Fig. 4.8: Relevant forces and torques in the equilibrium equation for axisymmetric problems.

$$\begin{aligned} -\frac{d}{dr}\left(r\,v_r\right) &= r\,p && (w\text{-direction}) \\ -\frac{d}{dr}\left(r\,m_{rr}\right) + m_{\theta\theta} + r\,v_r &= r\,q && (\varphi\text{-direction}) \end{aligned} \tag{4.17}$$

It can be confirmed whether $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by the correct procedure of transposition. Therefore, (4.15) and (4.17) are written in operator form:

$$\begin{Bmatrix} \kappa_{rr} \\ \kappa_{\theta\theta} \\ \gamma_r \end{Bmatrix} = \begin{bmatrix} 0 & d/dr \\ 0 & 1/r \\ d/dr & 1 \end{bmatrix} \begin{Bmatrix} w \\ \varphi \end{Bmatrix} \quad ; \quad \begin{bmatrix} 0 & 0 & -d/dr \\ -d/dr & 1/r & 1 \end{bmatrix} \begin{Bmatrix} r\,m_{rr} \\ r\,m_{\theta\theta} \\ r\,v_r \end{Bmatrix} = \begin{Bmatrix} r\,p \\ rq \end{Bmatrix} \tag{4.18}$$

Again the uneven derivatives change of sign, which means that the operators are correctly related.
After introduction of moments and forces given by:

$$\begin{aligned} &M_{rr} = r\,m_{rr} \quad ; \quad P = r\,p \quad ; \quad V_r = r\,v_r \\ &M_{\theta\theta} = r\,m_{\theta\theta} \quad ; \quad Q = rq \end{aligned} \tag{4.19}$$

the basic equations can be summarised as follows:

$$\boxed{\begin{aligned} \kappa_{rr} &= \frac{d\varphi}{dr} \\ \kappa_{\theta\theta} &= \frac{\varphi}{r} \\ \gamma_r &= \frac{dw}{dr} + \varphi \end{aligned}} \qquad (\textit{kinematic equations}) \tag{4.20a}$$

$$\boxed{\begin{aligned} M_{rr} &= r\,D\left(\kappa_{rr} + \upsilon\,\kappa_{\theta\theta}\right) \\ M_{\theta\theta} &= r\,D\left(\upsilon\,\kappa_{rr} + \kappa_{\theta\theta}\right) \\ V_r &= r\,D_\gamma\,\gamma \end{aligned}} \qquad (\textit{constitutive equations}) \tag{4.20b}$$

$$\boxed{\begin{aligned} &-\frac{dV_r}{dr} = P && (w\textit{-direction}) \\ &-\frac{dM_{rr}}{dr} + \frac{M_{\theta\theta}}{r} + V_r = Q && (\varphi\textit{-direction}) \end{aligned}} \qquad (\textit{equilibrium equations}) \tag{4.20c}$$

In (4.20)c the analogy with the beam theory easily can be recognised. When the distributed moment q is zero (and therefore also Q) and the radius r is infinitely large, the second equation states that the shear force V_r is the derivative of the moment M_{rr}.

Neglect of the shear deformation

The next step is to neglect the deformation caused by the shear force. It is assumed that $\gamma_r = 0$, so the third term in (4.20)a becomes:

$$\varphi = -\frac{dw}{dr}$$

which transforms the first two terms of (4.20)a into:

$$\kappa_{rr} = -\frac{d^2 w}{dr^2} \quad ; \quad \kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} \tag{4.21}$$

The third constitutive relation in (4.20)b disappears and in the second equilibrium equation of (4.20)c the shear force Q cannot exist, because φ is not an independent degree of freedom anymore. Therefore, the right-hand side becomes zero. From this equation the shear force is solved:

$$V_r = \frac{dM_{rr}}{dr} - \frac{M_{\theta\theta}}{r} \qquad (4.22)$$

Substitution of this result into the first equilibrium equation of (4.20)c delivers:

$$-\frac{d^2 M_{rr}}{dr^2} + \frac{d}{dr}\left(\frac{M_{\theta\theta}}{r}\right) = P \qquad (4.23)$$

The basic equations are reduced to:

$$\boxed{\begin{aligned} \kappa_{rr} &= -\frac{d^2 w}{dr^2} \\ \kappa_{\theta\theta} &= -\frac{1}{r}\frac{dw}{dr} \end{aligned}} \qquad (\textit{kinematic equations}) \qquad (4.24)\text{a}$$

$$\boxed{\begin{aligned} M_{rr} &= rD\left(\kappa_{rr} + \upsilon\kappa_{\theta\theta}\right) \\ M_{\theta\theta} &= rD\left(\upsilon\kappa_{rr} + \kappa_{\theta\theta}\right) \end{aligned}} \qquad (\textit{constitutive equations}) \qquad (4.24)\text{b}$$

$$\boxed{-\frac{d^2 M_{rr}}{dr^2} + \frac{d}{dr}\left(\frac{M_{\theta\theta}}{r}\right) = P} \qquad (\textit{equilibrium equation}) \qquad (4.24)\text{c}$$

The analysis of the axisymmetric plate without shear deformation can be summarised by showing how the scheme of Fig. 4.6 changes (see Fig. 4.9).

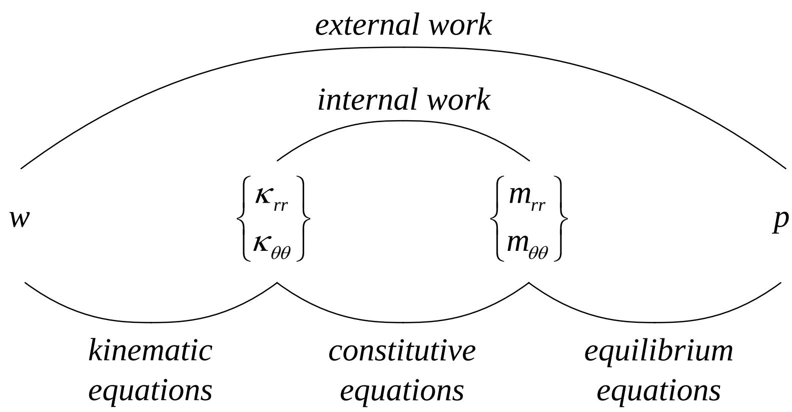


Fig. 4.9: Diagram displaying the relations between the quantities playing a role in the analysis of a plate subjected to bending without shear deformation.

4.2.2 Differential equation

The differential equation for plates without shear deformation can be derived from the basic equations (4.24) (with $P = pr$):

$$\boxed{D\left(\frac{\mathrm{d}^4 w}{\mathrm{d}r^4}+\frac{2}{r}\frac{\mathrm{d}^3 w}{\mathrm{d}r^3}-\frac{1}{r^2}\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}+\frac{1}{r^3}\frac{\mathrm{d}w}{\mathrm{d}r}\right)=p} \tag{4.25}$$

This differential equation will be derived once more from the more general case, which is formulated by the set (4.20). Because then it is easier to link up with the results for circular plates subjected to extension (section 3.5) and rectangular plates subjected to bending (section 4.1). Substitution of κ_{rr} and $\kappa_{\theta\theta}$ from (4.20)a into M_{rr} and $M_{\theta\theta}$ from (4.20)b leads to:

$$M_{rr}=D\left(r\frac{d\varphi}{dr}+\upsilon\varphi\right) \quad ; \quad M_{\theta\theta}=D\left(\varphi+\upsilon r\frac{d\varphi}{dr}\right)$$

With this result, the equilibrium equation in φ-direction from (4.20)c provides the following expression for the shear force V_r (The load term Q is zero since no shear deformation can occur):

$$V_r=D\left\{\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)-\frac{\varphi}{r}\right\}=D\left\{\frac{d}{dr}r\frac{d}{dr}-\frac{1}{r}\right\}\varphi$$

The operator between braces is exactly the previously obtained operator $\mathcal{L}$ for axisymmetric plates subjected to extension:

$$\boxed{\mathcal{L}=\frac{d}{dr}r\frac{d}{dr}-\frac{1}{r} \quad \text{or} \quad \mathcal{L}=r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r} \tag{4.26}$$

The expression for the shear force can now be briefly written as:

$$\boxed{V_r=D\mathcal{L}\varphi \quad \text{with} \quad \varphi=-\frac{dw}{dr}} \tag{4.27}$$

After division by r and substitution of $\mathcal{L}$ and φ, expression (4.27) delivers the following relation:

$$\boxed{v_r=-D\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)w} \tag{4.28}$$

The term between brackets is the operator of Laplace given by:

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

Comparison with (4.24)a shows that the two curvatures in the r - and θ -directions appear. This agrees with the definition of ∇^2 for rectangular plates in (4.14). Thus, expression (4.28) for the shear force becomes:

$$v_r = -D\frac{d}{dr}\nabla^2 w \tag{4.29}$$

The last step in the analysis is the introduction of the shear force given by (4.27) in the equilibrium equation for the w -direction of (4.20)c. The result is the required differential equation:

$$\boxed{D\frac{d}{dr}\mathcal{L}\frac{d}{dr}w = p} \tag{4.30}$$

Similarly, differential equation (4.30) can be worked out further. Substitution of $\mathcal{L}$ and division by r transforms the differential equation into:

$$\boxed{D\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)w = p} \tag{4.31}$$

In this formulation the Laplace operator appears twice, so that the biharmonic equation can be recognised again:

$$D\nabla^2\nabla^2 w = p \tag{4.32}$$

Now full correspondence is reached with the description for orthogonal coordinates provided in (4.14).
Working out equation (4.31) leads to the differential equation (4.25), which was directly obtained from the basic equations.

Remark
For the term:

$$\frac{1}{r}\frac{dw}{dr}$$

another derivation will be given, which is based on geometrical considerations and really shows what is happening in the plate.
Two concentric circles are considered, one with radius r and another with a slightly larger radius $r + dr$ (see Fig. 4.10). The tangent line in point A of the circle with radius r is called the t -axis. This axis is perpendicular to the radius. In order to show that d^2w/dt^2 is equal to

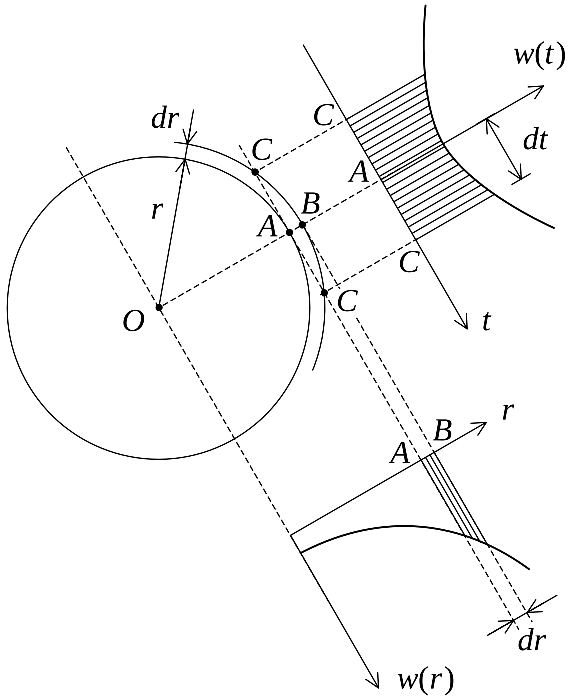


Fig. 4.10: Clarification of the curvature $\kappa_{\theta\theta}$ obtained by an alternative derivation.

$\left(dw/dr\right)/r$, the increase of the deflection of the points B and C situated on the second circle is considered. From the graph representing the deflection $w(t)$ as a function of t, it is clear that a curvature (i.e. a second derivative) develops in the t-direction, when w increases in the r-direction. When the deflection in point A is equal to w, it will increase in the r-direction to w_B. For w_B the Taylor expansion around point A holds:

$$w_B = w + \frac{dw}{dr}dr + \frac{1}{2}\frac{d^2w}{dr^2}\left(dr\right)^2 + \cdots \tag{4.33}$$

Going from point A in t-direction the deflection increases to w_C, for which the following Taylor expansion around point A is valid:

$$w_C = w + \frac{dw}{dt}dt + \frac{1}{2}\frac{d^2w}{dt^2}\left(dt\right)^2 + \cdots \tag{4.34}$$

In (4.33) all first three terms in the right-hand side can be different from zero. However, for small values of dr the following approximation holds:

$$w_B - w = \frac{dw}{dr}dr \tag{4.35}$$

where the higher order terms are neglected.
For reasons of symmetry the first derivative dw/dt will be zero, just like all higher order uneven derivatives. Therefore for small values of dr the increase can be approximated by:

$$w_C - w = \frac{1}{2}\frac{d^2 w}{dt^2}(dt)^2 \tag{4.36}$$

The displacements w_B and w_C are the same because the points B and C are situated on the same circle. Therefore from (4.35) and (4.36) it follows:

$$\frac{1}{2}\frac{d^2 w}{dt^2}(dt)^2 = \frac{dw}{dr}dr \tag{4.37}$$

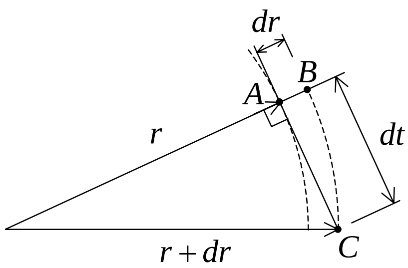


Fig. 4.11: Relation between dt, r and dr.

It is also possible to write $(dt)^2$ as a function of r. From triangle OAC of Fig. 4.11 the following relation yields:

$$(dt)^2 = (r + dr)^2 - r^2 = 2r\,dr + (dr)^2$$

For sufficiently small values of dr the term $(dr)^2$ can be neglected with respect to $2r\,dr$, so that it holds:

$$(dt)^2 = 2r\,dr$$

Substitution of this result into (4.37) gives:

$$\frac{1}{2}\frac{d^2 w}{dt^2}(2r\,dr) = \frac{dw}{dr}dr$$

Division of this equation by $r\,dr$ finally provides the required expression:

$$\frac{d^2 w}{dt^2} = \frac{1}{r}\frac{dw}{dr} \tag{4.38}$$

4.3 Axisymmetric applications

The solution of differential equation (4.31) for a uniformly distributed load p reads:

$$w = C_1 + C_2\, r^2 + C_3 \ln r + C_4\, r^2 \ln r + \frac{p\,r^4}{64D} \tag{4.39}$$

The last term is the particular solution. The four integration constants have to be determined from the boundary conditions. After that the moments and the shear force can be obtained from:

$$m_{rr} = -D\left(\frac{d^2w}{dr^2} + \frac{\upsilon}{r}\frac{dw}{dr}\right) \quad ; \quad m_{\theta\theta} = -D\left(\frac{1}{r}\frac{dw}{dr} + \upsilon\frac{d^2w}{dr^2}\right)$$
$$v_r = -D\frac{d}{dr}\nabla^2 w = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \tag{4.40}$$

For the formulation of the boundary conditions the derivatives of w are required:

$$\frac{dw}{dr} = \frac{p\,r^3}{16D} + 2\,C_2\,r + \frac{C_3}{r} + 2\,C_4\,r\ln r + C_4\,r$$
$$\frac{d^2w}{dr^2} = \frac{3p\,r^2}{16D} + 2\,C_2 - \frac{C_3}{r^2} + 3\,C_4 + 2\,C_4\ln r \tag{4.41}$$
$$\frac{d^3w}{dr^3} = \frac{3p\,r}{8D} + \frac{2C_3}{r^3} + \frac{2C_4}{r}$$

The expression for the shear force v_r is important:

$$v_r = -\frac{1}{2}pr - \frac{4D\,C_4}{r} \tag{4.42}$$

4.3.1 Simply supported circular plate with boundary moment

Fig. 4.12 shows a simply supported circular plate, which loaded by a boundary moment m_0. For the outer edge the radius is $r = a$, in the centre point $r = 0$. The boundary conditions are:

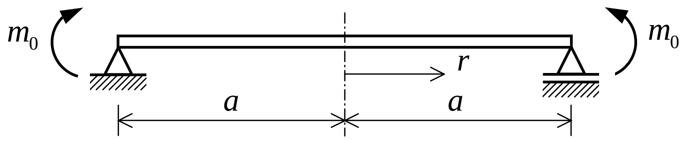


Fig. 4.12: Simply supported circular plate with edge moment.

$$r = 0 \quad \rightarrow \quad \begin{cases} \dfrac{dw}{dr} = 0 \\ v_r = 0 \end{cases} \qquad ; \qquad r = a \quad \rightarrow \quad \begin{cases} w = 0 \\ m_{rr} = m_0 \end{cases}$$

From the boundary conditions for $r = 0$ it follows:

$$C_3 = C_4 = 0$$

From the boundary conditions for $r = a$ it can be obtained:

$$C_1 = \frac{m_0\, a^2}{2(1+\upsilon)D} \quad ; \quad C_2 = -\frac{m_0}{2(1+\upsilon)D}$$

With the found results for the four constants, the final solution for the vertical deflection becomes:

$$\boxed{w = \frac{m_0}{2(1+\upsilon)D}\left(a^2 - r^2\right)} \tag{4.43}$$

The curvatures are:

$$\kappa_{rr} = -\frac{d^2 w}{dr^2} = \frac{m_0}{(1+\upsilon)D} \quad ; \quad \kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = \frac{m_0}{(1+\upsilon)D}$$

and the moments:

$$m_{rr} = D\left(\kappa_{rr} + \upsilon\,\kappa_{\theta\theta}\right) = m_0 \quad ; \quad m_{\theta\theta} = D\left(\upsilon\,\kappa_{rr} + \kappa_{\theta\theta}\right) = m_0$$

In this case, a homogeneous state of moments exists, because for each r the following expressions are valid:

$$m_{rr} = m_{\theta\theta} = m_0 \quad ; \quad v_r = 0$$

A homogeneous stress-state was also found for the circular plate subjected to extension as discussed in section 3.5. In Fig. 3.21 the load cases are shown for which it holds $\sigma_{rr} = \sigma_{\theta\theta}$.

4.3.2 Restrained circular plate with uniformly distributed load

Now a restrained plate is considered, loaded by a uniformly distributed load p (see Fig. 4.13). Again the outer edge of the plate is given by $r = a$ and the centre by $r = 0$. The boundary conditions are:

$$r = 0 \quad \rightarrow \quad \begin{cases} \dfrac{dw}{dr} = 0 \\ v_r \;= 0 \end{cases} \qquad ; \qquad r = a \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dr} = 0 \end{cases}$$

From the boundary conditions for $r = 0$ it is found:

$$C_3 = C_4 = 0$$

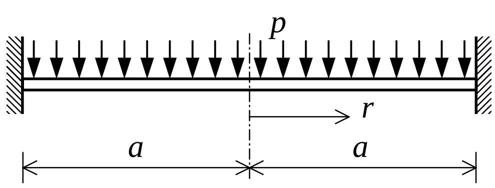


Fig. 4.13: Restrained circular plate with uniformly distributed load.

and from the boundary conditions for $r = a$ it follows:

$$C_1 = \frac{p\,a^4}{64D} \quad ; \quad C_2 = -\frac{p\,a^2}{32D}$$

With those four constants the following function for the deflection can be found:

$$\boxed{w = \frac{p}{64D}\left(a^2 - r^2\right)^2} \tag{4.44}$$

Of course, the maximum of the vertical deflection occurs in the centre of the plate, so for $r = 0$ this deflection becomes:

$$w_{\max} = \frac{pa^4}{64D}$$

The bending moments in this case are (see (4.40)):

$$m_{rr} = \frac{1}{16}pa^2\left((1+\upsilon)-(3+\upsilon)\frac{r^2}{a^2}\right) \quad ; \quad m_{\theta\theta} = \frac{1}{16}pa^2\left((1+\upsilon)-(1+3\upsilon)\frac{r^2}{a^2}\right) \tag{4.45)a}$$

and the shear force equals:

$$v_r = -\frac{1}{2}pr \tag{4.45)b}$$

Fig. 4.14 shows the graphical representation of these results.

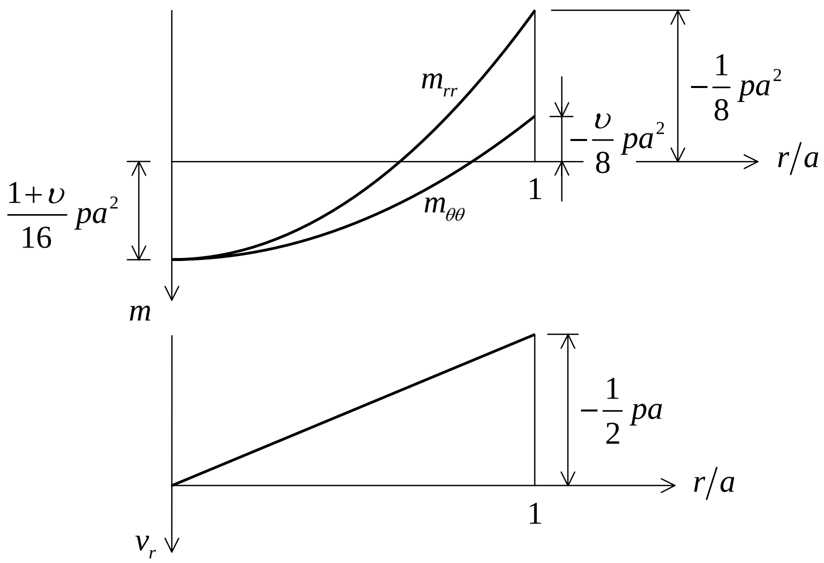


Fig. 4.14: Results for uniformly distributed load on restrained circular plate.

Remark 1
In the centre of the plate the moments m_{rr} and $m_{\theta\theta}$ are equal. At that spot the distinction between these moments disappears. At the restrained circumference the tangential moment $m_{\theta\theta}$ is just υ times the radial moment m_{rr}. This result also can be found for straight restrained edges.

Remark 2
For the considered load, the distribution of the shear force is comparable with the one in a beam. The expression for the shear force ($v_r = -pr/2$) can also directly be derived from the equilibrium of a plate part with radius r, for which it easily can be derived that (see Fig. 4.15):

$$2\pi r v_r + \pi r^2 p = 0$$

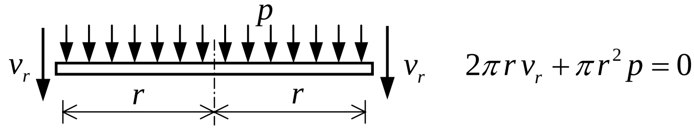


Fig. 4.15: Free-body diagram.

4.3.3 Simply supported circular plate with uniformly distributed load

The solution for the simply supported plate as drawn in Fig. 4.16 can easily be obtained from the solution of the restrained plate. Along the circumference of the restrained plate a radial moment is present equal to $m_{rr} = -pa^2/8$. For the simply supported plate this moment has to

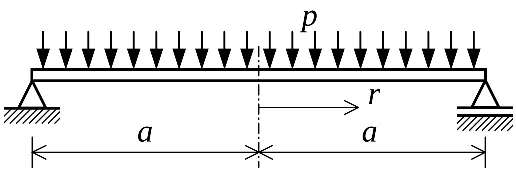


Fig. 4.16: Simply supported circular plate with uniformly distributed load.

be zero. Therefore, the solution for the simply supported plate can be found by superposition of the solution for the restrained plate and the solution for a circular plate with just having a boundary moment of magnitude $m_{rr} = +pa^2/8$. After superposition of the solutions it directly follows:

$$m_{rr} = \frac{3+\upsilon}{16} pa^2 \left(1 - \frac{r^2}{a^2}\right) \quad ; \quad m_{\theta\theta} = \frac{1}{16} pa^2 \left((3+\upsilon) - (1+3\upsilon)\frac{r^2}{a^2}\right)$$
$$v_r = -\frac{1}{2} p r \tag{4.46}$$

This means that in Fig. 4.14 the horizontal axis shifts upward over a distance of $pa^2/8$, leading to Fig. 4.17.

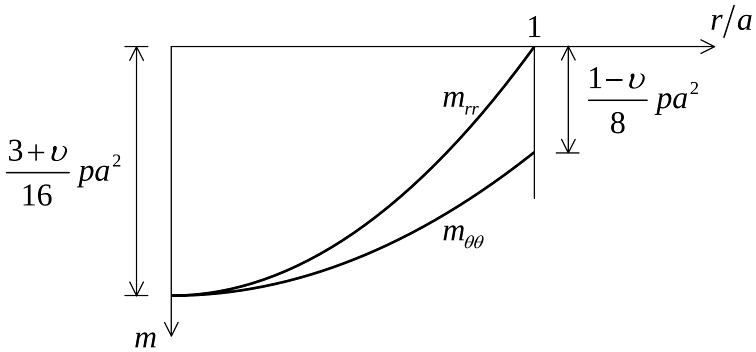


Fig. 4.17: Results for uniformly distributed load on simply supported circular plate.

4.3.4 Circular plate with point load in the centre

In this case the plate is only loaded in the centre. This load can be introduced as a boundary condition. Therefore, for all values r between 0 and a it holds:

$$w = C_1 + C_2\, r^2 + C_3 \ln r + C_4\, r^2 \ln r \tag{4.47}$$

The derivatives can be obtained from (4.41) with $p = 0$:

$$\frac{dw}{dr} = 2\,C_2\, r + \frac{C_3}{r} + 2\,C_4\, r \ln r + C_4\, r$$
$$\frac{d^2w}{dr^2} = 2\,C_2 - \frac{C_3}{r^2} + 3\,C_4 + 2\,C_4 \ln r$$
$$\frac{d^3w}{dr^3} = \frac{2C_3}{r^3} + \frac{2C_4}{r}$$

The expression (4.42) for the shear force becomes:

$$v_r = -\frac{4D\,C_4}{r}$$

Restrained plate

First the restrained case is considered as shown in Fig. 4.18. The boundary conditions are:

$$r = 0 \quad \rightarrow \quad \begin{cases} \dfrac{dw}{dr} = 0 \\ v_r = -\dfrac{F}{2\pi r} \end{cases} \qquad ; \qquad r = a \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dr} = 0 \end{cases}$$

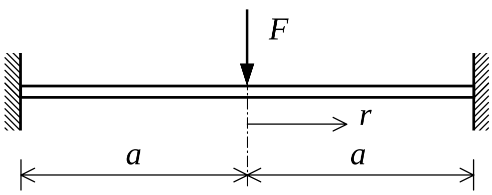
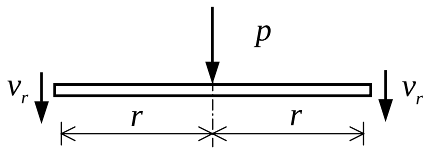


Fig. 4.18: Restrained circular plate with point load in centre.



$$2\pi r\, v_r + F = 0$$

Fig. 4.19: Free-body diagram.

The second boundary condition for $r = 0$ requires some explanation. It expresses that a small circular section with length $2\pi r$ has to transmit a total force F (see Fig. 4.19). This means that the shear force per unit of length equals $F/2\pi r$. The minus sign follows from the sign convention for the shear force. The boundary conditions for $r = 0$ respectively provide:

$$C_3 = 0 \quad ; \quad C_4 = \frac{F}{8\pi D}$$

From the boundary conditions along the outer edge, the two other constants can be determined:

$$C_2 = -\frac{F}{16\pi D} - \frac{F}{8\pi D}\ln a \quad ; \quad C_1 = \frac{F}{16\pi D}$$

Finally the solution becomes:

$$w = \frac{Fa^2}{16\pi D}\left(1 - \frac{r^2}{a^2}\right) + \frac{Fr^2}{8\pi D}\ln\left(\frac{r}{a}\right)$$

Also in this case the largest displacement occurs in the centre of the plate:

$$w_{\max} = \frac{Fa^2}{16\pi D}$$

Comparison of this result with the maximum vertical deflection for a uniformly distributed load shows that the deflection increases by a factor four when the full load ($F = \pi a^2 p$) is concentrated in the centre of the plate.
For the moments it is found:

$$m_{rr} = \frac{F}{4\pi}\left[(1+\upsilon)\ln\left(\frac{a}{r}\right) - 1\right] \quad ; \quad m_{\theta\theta} = \frac{F}{4\pi}\left[(1+\upsilon)\ln\left(\frac{a}{r}\right) - \upsilon\right] \qquad (4.48)\text{a}$$

The shear force is:

$$v_r = -\frac{F}{2\pi r} \qquad (4.48)\text{b}$$

These results are displayed in Fig. 4.20.

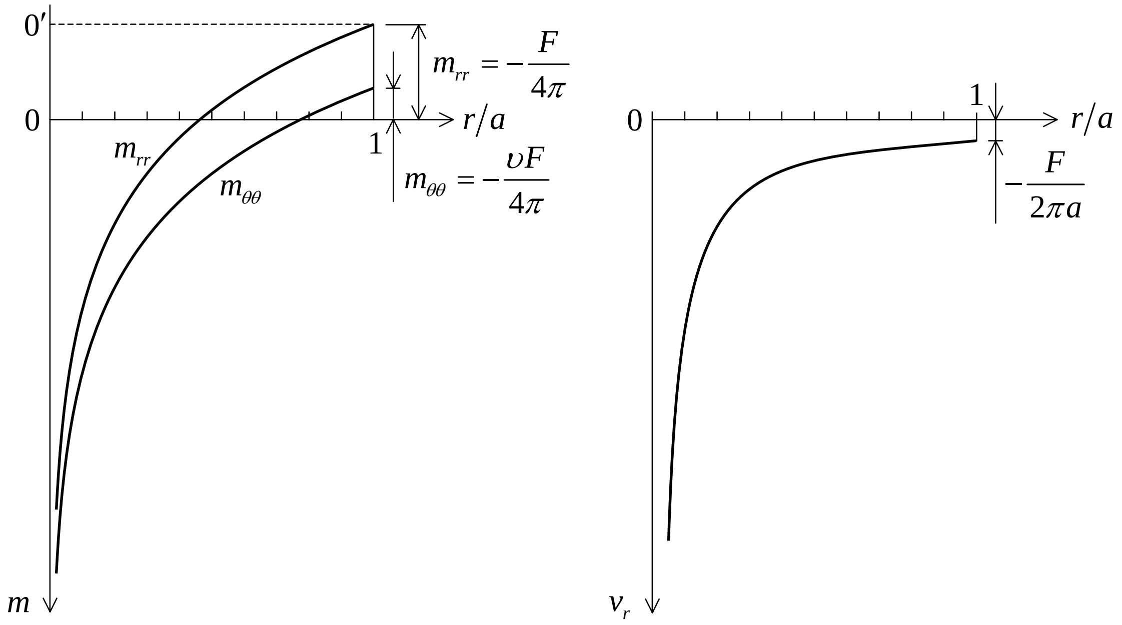


Fig. 4.20: Results for point load on restrained circular plate.

Simply supported plate
The solution for the simply supported plate can be found by shifting the horizontal axis in Fig. 4.20 for the moments in vertical direction, such that m_{rr} becomes zero at the edge $r = a$. The origin then shifts from 0 to $0'$. The graph for the shear force remains the same.

Remark 1
The most important result is the fact that the moments and the shear force in the centre of the plate are infinitely large. The bending and shear stresses become infinitely large too. The same holds for the vertical stress σ_{zz} just beneath the point load. In reality infinitely large stresses never develop because theoretical point loads do not exist. In the neighbourhood of a point load, assumptions of the plate-theory are not satisfied anymore. At a distance of approximately the plate thickness from the point load, the plate theory is valid again.

Remark 2
Here, the singular character of the moments and shear force has been derived for a circular plate. Also for other cases, these quantities will be very large in the neighbourhood of concentrated loads. It even can be stated that the behaviour in the neighbourhood of all concentrated loads is of the same type of character. For example, the formulae (4.48) can also be used for the calculation of the stresses in the neighbourhood of a concentrated load on a square plate. The only problem is the value to be assigned to the radius a. For this radius, for example the smallest distance to the edge can be chosen, or even better a sort of average distance to the edge. Moreover, when the radius r is very small, the exact value of a in relation (4.48)a for the moments is not that important anymore. For a load over a radius r_0 that is clearly smaller than a, as a first approximation the stresses can be calculated on basis of (4.48). The value of r_0 may rise to an upper limit that is different from case to case, but generally will be of the order of magnitude of $a/4$.

4.4 General solution procedure for circular plates

In literature, solutions to many problems can be found. In this respect the book "*Theory of plates and shells*" of S.P. Timoshenko and S. Woinowsky-Krieger can be mentioned. Generally the following axisymmetric situations are dealt with:

- only a part of the circular plate carries a uniformly distributed load;
- one or more circular line loads may occur;
- the plate suddenly may change of thickness;
- one or more circular line springs may be present.

In such cases the total plate ($0 \le r \le a$) is subdivided into a number concentric rings. A new ring is introduced when the distributed load suddenly changes in magnitude, or when a line load appears, or when a line spring is present, or when the thickness suddenly changes. For all plate parts a differential equation should hold, the solution of which contains four constants. When n plate parts can be distinguished, $4n$ unknowns have to be solved. So, $4n$ boundary conditions are required. This means 2 boundary conditions for $r = 0$ and 2 boundary conditions for $r = a$. On each of the $n-1$ transitions, 4 transition conditions have to be determined. Normally, this concerns the deflection w and its derivative dw/dr, as well as the moment m_{rr} and the shear force v_r. These last two quantities lead to transition conditions for:

$$\left(\frac{d^2w}{dr^2} + \upsilon \frac{dw}{dr} \right) \quad ; \quad \left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} \right)$$

A modern approach linking up with this theory is the numerical analysis of structures by the displacement method. Then the plate is divided into elements (above-mentioned concentric rings) and the load is concentrated in circular line loads.

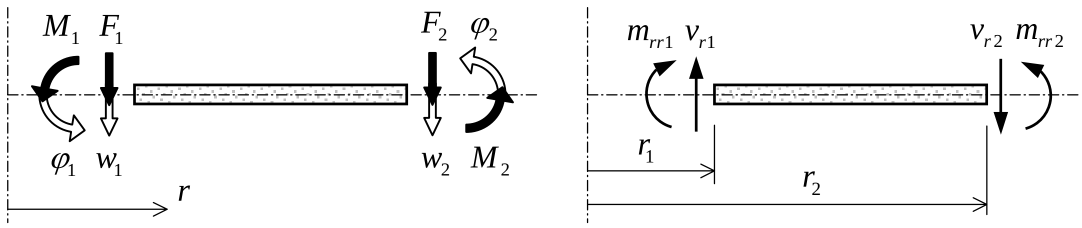


Fig. 4.21: Elemental degrees of freedom and generalised forces.

The connection between two elements is called a (circular) node. In this node two degrees of freedom are defined, w and $\varphi = dw/dr$. Per element i, two nodes 1 and 2 can be distinguished. With the four degrees of freedom w_1, φ_1, w_2 and φ_2 correspond four generalised nodal forces F_1, M_1, F_2 and M_2, respectively (see Fig. 4.21).
The stiffness matrix of such an element looks like:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} \tag{4.49}$$

This matrix will be symmetric if the generalised nodal forces are defined as follows:

$$\left.\begin{aligned} F_1 &= -2\pi\, r_1\, v_{r1} \\ M_1 &= -2\pi\, r_2\, m_{rr1} \end{aligned}\right\} \text{node 1} \quad ; \quad \left.\begin{aligned} F_2 &= -2\pi\, r_2\, v_{r2} \\ M_2 &= -2\pi\, r_2\, m_{rr2} \end{aligned}\right\} \text{node 2} \tag{4.50}$$

Now, the stiffness matrix can simply be determined column by column. This exercise will be explained for the first column. This column corresponds with the degree of freedom w_1. Assuming that this degree of freedom is set to 1 while the others remain zero, the element adopts the shape as drawn in Fig. 4.22. Substitution of these values for the four degrees of freedom into (4.49) and performing the matrix multiplication leads for this special case to:

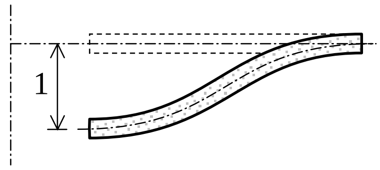


Fig. 4.22: deformed element for $w_1 = 1$ *and* $w_2 = \varphi_1 = \varphi_2 = 0$.

$$k_{11} = F_1 \quad ; \quad k_{21} = M_1 \quad ; \quad k_{31} = F_2 \quad ; \quad k_{41} = M_2$$

The left-hand sides just form the first column of the stiffness matrix. The right-hand sides (the generalised nodal forces) can be calculated for this special case. From the differential equation and the four boundary conditions for the displacements, the solution can be determined and after that the distribution of m_{rr} and v_r can be obtained. From (4.50) the generalised nodal forces can be calculated, and therefore also the first column of the stiffness matrix. The other three columns can be determined in a similar manner, by assigning the value 1 to the corresponding degree of freedom and the value zero to the remaining three.

Exercise
Show how the fourth column of stiffness matrix (4.49) can be determined.
- Which value do each of the four degrees of freedom get; sketch the special deformed shape;
- which differential equation applies, and which general solution;
- provide the four equations from which the still unknown coefficients can be solved.

5 Theory of elasticity in three dimensions

After the one-dimensional applications of chapter 2 and the two-dimensional plate problems of the chapters 3 and 4, a generalisation to three dimensions will be made.
In a space continuum the displacement of a point (x, y, z) in a cartesian coordinate system, can be decomposed into the components $u_x(x, y, z)$ in the x-direction, $u_y(x, y, z)$ in the y-direction and $u_z(x, y, z)$ in the z-direction. Per unit of volume the external loads P_x, P_y and P_z can be applied, which correspond with the degrees of freedom u_x, u_y and u_z, respectively. Regarding the internal quantities, it already was demonstrated that a surface element in a continuum is able to transmit a force per unit of area and that this force per unit of area was called a stress vector. In a space continuum, the stress vectors acting in arbitrary direction on three areas that are perpendicular to the cartesian coordinate axes x, y, z can be decomposed into three components along these three coordinate directions. Doing so, nine quantities appear indicated by $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$ (acting on the area perpendicular to the x-axis), $\sigma_{yx}, \sigma_{yy}, \sigma_{yz}$ (acting on the area perpendicular to the y-axis) and $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ (acting on the area perpendicular to the z-axis). Again it can be seen that the first subscript of the stress components (often just called stresses) indicates the direction of the normal on the area, and the second subscript the direction of the component of the stress vector. Similarly as in the plate theory, a stress component is called a normal stress when the two indices are equal $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$. When the indices are different they are called shear stresses $(\sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy})$. Also in this case the sign convention holds that a stress component is positive when it is working in positive coordinate direction on an area with its normal in positive coordinate direction.
With the defined internal stress components, internal deformation components correspond. These are known as specific strains caused by the normal stresses and changes of the right angle due to the shear stresses.
As indicated in Fig. 5.1, the specific strains associated with the normal stresses are called $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, respectively. The shear deformations consist out of three pairs of equal angular

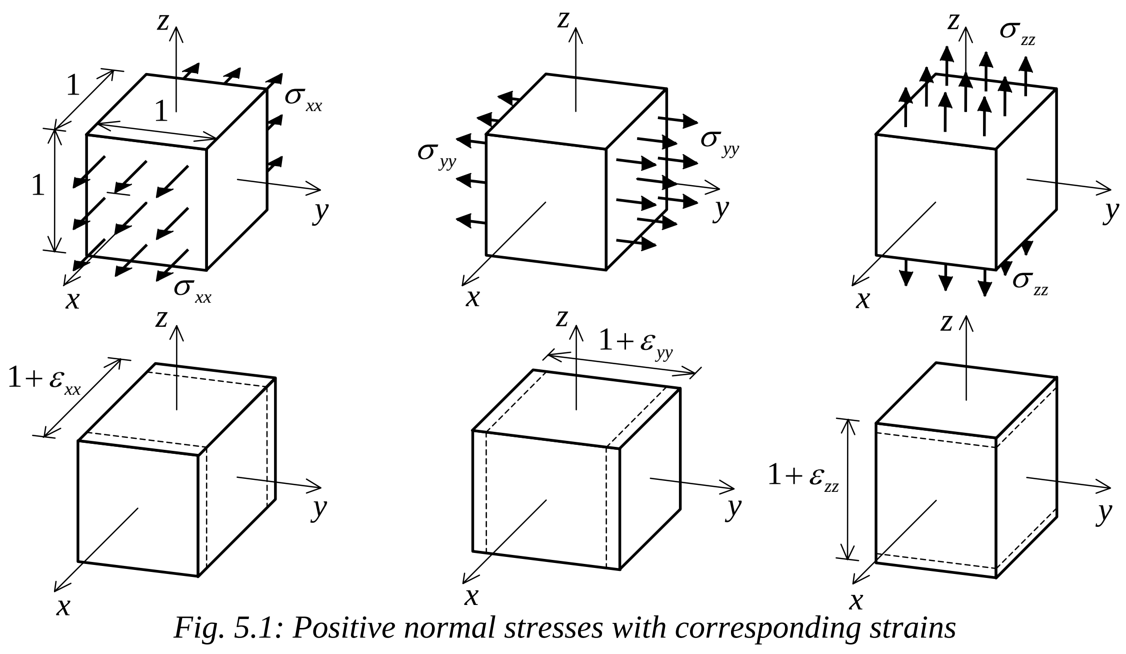


Fig. 5.1: Positive normal stresses with corresponding strains in a three-dimensional continuum.

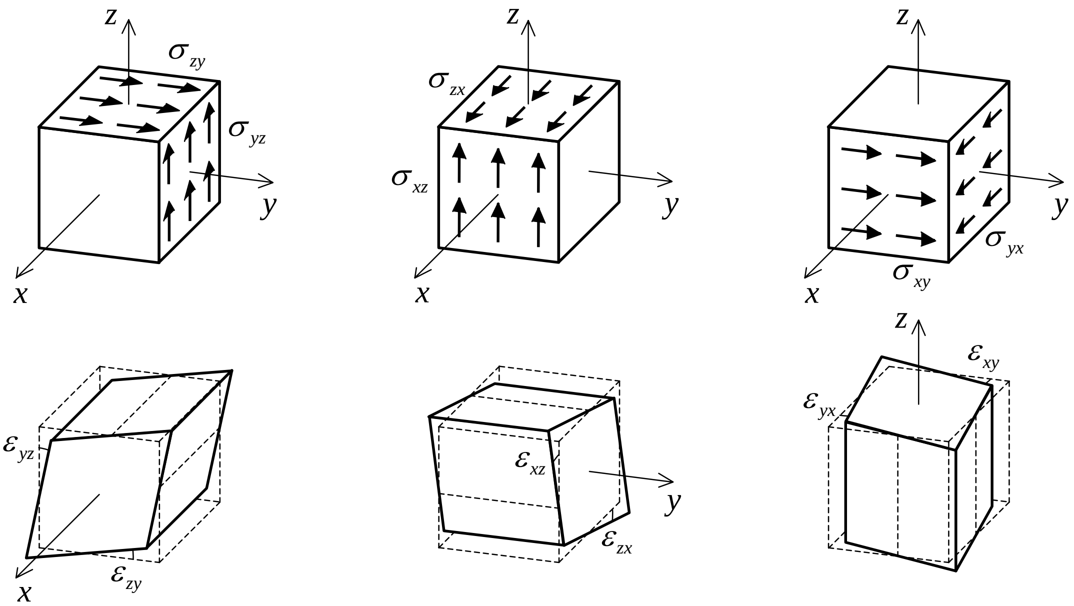


Fig. 5.2: Positive shear stresses with corresponding strains in a three-dimensional continuum.

changes $\varepsilon_{yz} = \varepsilon_{zy}$, $\varepsilon_{xz} = \varepsilon_{xz}$, $\varepsilon_{xy} = \varepsilon_{yx}$ (see Fig. 5.2). As done before, the shear deformations can be used, which are defined by: $\gamma_{yz} = 2\varepsilon_{yz}$, $\gamma_{zx} = 2\varepsilon_{zx}$, $\gamma_{xy} = 2\varepsilon_{xy}$. Then the scheme of relations as shown in Fig. 5.3 can be set up.

Since six stress components are present and only three load components (i.e. three equilibrium equations) a three-dimensional stress problem is statically indeterminate to the third degree.

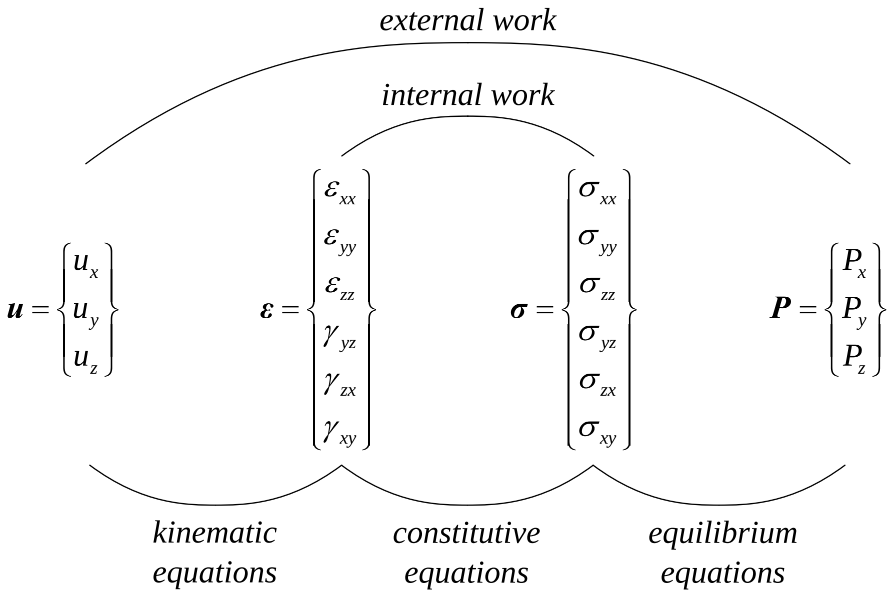


Fig. 5.3: Diagram displaying the relations between the quantities playing a role in the analysis of three-dimensional problems.

5.1 Basic equations

Subsequently the three categories of basic equations will be formulated in the following order: kinematic equations, constitutive equations and equilibrium equations.

Kinematic equations
In section 3.1, the kinematic equations for a plate were derived. Similarly by considerations in three directions, the kinematic equations for a space continuum are found:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= u_{x,x} \quad ; \quad \gamma_{yz} = 2\,\varepsilon_{yz} = u_{y,z} + u_{z,y} \\ \varepsilon_{yy} &= u_{y,y} \quad ; \quad \gamma_{zx} = 2\,\varepsilon_{zx} = u_{z,x} + u_{x,z} \\ \varepsilon_{zz} &= u_{z,z} \quad ; \quad \gamma_{xy} = 2\,\varepsilon_{xy} = u_{x,y} + u_{y,x} \end{aligned}} \tag{5.1}$$

In this notation, the subscript "$,x$" means differentiation with respect to x, etc.
In addition to deformations, a volume particle can also be subjected to a displacement as a rigid body. Six displacement components exist. Three of them are pure translations u_x, u_y, u_z in x-, y-, z- directions, respectively. The other three are rotations about the x-, y-, z- axes. They are called $\omega_{yz}, \omega_{zx}, \omega_{xy}$, respectively.

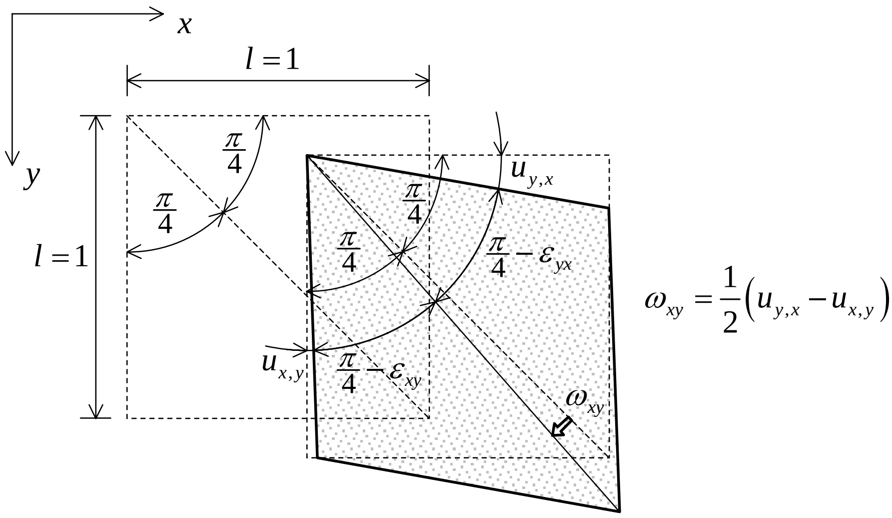


Fig. 5.4: Rotation about z-axis.

Fig. 5.4 shows the rotation ω_{xy} about the z-axis. Doing so, for the three rotations it is found (anticlockwise is positive):

$$\boxed{\begin{aligned} \omega_{yz} &= \frac{1}{2}\left(u_{z,y} - u_{y,z}\right) \\ \omega_{zx} &= \frac{1}{2}\left(u_{x,z} - u_{z,x}\right) \\ \omega_{xy} &= \frac{1}{2}\left(u_{y,x} - u_{x,y}\right) \end{aligned}} \tag{5.2}$$

Constitutive equations

Hooke's law is valid for an isotropic linear-elastic material. The stress-strain relations are:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\upsilon}{E}\left(\sigma_{yy}+\sigma_{zz}\right) \quad ; \quad 2\varepsilon_{yz} = \frac{2(1+\upsilon)}{E}\sigma_{yz} \\
\varepsilon_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\upsilon}{E}\left(\sigma_{zz}+\sigma_{xx}\right) \quad ; \quad 2\varepsilon_{zx} = \frac{2(1+\upsilon)}{E}\sigma_{zx} \\
\varepsilon_{zz} &= \frac{1}{E}\sigma_{zz} - \frac{\upsilon}{E}\left(\sigma_{xx}+\sigma_{yy}\right) \quad ; \quad 2\varepsilon_{xy} = \frac{2(1+\upsilon)}{E}\sigma_{xy}
\end{aligned}
\tag{5.3}
$$

where E is the *modulus of elasticity* (*Young's modulus*) and υ *Poisson's ratio* (*lateral contraction coefficient*). The term $2(1+\upsilon)/E$ is the reciprocal quantity of the shear modulus G. The matrix formulation of the constitutive equations read:

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}
\begin{bmatrix}
1 & -\upsilon & -\upsilon & 0 & 0 & 0 \\
 & 1 & -\upsilon & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 \\
 & & & 2(1+\upsilon) & 0 & 0 \\
 & \mathit{symmetrical} & & & 2(1+\upsilon) & 0 \\
 & & & & & 2(1+\upsilon)
\end{bmatrix}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}
\tag{5.4)a}
$$

or briefly:

$$
\boldsymbol{\varepsilon} = \boldsymbol{C}\,\boldsymbol{\sigma} \tag{5.4)b}
$$

where $\boldsymbol{C}$ is called the *flexibility matrix* or *compliance matrix*.

Through inversion, the stiffness formulation of the constitutive equations appears:

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \frac{(1-\upsilon)E}{(1+\upsilon)(1-2\upsilon)}
\begin{bmatrix}
1 & \frac{\upsilon}{1-\upsilon} & \frac{\upsilon}{1-\upsilon} & 0 & 0 & 0 \\
 & 1 & \frac{\upsilon}{1-\upsilon} & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 \\
 & & & \frac{1-2\upsilon}{2(1-\upsilon)} & 0 & 0 \\
 & & & & \frac{1-2\upsilon}{2(1-\upsilon)} & 0 \\
 & \mathit{symmetrical} & & & & \frac{1-2\upsilon}{2(1-\upsilon)}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}
\tag{5.5)a}
$$

or briefly:

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\varepsilon}} \tag{5.5)b$$

where $\boldsymbol{D}$ is called the *stiffness matrix* or *rigidity matrix*.
In this formulation the constitutive equations are normally used in the finite element programmes with spatial elements.

Equilibrium equations
In Fig. 5.5 the equilibrium in x-direction is considered. The edges of the drawn cube have unit length. Similarly the equilibrium in the y- and z-directions can be set up. Doing so, it can be derived:

$$\boxed{\begin{aligned} \sigma_{xx,x} + \sigma_{yx,x} + \sigma_{zx,z} + P_x &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z} + P_y &= 0 \\ \sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + P_z &= 0 \end{aligned}} \tag{5.6)a$$

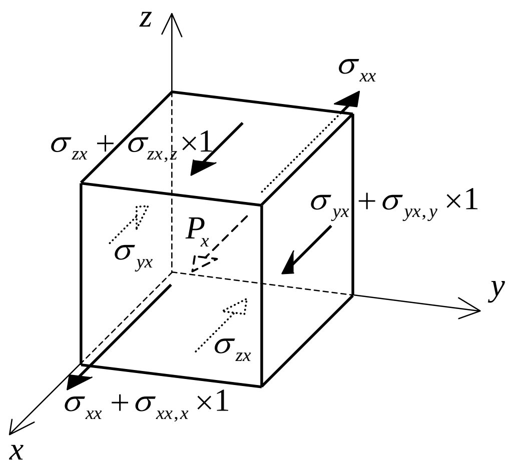


Fig. 5.5: Forces in x-direction.

Equilibrium of moments about the x-, y- and z-axis leads to:

$$\boxed{\sigma_{yz} = \sigma_{zy} \quad ; \quad \sigma_{zx} = \sigma_{xz} \quad ; \quad \sigma_{xy} = \sigma_{yx}} \tag{5.6)b$$

Check with matrices of differential operators
For the kinematic equations it can be written:

$$\boxed{\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}} \tag{5.7a}$$

or briefly:

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{B}}\, \boldsymbol{u}} \tag{5.7b}$$

Likewise, the equilibrium equations (5.6)a in this notation read:

$$\boxed{\begin{bmatrix} -\frac{\partial}{\partial x} & 0 & 0 & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial y} \\ 0 & -\frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}} \tag{5.8a}$$

or briefly:

$$\boxed{\boldsymbol{\mathcal{B}}'\boldsymbol{\sigma} = \boldsymbol{P}} \tag{5.8b}$$

Also in this case $\mathcal{B}'$ can be obtained by transposition of $\mathcal{B}$ plus the introduction of a minus sign for all uneven derivatives.

5.2 Solution procedures and boundary conditions

For the *force method*, one would expect three compatibility conditions. However, in literature continuously six compatibility conditions are derived. But these conditions are linearly independent, so that from the six conditions also three identities can be derived, the so-called identities of Bianchi. This means that in the three-dimensional analysis an extra complication arises, which is not present in the previously discussed one- and two-dimensional problems. In this Course, no further attention will be paid to this matter.
In the *displacement method*, the kinematic and the constitutive equations are substituted in the equilibrium equations. This approach results in a set of three simultaneous partial differential equations in the three components u_x, u_y and u_z of the displacement field:

$$\boxed{\begin{aligned} \cdots u_x + \cdots u_y + \cdots u_z &= P_x \\ \cdots u_x + \cdots u_y + \cdots u_z &= P_y \\ \cdots u_x + \cdots u_y + \cdots u_z &= P_z \end{aligned}} \tag{5.9}$$

The positions indicated by the three dots are occupied by differential operators multiplied by the stiffness terms from (5.5). An alternative description of these three differential equations will be provided in section 5.5.

Remark
In this course, no general applications will be discussed for three-dimensional stress-states, which are described by the system of differential equations given by (5.9). Only one special case will be highlighted, the torsion of bars. Chapter 6 is completely dedicated to this problem. For the case of torsion, it appears that the three-dimensional stress-state can be reduced to a two-dimensional problem.

Boundary conditions
The general goal of the theory of elasticity can be described as follows:
The calculation of displacements, deformations and stresses inside a body, which is subjected to known volume forces and certain known conditions at its outer surface.
The most frequently appearing boundary conditions are:

Kinematic boundary conditions
This boundary condition occurs when at a specific part of the outer surface (say S_u), such a provision is made that the points of that part are subjected to a prescribed displacement. For example, a part of the body can be glued completely to a rigid supporting block. Then the displacements for the glued surface are zero (in other words: it is prescribed that the displacements are zero). The formulae for this type of boundary condition read:

$$\boxed{\left.\begin{aligned} u_x &= u_x^o \\ u_y &= u_y^o \\ u_z &= u_z^o \end{aligned}\right\} \text{ on } S_u} \tag{5.10}$$

where u_x^o, u_y^o and u_z^o are prescribed (for example zero).

Dynamic boundary conditions

This boundary condition occurs when at a specific part of the outer surface (say S_p) a certain surface load is acting. For that case, three relations can be formulated between the stress components with respect to the cartesian coordinate system. When the unit outward-pointing normal on the surface has the components e_x, e_y and e_z, the formulation of the boundary condition becomes:

$$\boxed{\left.\begin{aligned} \sigma_{xx}e_x + \sigma_{yx}e_y + \sigma_{zx}e_z &= p_x \\ \sigma_{xy}e_x + \sigma_{yy}e_y + \sigma_{zy}e_z &= p_y \\ \sigma_{xz}e_x + \sigma_{yz}e_y + \sigma_{zz}e_z &= p_z \end{aligned}\right\} \quad \text{on } S_p} \tag{5.11}$$

where p_x, p_y and p_z are prescribed. On an unloaded part of the surface p_x, p_y and p_z are zero of course.

The boundary conditions (5.11) are a generalisation into three dimensions of the corresponding conditions for a plate loaded in its plane. The derivation is performed analogously. A triangular surface element ABC with unit area as shown in Fig. 5.6 is

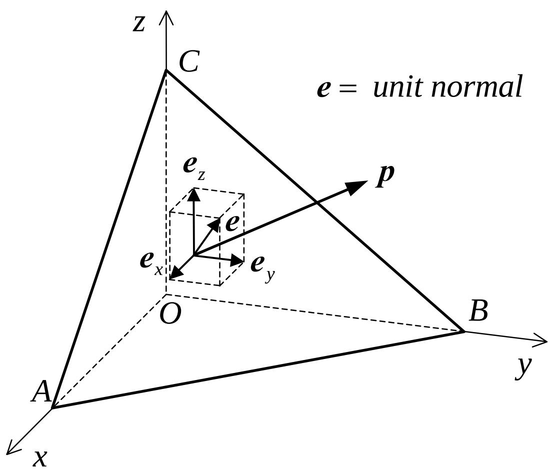


Fig. 5.6: Derivation of dynamic boundary conditions.

considered, on which a vector $\boldsymbol{p}$ is acting with components p_x, p_y and p_z. As mentioned before, the unit outward-pointing normal on ABC has the components e_x, e_y and e_z. From elementary stereometric principles it follows that the areas of triangles OBC, OAC and OAB are equal to e_x, e_y and e_z, respectively. By considering the equilibrium of the tetrahedron in the directions x, y and z, the three conditions of (5.11) can be derived.

Exercises

1. Find the deformation field corresponding with the following displacement field:
   $u_x = a + by \quad ; \quad u_y = c - b\,x \quad ; \quad u_z = d$
2. Determine the displacement field corresponding with the following (homogeneous) deformation field, when also is given that $u_z = 0$:
   $\varepsilon_{xx} = a \quad ; \quad \varepsilon_{yy} = b \quad ; \quad \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$
3. Is the following stress field possible, for a body in equilibrium without being subjected to volume loads?
   $\sigma_{xx} = a\,x^2 \quad ; \quad \sigma_{xy} = \sigma_{yx} = -2\,a\,x\,y \quad ; \quad \sigma_{yy} = a\,y^2 \quad ; \quad \sigma_{xz} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$
4. When in a body, which is in equilibrium, the following stresses are present (k, ρ and g are constants): $\sigma_{xx} = \sigma_{yy} = k\rho g\,z$; $\sigma_{zz} = \rho\,g\,z$; $\sigma_{xy} = \sigma_{xz} = \sigma_{yx} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0$.
   Which volume force is acting on the body?

5.3 Alternative formulation of the constitutive equations

In section 5.1 Hooke's law was presented for the description of the behaviour of isotropic linear-elastic material. A relation was formulated between the six stress components σ_{xx}, σ_{yy}, σ_{zz}, σ_{yz}, σ_{zx}, σ_{xy} and the associated deformations ε_{xx}, ε_{yy}, ε_{zz}, ε_{yz}, ε_{zx}, ε_{xy}. This relation was presented in both flexibility and stiffness formulation. It appeared that the two elastic constants E and υ were sufficient for a unique description of the material behaviour. Sometimes it is advantageous, to adapt the description such that in the total deformation distinction can be made between the *change of volume* and the *change of shape*. For example for the behaviour of soil this may be important, where the change of volume is prevented by the pore water while the change of shape can take place unhampered. Then instead of the constants E and υ, two other constants are introduced. Another example is rubber, which is incompressible. This means that change of volume is zero and the value of υ is practically 0.5. In $(5.5)_a$ the term $(1-2\upsilon)$ that appears in the denominator makes the relation between stresses and strains undetermined. The splitting-up of the deformations causing a change of volume and a change of shape may simplify the description of the non-linear behaviour of materials. Among other things this is important for concrete and soil. In this section the alternative description of Hooke's law will be summarised. To start with, the law will be split up in a separate *law for the change of volume* and a *law for the change of shape*. This will be done in both the flexibility and stiffness formulations. Then two other material constants will be introduced, they are the *compression modulus* K and the *shear modulus* G. The starting point of the derivation is formed by the basic equations (5.4) and (5.5). Finally, for the two description methods, the two separate laws are combined to one total law of Hooke. Further, it appears that for the stiffness formulation another alternative exists, where the constants K and G are replaced by the so-called *constants of Lamé* λ and μ.

5.3.1 Separate laws of Hooke for the change of volume and shape

From the occurring stress-state given by the stress components σ_{xx}, σ_{yy}, σ_{zz}, σ_{yz}, σ_{zx}, σ_{xy} the so-called *hydrostatic stress* σ_0 is split off. The hydrostatic stress is defined by:

$$\boxed{\sigma_0 = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)} \tag{5.12}$$

The remaining stress components are the deviator stresses s_{xx}, s_{yy}, s_{zz}, σ_{yz}, σ_{zx}, σ_{xy}. For the first three components it holds:

$$\boxed{\begin{aligned} s_{xx} &= \sigma_{xx} - \sigma_0 = \frac{1}{3}\left(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}\right) \\ s_{yy} &= \sigma_{yy} - \sigma_0 = \frac{1}{3}\left(2\sigma_{yy} - \sigma_{zz} - \sigma_{xx}\right) \\ s_{zz} &= \sigma_{zz} - \sigma_0 = \frac{1}{3}\left(2\sigma_{zz} - \sigma_{xx} - \sigma_{yy}\right) \end{aligned}} \tag{5.13}$$

Analogously a component e_0 is split off from the existing deformations ε_{xx}, ε_{yy}, ε_{zz}, ε_{yz}, ε_{zx}, ε_{xy}, which is equal to one third of the *change of volume* e:

$$\boxed{e_0 = \frac{1}{3}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = \frac{1}{3}e} \tag{5.14}$$

Then the remaining part is formed by the *deviator deformations* e_{xx}, e_{yy}, e_{zz}, ε_{yz}, ε_{zx}, ε_{xy}. The first three components are:

$$\boxed{\begin{aligned} e_{xx} &= \varepsilon_{xx} - e_0 = \frac{1}{3}\left(2\varepsilon_{xx} - \varepsilon_{yy} - \varepsilon_{zz}\right) \\ e_{yy} &= \varepsilon_{yy} - e_0 = \frac{1}{3}\left(2\varepsilon_{yy} - \varepsilon_{zz} - \varepsilon_{xx}\right) \\ e_{zz} &= \varepsilon_{zz} - e_0 = \frac{1}{3}\left(2\varepsilon_{zz} - \varepsilon_{xx} - \varepsilon_{yy}\right) \end{aligned}} \tag{5.15}$$

No change of volume is associated with the six deviator deformations. It just changes the form (shape) of a material particle.

Flexibility relations

A relation can be established between e and σ_0 by adding up the first three equations of $(5.4)_a$. This delivers:

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{3(1-2\upsilon)}{E}\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

or:

$$\boxed{e = \frac{1}{K}\sigma_0}$$ (*Hooke's law for the change of volume in flexibility formulation*) (5.16)

with:

$$\boxed{K = \frac{E}{3(1-2\upsilon)}}$$ (*modulus of compression*) (5.17)

Relation (5.16) is Hooke’s law for the change of volume. A relation can also be derived between the deviator deformations e_{xx}, e_{yy}, e_{zz} and the deviator stresses s_{xx}, s_{yy}, s_{zz}. For example from (5.15) it is known that $e_{xx} = \varepsilon_{xx} - e_0$, for both ε_{xx} and e_0 the relation with the stresses is known, so that:

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \upsilon\sigma_{yy} - \upsilon\sigma_{zz}\right) - \frac{1-2\upsilon}{3E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

or:

$$e_{xx} = \frac{1+\upsilon}{E}\left(\frac{2}{3}\sigma_{xx} - \frac{1}{3}\sigma_{yy} - \frac{1}{3}\sigma_{zz}\right)$$

This expression can be briefly written as:

$$e_{xx} = \frac{1}{2G}s_{xx}$$

where G is the *shear modulus*. A similar derivation holds for the relations between e_{yy} and s_{yy}, and between e_{zz} and s_{zz}. The shear modulus also establishes a relation between the shear deformations γ_{yz}, γ_{zx}, γ_{xy} and the shear stresses σ_{yz}, σ_{zx}, σ_{yx}. When for the shear deformations the quantities ε_{yz}, ε_{zx}, ε_{xy} are used the factor $1/2G$ appears again. Therefore, for all six deviator deformations and stresses it holds:

$$\boxed{\begin{array}{lll} e_{xx} = \dfrac{1}{2G}s_{xx} & ; & \varepsilon_{yz} = \dfrac{1}{2G}\sigma_{yz} \\ e_{yy} = \dfrac{1}{2G}s_{yy} & ; & \varepsilon_{zx} = \dfrac{1}{2G}\sigma_{zx} \\ e_{zz} = \dfrac{1}{2G}s_{zz} & ; & \varepsilon_{xy} = \dfrac{1}{2G}\sigma_{xy} \end{array}}$$ (*Hooke's law for the change of shape in flexibility formulation*) (5.18)

with:

$$\boxed{G = \frac{E}{2(1+\upsilon)}} \qquad (\textit{shear modulus}) \tag{5.19}$$

Stiffness relations
The two components (5.16) and (5.18) of Hooke's law can also be derived in inverse form as stiffness relations. By addition of the first three equations in $(5.5)_a$, and division of the result by three, it is found:

$$\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = \frac{E}{3(1-2\upsilon)}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$$

which is just equal to:

$$\boxed{\sigma_0 = K\, e} \qquad (\textit{Hooke's law for the change of volume in stiffness formulation}) \tag{5.20}$$

The relation between the deviator stresses s_{xx}, s_{yy}, s_{zz} and the deviator strains e_{xx}, e_{yy}, e_{zz} can simply be obtained. For s_{xx} it is known that $s_{xx} = \sigma_{xx} - \sigma_0$. Substitution of σ_{xx} and σ_0 as functions of the deformations then yields:

$$s_{xx} = \frac{(1-\upsilon)E}{(1+\upsilon)(1-2\upsilon)}\left(\varepsilon_{xx} + \frac{\upsilon}{1-\upsilon}\varepsilon_{yy} + \frac{\upsilon}{1-\upsilon}\varepsilon_{zz}\right) - \frac{E}{3(1-2\upsilon)}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$$

After some elaboration this reduces to:

$$s_{xx} = \frac{E}{(1+\upsilon)}\left(\frac{2}{3}\varepsilon_{xx} - \frac{1}{3}\varepsilon_{yy} - \frac{1}{3}\varepsilon_{zz}\right)$$

which briefly can be written as:

$$s_{xx} = 2G\, e_{xx}$$

Analogously similar expressions are found for s_{yy} and s_{zz}. Therefore, Hooke's law for the change of shape in stiffness form reads:

$$\boxed{\begin{array}{lcl} s_{xx} = 2G\, e_{xx} & ; & \sigma_{yz} = 2G\, \varepsilon_{yz} \\ s_{yy} = 2G\, e_{yy} & ; & \sigma_{zx} = 2G\, \varepsilon_{zx} \\ s_{zz} = 2G\, e_{zz} & ; & \sigma_{xy} = 2G\, \varepsilon_{xy} \end{array}} \qquad (\textit{Hooke's law for the change of shape in stiffness formulation}) \tag{5.21}$$

5.3.2 Hooke’s law for total deformations and stresses

With the two separate laws of Hooke on basis of K and G for the change of volume and shape, respectively, a general law on basis of K and G can be formulated for the total deformations (in flexibility formulation), or for the total stresses (in stiffness formulation).

Flexibility relations

The total strains ε_{xx}, ε_{yy}, ε_{zz}, ε_{yz}, ε_{zx}, ε_{xy} are the sum of the average strain $e_0 = \frac{1}{3}e$ and the deviator deformations e_{xx}, e_{yy}, e_{zz}, ε_{yz}, ε_{zx}, ε_{xy}. With the relations (5.16) and (5.18) it then directly can be found:

$$\boxed{\begin{array}{lcl} \varepsilon_{xx} = \dfrac{\sigma_0}{3K} + \dfrac{s_{xx}}{2G} & ; & \varepsilon_{yz} = \dfrac{\sigma_{yz}}{2G} \\ \varepsilon_{yy} = \dfrac{\sigma_0}{3K} + \dfrac{s_{yy}}{2G} & ; & \varepsilon_{zx} = \dfrac{\sigma_{zx}}{2G} \\ \varepsilon_{zz} = \dfrac{\sigma_0}{3K} + \dfrac{s_{zz}}{2G} & ; & \varepsilon_{xy} = \dfrac{\sigma_{xy}}{2G} \end{array}} \quad \text{(\textit{Hooke's law in K and G for the total deformations in flexibility formulation})} \tag{5.22}$$

Stiffness relations

The total stresses σ_{xx}, σ_{yy}, σ_{zz}, σ_{yz}, σ_{zx}, σ_{xy} are the sum of the hydrostatic stress σ_0 and the deviator stresses s_{xx}, s_{yy}, s_{zz}, σ_{yz}, σ_{zx}, σ_{xy}. With the relation (5.20) and (5.21), for the total stresses it then directly can be found:

$$\boxed{\begin{array}{lcl} \sigma_{xx} = K\,e + 2G\,e_{xx} & ; & \sigma_{yz} = 2G\,\varepsilon_{yz} \\ \sigma_{yy} = K\,e + 2G\,e_{yy} & ; & \sigma_{zx} = 2G\,\varepsilon_{zx} \\ \sigma_{zz} = K\,e + 2G\,e_{zz} & ; & \sigma_{xy} = 2G\,\varepsilon_{xy} \end{array}} \quad \text{(\textit{Hooke's law in K and G for the total stresses in stiffness formulation})} \tag{5.23}$$

This law for the total stresses in stiffness formulation, can also be represented in a different manner. The three deviator deformations e_{xx}, e_{yy}, e_{zz} are replaced by $\varepsilon_{xx} - \frac{1}{3}e$, $\varepsilon_{yy} - \frac{1}{3}e$, $\varepsilon_{zz} - \frac{1}{3}e$, respectively. The law then becomes:

$$\boxed{\begin{array}{lcl} \sigma_{xx} = \lambda\,e + 2\mu\,\varepsilon_{xx} & ; & \sigma_{yz} = 2\mu\,\varepsilon_{yz} \\ \sigma_{yy} = \lambda\,e + 2\mu\,\varepsilon_{yy} & ; & \sigma_{zx} = 2\mu\,\varepsilon_{zx} \\ \sigma_{zz} = \lambda\,e + 2\mu\,\varepsilon_{zz} & ; & \sigma_{xy} = 2\mu\,\varepsilon_{xy} \end{array}} \quad \text{(\textit{Hooke's law in } \lambda \text{ \textit{and} } \mu \text{ \textit{for the total stresses in stiffness formulation}})} \tag{5.24}$$

where λ and μ are called the Lamé constants. These constants can be expressed in K and G and also in E and υ. The following expressions are valid:

$$\lambda = K - \frac{2}{3}G = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} \quad ; \quad \mu = G = \frac{E}{2(1+\upsilon)} \tag{5.25}$$

The quantity μ is identical to G, but it is customary to use μ in combination with λ.

5.3.3 The displacement method in the description of Lamé

In section 5.2 it has been discussed that the displacement method for three-dimensional problems amounts to the simultaneous solution of three partial differential equations in u_x, u_y and u_z (see (5.9)).
These three differential equations can be formulated very concisely, if Hooke's law is expressed in the Lamé constants λ and μ.
The three sets basic equations then are:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= u_{x,x} \quad ; \quad \varepsilon_{yz} = \frac{1}{2}\left(u_{y,z} + u_{z,y}\right) \\ \varepsilon_{yy} &= u_{y,y} \quad ; \quad \varepsilon_{zx} = \frac{1}{2}\left(u_{z,x} + u_{x,z}\right) \\ \varepsilon_{zz} &= u_{z,z} \quad ; \quad \varepsilon_{xy} = \frac{1}{2}\left(u_{x,y} + u_{y,x}\right) \end{aligned}} \qquad (\textit{kinematic equations})$$

$$\boxed{\begin{aligned} \sigma_{xx} &= \lambda e + 2\mu\varepsilon_{xx} \quad ; \quad \sigma_{yz} = 2\mu\varepsilon_{yz} \\ \sigma_{yy} &= \lambda e + 2\mu\varepsilon_{yy} \quad ; \quad \sigma_{zx} = 2\mu\varepsilon_{zx} \\ \sigma_{zz} &= \lambda e + 2\mu\varepsilon_{zz} \quad ; \quad \sigma_{xy} = 2\mu\varepsilon_{xy} \end{aligned}} \qquad (\textit{constitutive equations})$$

$$\boxed{\begin{aligned} \sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} + P_x &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z} + P_y &= 0 \\ \sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + P_z &= 0 \end{aligned}} \qquad (\textit{equilibrium equations})$$

By downward substitution the three equilibrium equations are transformed into the so-called equations of Navier:

$$\boxed{\begin{aligned} (\lambda + \mu)e_{,x} + \mu\nabla^2 u_x + P_x &= 0 \\ (\lambda + \mu)e_{,y} + \mu\nabla^2 u_y + P_y &= 0 \\ (\lambda + \mu)e_{,z} + \mu\nabla^2 u_z + P_z &= 0 \end{aligned}} \qquad (\textit{equations of Navier}) \tag{5.26}$$

where the volume strain e is a function of the displacements and ∇^2 is the Laplace operator for three dimensions, i.e.:

$$e = u_{x,x} + u_{y,y} + u_{z,z} \quad ; \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Use of tensor notation

A big advantage in the formulation of above relations can be obtained by the use of the index notation including summation convention. To start with, the coordinate axes x, y, z are indicated by x_1, x_2, x_3, respectively. The displacement components then are u_1, u_2, u_3. The stress components are σ_{ij} $(i,\ j = 1, 2, 3)$ and the strain components are ε_{ij} $(i,\ j = 1, 2, 3)$. The notation for partial differentiation is:

$$a_{,i} = \frac{\partial a}{\partial x_i}$$

The summation convention of Einstein requires that when in an expression one subscript appears twice, a summation has to be carried out with respect to this index from 1 to 3, i.e.:

$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

Another useful quantity is the Kronecker delta, defined by:

$$\delta_{ij} = \begin{cases} 1 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases}$$

The three sets of basic equations now become:

$$\boxed{\begin{array}{ll} e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) & (i,\ j = 1, 2, 3) \\ \sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} & (i,\ j = 1, 2, 3) \\ \sigma_{ij,i} + P_j = 0 & (j = 1, 2, 3) \end{array}} \quad \begin{array}{l} (\textit{kinematic equations}) \\ (\textit{constitutive equations}) \\ (\textit{equilibrium equations}) \end{array} \qquad (5.27)$$

Downward substitution again provides the equations of Navier:

$$\boxed{(\lambda + \mu)\,u_{j,ji} + \mu\,u_{i,jj} + P_i = 0 \quad (i = 1, 2, 3)} \qquad (5.28)$$

The boundary conditions are:

$$\boxed{\begin{array}{lll} u_i = u_i^0 & \text{on } S_p & (i = 1, 2, 3) \\ \sigma_{ij} e_i = p_j & \text{on } S_0 & (j = 1, 2, 3) \end{array}} \qquad (5.29)$$

The advantage of this notation is that the whole system of equations can be written very concisely and simple. Substitution of one equation into another can be done as well. In literature this notation is used intensively.

6 Torsion of bars

6.1 Problem definition

During the civil engineering training at the university, the student is thoroughly introduced in the behaviour of bar structures. The student has been familiarised with the basic cases of *extension*, *bending*, *shear*, *torsion* and their combinations. For each basic case, two external quantities can be identified, namely a specific deformation and a corresponding generalised stress resultant. For the case of extension they are ε and N, for bending κ and M, for shear γ and V and for torsion θ and M_t.
From each of the four basic cases, for the designer always two specifications are important. First, he has to know the *stiffness*. This is the relation between the specific deformation and the corresponding stress resultant. For the several cases the relations are:

$N = EA\varepsilon$ (*extension*)
$M = EI\kappa$ (*bending*)
$V = GA_s\gamma$ (*shear*)
$M_t = GI_t\theta$ (*torsion*)

Here is EA the axial stiffness, EI the bending stiffness, GA_s the shear stiffness and GI_t the torsional stiffness. The quantity E is the *modulus of elasticity* (also called *Young's modulus*) and G is the *shear modulus*. The quantities A, I, A_s and I_t follow from the shape of the cross-section of the bar. The area A and the bending moment of inertia I of the cross-section do not need extra explanation. The quantity A_s is the cross-sectional area to be applied for shear; only for circular cross-sections this area is equal to A. The quantity I_t is called the torsional moment of inertia. During previous courses a lot of attention is paid to the determination of A and I, and to a less extend to A_s. Compared to this, the torsional problem was summarily dealt with. In previous lectures, only for a number simple cases a solution has been derived, but a generally valid analysis has not been provided up to now.
As mentioned, a second quantity in each of the basic cases is important for the designer. This is information about the *stress distribution* over the cross-section. For the case of extension the stress is constant, for bending the stress varies linearly, and for shear the stresses can be derived from the stress distribution for bending via an equilibrium consideration. The stress distribution for torsion has been derived only for the above-mentioned simple special cases. A generally valid procedure has not been presented yet. In summary, for the stress calculation the following is known:

$N = \sigma A$ (*extension*)

$M = \sigma W$ (*bending*)

$V = \sigma \dfrac{bI}{S}$ (*shear*)

$M_t = \ ?$ (*torsion*)

Here is A the cross-sectional area, W the section factor, I the bending moment of inertia, b the width subjected to the shear stress σ and S the static moment of a part of the cross-section. The stiffness problems and the stress distributions are shown schematically in Fig. 6.1. For the case of torsion, the only thing that can be established is that the torsional moment M_t has to be obtained from the integration over the cross-section of the product of the shear stress σ and the lever arm r.

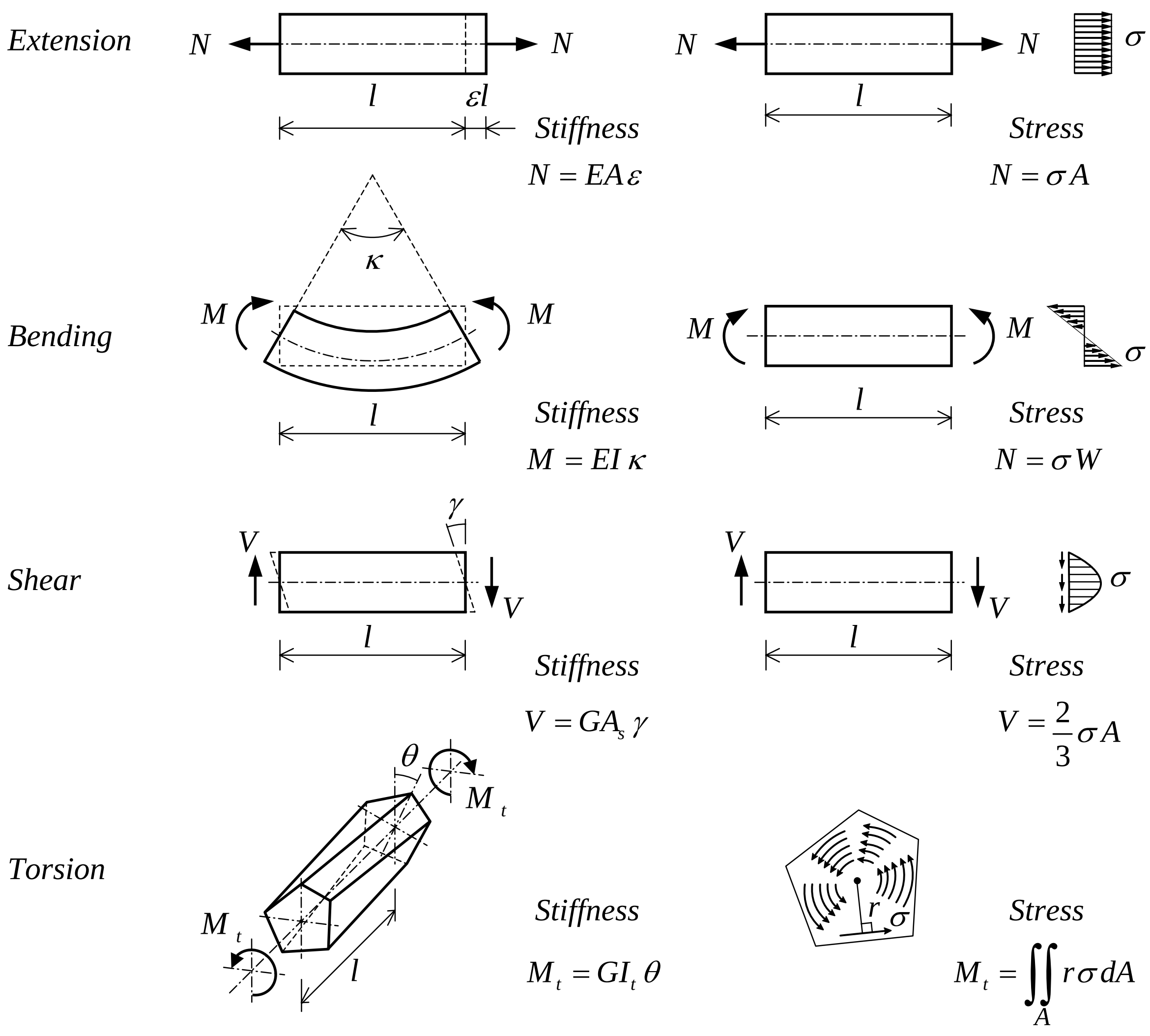
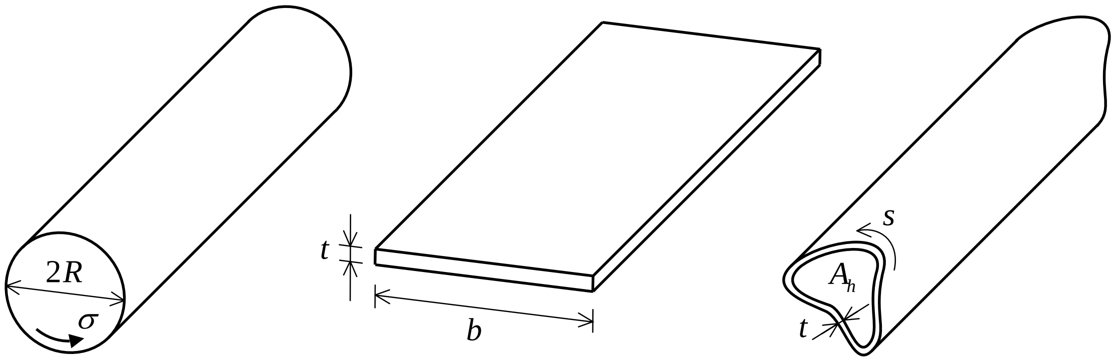


Fig.6.1: Definition of stiffness and stress distribution for the four basic load cases of a bar.

Before this problem definition is concluded, the three simple special cases of torsion are mentioned, for which the solution was generated in previous courses. It only concerned prismatic bars of circular cross-section, strip-shaped cross-section and thin-walled hollow cross-section. The found relations for the torsional moments of inertia I_t and the maximum occurring stresses are indicated in Fig. 6.2.

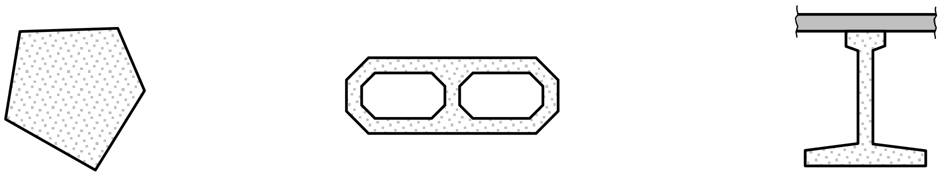
The main goal of this chapter is to offer a generally valid theory for prismatic bars with arbitrarily shaped cross-sections. These cross-sections may be solid but may contain holes as well. In the case of hollow cross-sections, the wall thickness not necessarily needs to be small. Attention is also paid to the possibility of cross-sections composed out of two different materials. Fig. 6.3 provides an overview of the cross-sections to be considered.



$$I_t = \frac{1}{2}\pi R^4 \qquad I_t = \frac{1}{3} b t^3 \qquad I_t = \frac{4A_h^2}{\oint \frac{ds}{t}} = \frac{4t\,A_h^2}{C}$$

$$\sigma = \frac{M_t}{\frac{1}{2}\pi R^3} \qquad \sigma = \frac{M_t}{\frac{1}{3} b t^2} \qquad \sigma = \frac{M_t}{2t A_h}$$

Fig. 6.2: Torsional moment of inertia I_t and shear stress σ for simple special cases.

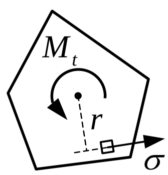


Solid cross-section of arbitrary shape — *Cross-section with holes thick- and thin-walled* — *cross-section composed of several materials*

Fig. 6.3: A general theory is required to analyse cross-sections that are common in the engineering practice.

The approach to be followed is summarised in Fig. 6.4. By definition, the torsional moment M_t is equal to the product of GI_t and the specific torsion θ. However, M_t is also equal to the integral over the cross-sectional area of the shear stress σ times the arm r. This means that a recipe can be formulated for the calculation of GI_t, provided that a value of θ is adopted. For this assumed deformation the stresses σ are determined. The torsional moment in the cross-section then can be obtained by calculation of the integral for $r\sigma$. Because the torsional stiffness GI_t is equal to the torque M_t for $\theta = 1$, the torsional moment of inertia is known too (see Fig. 6.4). Since the stress distribution is known, the largest stress and its position in the cross-section are fixed as well.

RECIPE for the calculation of GI_t
- *find σ for given θ*
- *determine GI_t from:* $GI_t = \frac{1}{\theta}\iint_A r\sigma\, dA$



$$\left.\begin{aligned} M_t &= GI_t\theta \\ M_t &= \iint_A r\sigma\, dA \end{aligned}\right\} \rightarrow \qquad GI_t\theta = \iint_A r\sigma\, dA$$

Fig. 6.4: The calculation of the torsional stiffness is formulated as a stress problem for an imposed deformation θ.

6.2 Basic equations and boundary conditions

De Saint-Venant has published the theory for torsion in 1855. This theory is correct if at the ends of the bar certain conditions are satisfied. These conditions prescribe that the torsional moments have to be applied via a certain distribution of shear stresses over the cross-section, and that no normal stresses are generated in axial direction at the ends (a dynamic boundary condition). This last condition implies that an eventual distribution of displacements in axial direction can be generated without restrictions, because at the surface where the (surface) load has been prescribed, no kinematic boundary condition can be imposed at the same time.
The right-handed coordinate system is chosen such that the x-direction is parallel to or coincides with the bar axis. So, the y-axis and z-axis are situated in the cross-section (see Fig. 6.5). The figures are drawn in such a manner that the x-axis is pointing backwards. In a

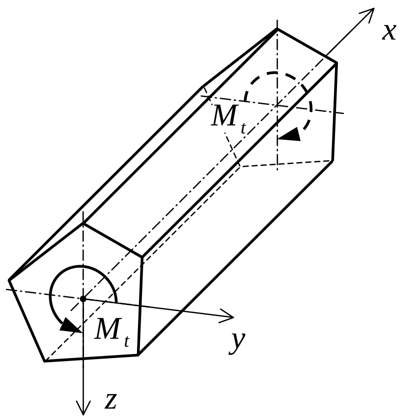


Fig. 6.5: Choice of coordinate system.

three-dimensional stress state, normally three displacements are generated, and in the three corresponding directions a volume load may be applied. Generally, six different stresses with six corresponding strains are present as well. The kinematic, constitutive and equilibrium equations are already provided in chapter 5. Using the brief notation for differentiation, they can be summarised as follows:

Kinematic equations

$$\begin{aligned} \varepsilon_{xx} &= u_{x,x} \quad ; \quad \gamma_{yz} = u_{y,z} + u_{z,y} \\ \varepsilon_{yy} &= u_{y,y} \quad ; \quad \gamma_{zx} = u_{z,x} + u_{x,z} \\ \varepsilon_{zz} &= u_{z,z} \quad ; \quad \gamma_{xy} = u_{x,y} + u_{y,x} \end{aligned} \tag{6.1}$$

Constitutive equations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \frac{(1-\upsilon)E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1 & \frac{\upsilon}{1-\upsilon} & \frac{\upsilon}{1-\upsilon} \\ & 1 & \frac{\upsilon}{1-\upsilon} \\ \text{symm.} & & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} \quad ; \quad \begin{Bmatrix} \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = G \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ \text{symm.} & & 1 \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} \tag{6.2}$$

Equilibrium equations

$$\begin{aligned}
&\sigma_{xx,x}+\sigma_{yx,y}+\sigma_{zx,z}+P_x=0\\
&\sigma_{xy,x}+\sigma_{yy,y}+\sigma_{zy,z}+P_y=0\\
&\sigma_{xz,x}+\sigma_{yz,y}+\sigma_{zz,z}+P_z=0
\end{aligned}\tag{6.3}$$

In the case of torsion the volume forces P_x, P_y and P_z are absent.

Distribution of displacements and stresses

De Saint-Venant succeeded to indicate a displacement field with enough freedom to allow for displacements at the ends, and from which a stress field follows that satisfies all requirements for equilibrium in the volume, along the circumference and at the ends of the bar. In the theory of elasticity only one unique solution can exist, which is in equilibrium with the external load and which satisfies the boundary conditions. Therefore, the solution of De Saint-Venant has to be the correct one.

The displacement field in question will be described now. De Saint-Venant stated that for torsion, the shape of the cross-section is not affected by the deformations. Regarding the displacements u_y and u_z in the plane of the cross-section, the displacement field manifests itself as a rotation about the x-axis as a rigid body.

This rotation is indicated by the symbol φ. This φ is identical to the rotation ω_{yz} as discussed in chapter 5. Further, it can be stated that the displacement u_x can be different from zero and may have a certain distribution over the cross section. However, this distribution is the same for all cross sections. This means that the displacement field is independent of x.

The fact that an arbitrary distribution of u_x can occur over the cross section means that an initially unloaded flat cross-section starts to warp as soon as a torque is applied. Such a displacement field for a bar with square cross-section is drawn in Fig. 6.6. Possible

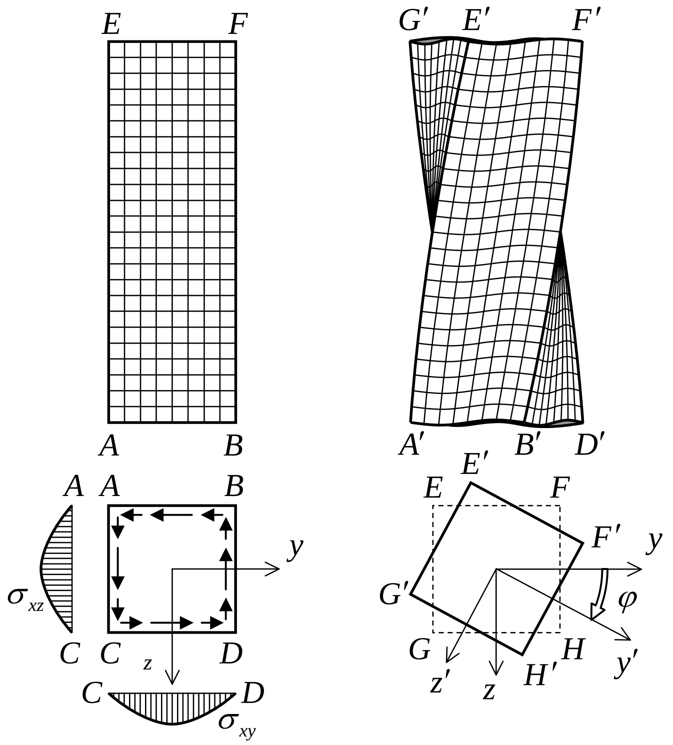


Fig. 6.6: The displacement field is composed of a rotation φ of the cross-section and a warping of the cross-section (the magnitude of φ is very exaggerated).

distributions of the shear stresses have been sketched. Along the edges AC and BD the shear stress σ_{xy} has to be zero. This means that the stress in the points A and B and C and D is zero, but between those points along the edges AB and CD the stress is allowed to increase. The same holds for the shear angle γ_{xy}, which causes the originally rectangular lateral surface $ABEF$ to deform into $A'B'E'F'$ (see Fig. 6.6). The other lateral surfaces experience the same deformation, which makes it plausible that after deformation the originally flat cross-sections are warped.

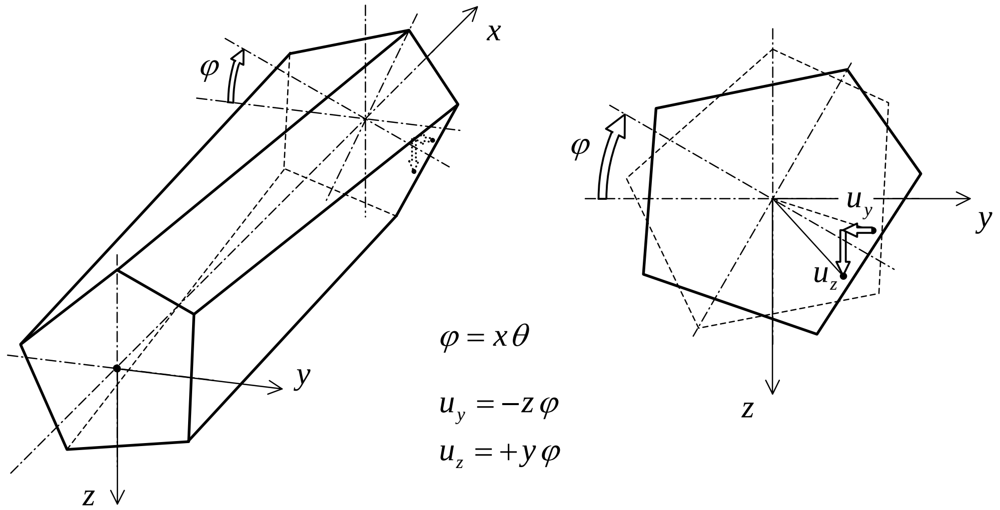


Fig. 6.7: The displacement field described by u_y and u_z.

After this qualitative description of the displacement field, a quantitative formulation will be provided. As a result of the rotation φ, the displacements u_y and u_z in a point x, y, z of the cross-section are equal to (see Fig. 6.7):

$$u_y = -z\,\varphi$$
$$u_z = +y\,\varphi$$

The rotation φ depends on the specific torsion θ. For constant θ, from $d\varphi/dx = \theta$ it follows:

$$\varphi = \theta\, x$$

where it has been used that $\varphi = 0$ for $x = 0$. So, the displacements u_y and u_z become:

$$\begin{aligned} u_y &= -x\,z\,\theta \\ u_z &= +x\,y\,\theta \end{aligned} \qquad (6.4)_a$$

The warping displacement u_x is independent of x, it will increase linearly with the specific torsion θ. Therefore, it can be written:

$$u_x = \psi(y,z)\theta \qquad (6.4)_b$$

where the so-called warping function ψ describes the displacement distribution over the cross-section for $\theta = 1$. In order to check whether this displacement field is suitable for the considered torsion problem, it is substituted into the kinematic equation (6.1). The result is:

$$
\begin{aligned}
\varepsilon_{xx} &= 0 \quad ; \qquad \gamma_{yz} = 0 \\
\varepsilon_{yy} &= 0 \quad ; \qquad \boxed{\gamma_{zx} = \left(\psi_{,z} + y\right)\theta \neq 0} \\
\varepsilon_{zz} &= 0 \quad ; \qquad \boxed{\gamma_{xy} = \left(\psi_{,y} - z\right)\theta \neq 0}
\end{aligned}
\tag{6.5}
$$

Only γ_{zx} and γ_{xy} appear to become unequal to zero. Then from the constitutive equations it follows that only the stresses σ_{zx} and σ_{xy} are different from zero too:

$$
\begin{aligned}
\sigma_{xx} &= 0 \quad ; \qquad \sigma_{yz} = 0 \\
\sigma_{yy} &= 0 \quad ; \qquad \boxed{\sigma_{zx} = G\left(\psi_{,z} + y\right)\theta \neq 0} \\
\sigma_{zz} &= 0 \quad ; \qquad \boxed{\sigma_{xy} = G\left(\psi_{,y} - z\right)\theta \neq 0}
\end{aligned}
\tag{6.6}
$$

The stresses σ_{zx} and σ_{xy} are the shear stresses in the cross-section that correspond with the torsional moment in the cross-section. So, the chosen displacement field satisfies the requirements. Of the three equilibrium equations (6.3), only the first one is important for the equilibrium of the bar. The second and third one are satisfied automatically, because differentiation of ψ with respect to x yields zero, i.e.:

$$
\begin{aligned}
&0 \ \boxed{+\sigma_{yx,y} + \sigma_{zx,z} = 0} \\
&0 \ + \ 0 \ + \ 0 \ = 0 \\
&0 \ + \ 0 \ + \ 0 \ = 0
\end{aligned}
\tag{6.7}
$$

In Fig. 6.8 it is demonstrated how the remaining equilibrium equation can be interpreted. An elementary cube of material is considered the edges of which have unit length and are parallel to the coordinate directions. In the face coinciding with the cross-section the shear stresses σ_{xy} and σ_{xz} are present. In the x-direction, on the faces with constant y and z, their counterparts σ_{yx} and σ_{zx} can be found. As shown these stresses increase in y and z direction, respectively. The requirement that the cube is in equilibrium in x-direction, directly leads to the obtained equilibrium equation.

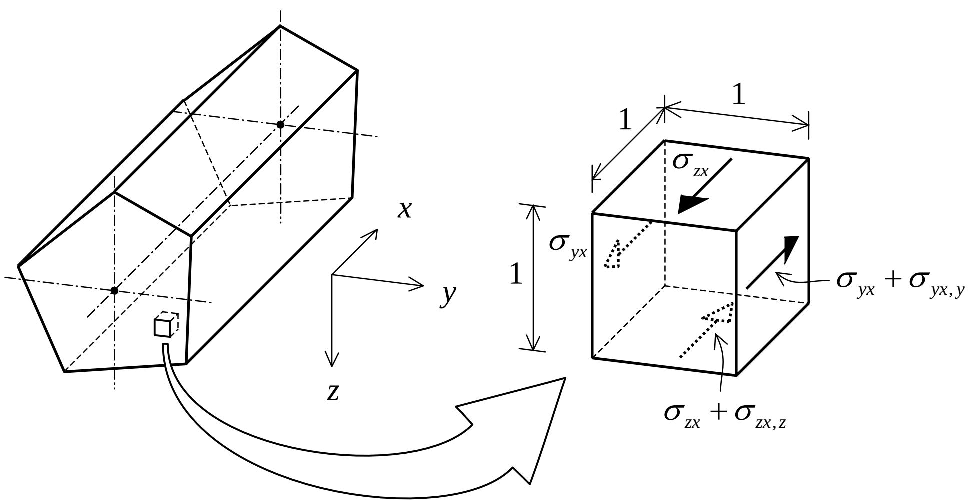


Fig. 6.8: Interpretation of the equilibrium equation in x-direction.

Summarising, for the special case of the De Saint-Venant torque, the general three-dimensional kinematic, constitutive and equilibrium equations reduce to:

$$\boxed{\begin{aligned}\gamma_{zx} &= \left(\psi_{,z} + y\right)\theta \\ \gamma_{xy} &= \left(\psi_{,y} - z\right)\theta\end{aligned}} \qquad (\textit{kinematic equations}) \tag{6.8}$$

$$\boxed{\left.\begin{aligned}\sigma_{zx} &= G\gamma_{zx} \\ \sigma_{xy} &= G\gamma_{xy}\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned}\gamma_{zx} &= \frac{1}{G}\sigma_{zx} \\ \gamma_{xy} &= \frac{1}{G}\sigma_{xy}\end{aligned}\right.} \qquad (\textit{constitutive equations}) \tag{6.9}$$

$$\boxed{\sigma_{yx,y} + \sigma_{zx,z} = 0} \qquad (\textit{equilibrium equation}) \tag{6.10}$$

The problem contains only one degree of freedom, the warping function $\psi(y,z)$; this corresponds with the fact that just one equilibrium equation is present. No volume load in x-direction exists. Only two stresses and their corresponding strains are different from zero, therefore just two kinematic equations and two constitutive equations remain.
The scheme of relations as depicted in Fig. 6.9 is applicable. The two stresses $\sigma_{xz} = \sigma_{zx}$ and

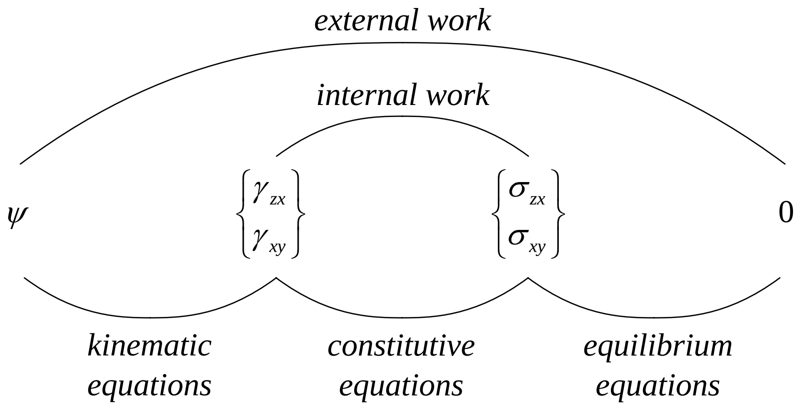


Fig. 6.9: Diagram displaying the relations between the quantities playing a role in the analysis of the De Saint-Venant torsion.

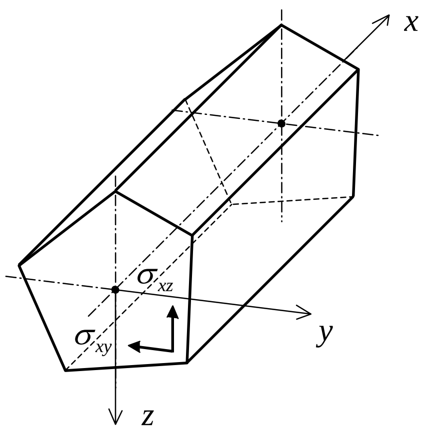


Fig. 6.10: A torsional moment will generate shear stresses σ_{xz} and σ_{xy}.

$\sigma_{xy} = \sigma_{yx}$, which play a role in the assumed displacement field of the problem, are exactly the shear stresses in the cross-section that are caused by the torsional moment. This has been depicted in Fig. 6.10.

Dynamic boundary conditions
The two stresses σ_{xz} and σ_{xy} can also satisfy the dynamic boundary conditions along the circumference of the bar. Generally a normal stress σ_{nn} and two shear stresses σ_{ns} and σ_{nx} are present on the cylindrical surface (see Fig. 6.11). The stresses σ_{nn} and σ_{ns} follow via

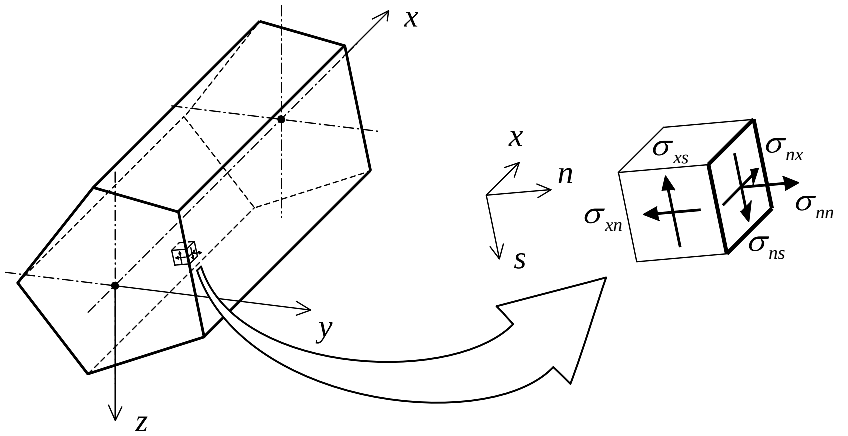


Fig. 6.11: The stresses resulting from the displacement field of De Saint-Venant can satisfy the condition that the stresses σ_{nn}, σ_{ns} and σ_{nx} along the outer surface are zero.

a transformation from the stresses σ_{yy}, σ_{yz} and σ_{zz}. Because all these stresses are zero, the stresses σ_{nn} and σ_{ns} will be zero too. The shear stress σ_{nx} is equal to σ_{xn}, which is situated in the plane of the cross-section. The shear stresses σ_{xn} and σ_{xs} can be obtained via a transformation in the same plane of the shear stresses σ_{xz} and σ_{xy}. The stresses σ_{xs} and σ_{xn} are tangent and normal to the circumference of the cross-section, respectively. The stress σ_{xn} has to be zero, because σ_{nx} cannot occur on the stress-free cylindrical surface. In other words, a completely stress-free cylindrical surface can be realised by requiring that σ_{nx} is zero, i.e.:

$$\boxed{\sigma_{xn} = 0} \qquad (\textit{dynamic boundary condition}) \qquad (6.11)$$

Solution strategies
After the formulation of the three sets of basic equations, the next step is the establishment of the solution procedure for these equations. Again the two strategies of the displacement and force method can be followed. Both methods will be discussed and it will become clear that the displacement method leads to a simple and concise formulation for both solid cross-sections and cross-sections with holes. The force method provides a simple formulation only for solid cross-sections, for cross-sections with holes the formulation becomes rather complicated. Nevertheless, in the past the force method was used in the classical approach of the torsional problem. The reason was that for this method a number of analogies exist that provided a lot of insight into the problem. Nowadays in the computer age, no clear preference

for one of the methods exists and both methods can be applied. However, in this course most of the attention is paid to the force method, because this method links up with the visual imagination of the engineer.

6.3 Displacement method

Differential equation

In the procedure of the displacement method, successive substitutions take place from the kinematic equations towards the equilibrium equation. Here the constitutive equations are used in stiffness formulation. Doing so, the equilibrium equation is transformed into a differential equation for the unknown degree of freedom ψ. The procedure can be summarised by:

$$\begin{aligned}
&\gamma_{zx} = \left(\psi_{,z} + y\right)\theta \\
&\gamma_{xy} = \left(\psi_{,y} - z\right)\theta
\end{aligned} \quad \text{(\textit{kinematic equations})}$$

$$\begin{aligned}
&\sigma_{zx} = G\gamma_{zx} \\
&\sigma_{xy} = G\gamma_{xy}
\end{aligned} \quad \text{(\textit{constitutive equations})}$$

$$\sigma_{yx,y} + \sigma_{zx,z} = 0 \quad \text{(\textit{equilibrium equation})}$$

$$\boxed{G\left(\psi_{,yy} + \psi_{,zz}\right)\theta = 0} \qquad (6.12)$$

Because G and θ are constants, the found differential equation simply states that the Laplace operator of ψ is equal to zero:

$$\boxed{\psi_{,yy} + \psi_{,zz} = 0} \qquad (6.13)$$

Boundary condition

For the solution of differential equation (6.13) it is required to reformulate the boundary condition $\sigma_{xn} = 0$ in terms of ψ. This can be done as follows.

The condition $\sigma_{xn} = 0$ implies that the deformation $\gamma_{xn} = 0$ too. For this deformation it can be written:

$$\gamma_{xn} = u_{x,n} + u_{n,x}$$

The displacement u_n can simply be expressed in u_y and u_z by the following transformation formula (also see Fig. 6.12):

$$u_n = u_y \cos\alpha + u_z \sin\alpha$$

The expression for the deformation becomes:

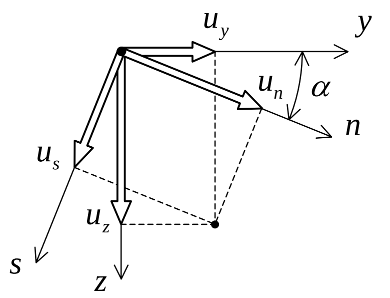


Fig. 6.12: Transformation of displacements in the plane of the cross-section.

$$\gamma_{xn} = u_{x,n} + u_{y,x} \cos\alpha + u_{z,x} \sin\alpha$$

Finally, the relations for u_x, u_y and u_z given by (6.4) are substituted, they read:

$$u_x = \psi\,\theta \quad ; \quad u_y = -x\,z\,\theta \quad ; \quad u_z = x\,y\,\theta$$

The requirement that γ_{xn} is zero delivers the relation:

$$\left(\psi_{,n} - z \cos\alpha + y \sin\alpha\right)\theta = 0$$

or differently written, it delivers the condition for the slope of ψ perpendicular to the edge:

$$\boxed{\psi_{,n} = z \cos\alpha - y \sin\alpha} \tag{6.14}$$

In each point of the edge the values of y, z and α are known, so that $\psi_{,n}$ is prescribed along the entire circumference. The solution of differential equation (6.13) together with boundary condition (6.14) is classified as a problem of the *Neumann type*. Now from a mathematical point of view, the warping function ψ is determined and can be solved. After that γ_{zx} and γ_{xy} can be solved from the kinematic equations, which also determine the values of the stresses. For obtaining a correct solution for ψ, in one point of the cross-section a value of ψ has to be prescribed in order to prevent a rigid body movement of the body in x-direction.

Hollow Cross-sections

When the cross-section is not solid but contains one or more holes, the procedure is not essentially more difficult. Then along the circumference of each hole the dynamic boundary condition $\sigma_{xn} = 0$ applies too. This means that along the holes the condition (6.14) for $\psi_{,n}$ has to be imposed.

6.4 Force method

In the force method a solution for the stresses is sought that a priori satisfies the equilibrium equations and dynamic boundary conditions. Because one equilibrium equation exists for the two unknown stresses σ_{xy} and σ_{xz}, the problem is statically indeterminate to the first degree.

Therefore, only one stress function has to be introduced, which just like the stresses is a function of y and z. In this case, a stress function that meets the conditions is defined by:

$$\sigma_{xy} = \phi_{,z} \quad ; \quad \sigma_{xz} = -\phi_{,y} \tag{6.15}$$

These relations between the stresses and the redundant ϕ guarantee that the equilibrium equation $\sigma_{yx,y} + \sigma_{zx,z} = 0$ is satisfied automatically. For the determination of ϕ a compatibility condition has to be formulated. This condition is found by elimination of the degree of freedom ψ from the two kinematic equations:

$$\gamma_{zx} = \left(\psi_{,z} + y\right)\theta \quad ; \quad \gamma_{xy} = \left(\psi_{,y} - z\right)\theta$$

When both equations are differentiated with respect to y and z respectively, the two equations contain the term $\psi_{,yz}$, which easily can be eliminated. The result reads:

$$\gamma_{xz,y} - \gamma_{xy,z} = 2\theta \tag{6.16}$$

where $\gamma_{zx,y}$ is replaced by $\gamma_{xz,y}$. It appears that the deformations γ_{xz} and γ_{xy} cannot obtain independently any value, they are coupled.

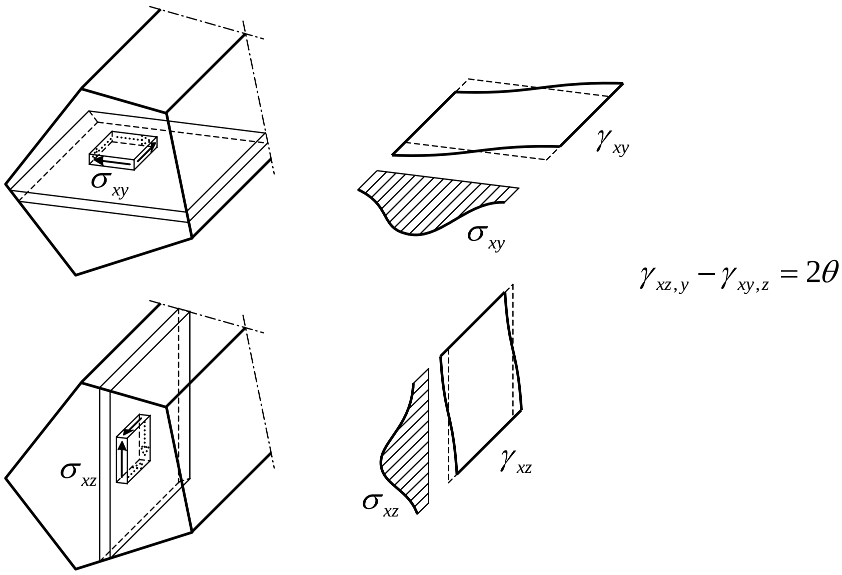


Fig. 6.13: The deformations γ_{xy} and γ_{xz} have to be compatible.

On basis of Fig. 6.13, a physical interpretation of the compatibility condition can be given. In a horizontal slice of the bar the shear stresses σ_{xy} generate the shear angles γ_{xy}. The originally rectangular slice deforms into another shape. At the same time a vertical rectangular slice deforms under the influence of the shear stresses σ_{xz}, which initiate the shear angles γ_{xz}. All those horizontal and vertical slices have to fit precisely during deformation, such that a continuous warped cross-section is maintained. This means that there has to be a relation between the deformations γ_{xy} and γ_{xz}.
The solution strategy now is the successive substitution from the equilibrium equation up to the compatibility equation. In this case, the constitutive equations are given in flexibility formulation, i.e.:

$$\boxed{-\frac{1}{G}\left(\phi_{,yy}+\phi_{,zz}\right)=2\theta}\tag{6.17}$$

$$\gamma_{xz,y}-\gamma_{xy,z}=2\theta \qquad (\textit{compatibility equation})$$

$$\gamma_{xy}=\frac{1}{G}\sigma_{xy}$$
$$\gamma_{xz}=\frac{1}{G}\sigma_{xz} \qquad (\textit{constitutive equations})$$

$$\sigma_{xy}=+\phi_{,z}$$
$$\sigma_{xz}=-\phi_{,y} \qquad (\textit{equilibrium equations})$$

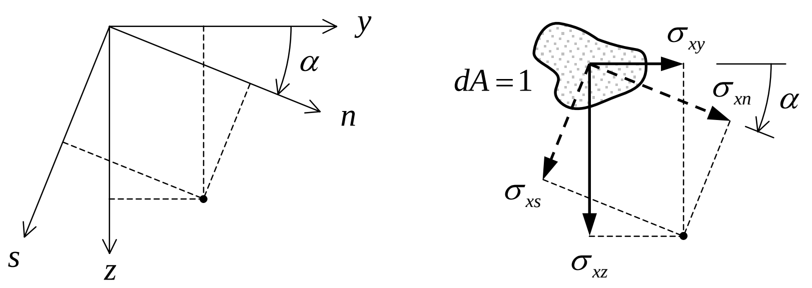
It can be seen that the force method results in a Laplace equation too. However, in this case the right-hand side is not equal to zero.

Remark
In (6.15) the stress function is defined in such a manner that the stress σ_{xy}, which is acting in y-direction is equal to the derivative of ϕ in the z-direction perpendicular to that. Likewise, the value of the stress σ_{xz} is equal to the derivative of ϕ in perpendicular direction (except for the sign). It can be shown that this property also holds for the shear stresses σ_{xn} and σ_{xs} in arbitrarily chosen orthogonal directions n and s (see Fig. 6.14):

$$\sigma_{xn}=+\phi_{,s} \quad ; \quad \sigma_{xs}=-\phi_{,n}\tag{6.18}$$

To prove this, the stresses σ_{xn} and σ_{xs} and also $\phi_{,n}$ and $\phi_{,s}$ will be expressed in $\phi_{,y}$ and $\phi_{,z}$. From the results, relation (6.18) can be confirmed.
Fig. 6.14 shows that the coordinates and the shear stresses in the cross-section transform by the same rule. In the expression for the stresses, σ_{xy} and σ_{xz} are replaced by respectively $\phi_{,z}$ and $-\phi_{,y}$. This results in:

$$\begin{aligned}\sigma_{xn}&=+\phi_{,z}\cos\alpha-\phi_{,y}\sin\alpha\\ \sigma_{xs}&=-\phi_{,z}\sin\alpha+\phi_{,y}\cos\alpha\end{aligned}\tag{6.19}$$



$$n=+y\cos\alpha+z\sin\alpha \qquad \sigma_{xn}=+\sigma_{xy}\cos\alpha+\sigma_{xz}\sin\alpha$$
$$s=-y\sin\alpha+z\cos\alpha \qquad \sigma_{xs}=-\sigma_{xy}\sin\alpha+\sigma_{xz}\cos\alpha$$

Fig. 6.14: Transformation of coordinates and shear stresses in the cross-section.

By using the chain rule, $\phi_{,n}$ and $\phi_{,s}$ can be expressed in $\phi_{,y}$ and $\phi_{,z}$:

$$\phi_{,n} = \phi_{,y}\, y_{,n} + \phi_{,z}\, z_{,n}$$
$$\phi_{,s} = \phi_{,y}\, y_{,s} + \phi_{,z}\, z_{,s}$$

To determine the derivatives of y and z with respect to n and s, the coordinate transformations of Fig. 6.14 are inverted:

$$y = n\cos\alpha - s\sin\alpha$$
$$z = n\sin\alpha + s\cos\alpha$$

This leads to the expressions:

$$\begin{aligned} \phi_{,n} &= +\phi_{,y}\cos\alpha + \phi_{,z}\sin\alpha \\ \phi_{,s} &= -\phi_{,y}\sin\alpha + \phi_{,z}\cos\alpha \end{aligned} \qquad (6.20)$$

Comparison of (6.19) with (6.20) shows that relation (6.18) is generally valid.

Boundary conditions
The dynamic boundary condition along the circumference of a solid cross-section reads:

$$\sigma_{xn} = 0$$

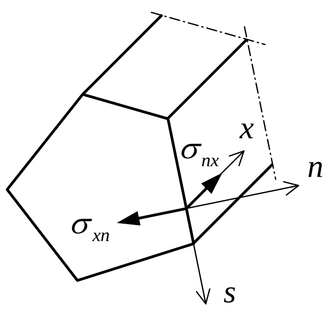
where n is normal to the edge and pointing outward (see Fig. 6.15). On basis of (6.18) it then follows that:

$$\phi_{,s} = 0$$

The derivative in the direction of the circumference is equal to zero. This means that ϕ has a constant value along the circumference. For a solid section this constant value can be set to zero without any loss of general validity, for the stresses are obtained by differentiation of ϕ so that the constant disappears. Therefore, as boundary condition it will be prescribed:

$$\phi = 0 \qquad (6.21)$$

The found differential equation and corresponding boundary condition given by:



$$\boxed{\begin{array}{c} \sigma_{xn} = 0 \;\; \rightarrow \;\; \phi_{,s} = 0 \;\; \rightarrow \\ \phi = \text{ constant } \;\; \rightarrow \;\; \text{choose } \phi = 0 \end{array}}$$

Fig. 6.15: Boundary condition along the circumference of the bar.

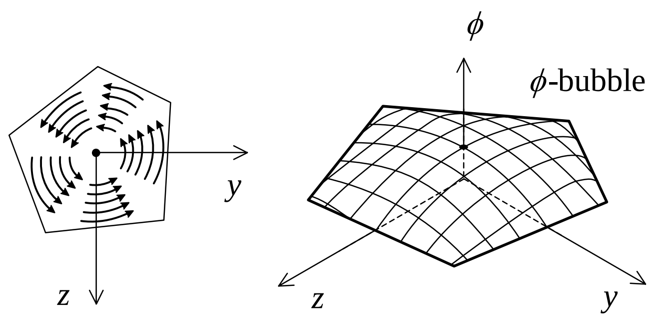
$$\boxed{\begin{array}{l} -\dfrac{1}{G}\left(\phi_{,yy}+\phi_{,zz}\right)=2\theta \\ \phi=0 \end{array}} \qquad \begin{array}{l} (\textit{differential equation}) \\ (\textit{boundary condition}) \end{array} \tag{6.22}$$

determine in mathematical sense the torsional problem in the force method. In this way the problem is written in the so-called Dirichlet formulation. The stress function ϕ can be solved from the set (6.22), after which the stresses can be found by the derivatives of the function ϕ:

$$\boxed{\begin{array}{l} \sigma_{xy}=+\phi_{,z} \\ \sigma_{xz}=-\phi_{,y} \end{array}} \tag{6.23}$$

The ϕ-bubble

From the simple torsional problem of the circular cross-section, as discussed in previous courses, it is known that the shear stresses are zero in the centre of the cross-section and that the “round-going” stresses increase in radial direction. This pattern can be expected for cross-sections of arbitrary shape too. In the point where the stresses σ_{xy} and σ_{xz} are zero, the derivatives $\phi_{,z}$ and $\phi_{,y}$ have to be zero. At that position the function ϕ obtains an extreme value, while ϕ is zero on the edge. When a section is made through the distribution of ϕ perpendicular to the cross-section a sort of hood covering of the cross-section can be noticed, which will be called the “ϕ-bubble” (see Fig. 6.16). The slopes of the ϕ-bubble determine the magnitude of the stresses. Indeed it can be observed that the stresses increase towards the edge.



on circumference $\rightarrow \quad \phi=0$

in point where $\left.\begin{array}{l}\phi_{,y}=0 \\ \phi_{,z}=0\end{array}\right\} \rightarrow \quad \phi=\phi_{\max}$

Fig. 6.16: The distribution of ϕ over the cross-section is called a “ϕ-bubble”.

Check of the shear forces

It was shown that σ_{xy} and σ_{xz} are the only stresses present, and how they can be determined. In general, the resultants of these stresses over the cross-section may be a torsional moment M_t and the shear forces V_y and V_z. In the case of torsion, the stress distribution should be statically equivalent with a torsional moment M_t, while the shear forces V_y and V_z are zero. Therefore, the values of the shear forces will be checked.
The horizontal shear force equals:

$$V_y = \iint_A \sigma_{xy}\, dA = \iint_A \frac{\partial\phi}{\partial z} dy dz$$

First, integration is carried out in z -direction and then in y -direction (see Fig. 6.17):

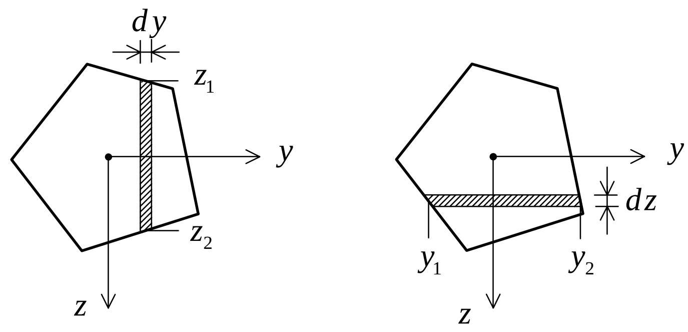


Fig. 6.17: Integration paths required for the conformation that V_y and V_z are zero.

$$V_y = \int \left\{ \int_{z_1}^{z_2} \frac{\partial\phi}{\partial z} dz \right\} dy = \int \left\{ \int_{z_1}^{z_2} d\phi \right\} dy = \int \left(\phi_2 - \phi_1\right) dy = 0$$

At the edge the values of ϕ_1 and ϕ_2 are both zero, which means that V_y is zero too.
For the vertical shear force a similar approach is adopted. In this case the integration is first carried out in y -direction:

$$V_z = -\iint_A \sigma_{xz}\, dA = -\iint_A \frac{\partial\phi}{\partial y} dy dz \quad \rightarrow \quad V_z = -\int \left\{ \int_{y_1}^{y_2} d\phi \right\} dz = -\int \left(\phi_2 - \phi_1\right) dz = 0$$

The resulting torsional moment

During the problem definition in section 6.1, the resulting moment was written as:

$$M_t = \iint_A r\, \sigma\, dA$$

This integral can be worked out into more detail by introducing the shear stresses σ_{xy} and σ_{xz} with their arms z and y, respectively. As can be observed in Fig. 6.18, positive values of σ_{xy} deliver moments that reduce the torque and the positive values of σ_{xz} increase the torque (in the first quadrant). Therefore, the expression of the torsional moment becomes:

$$M_t = \iint \left(y\, \sigma_{xz} - z\, \sigma_{xy} \right) dA \tag{6.24}$$

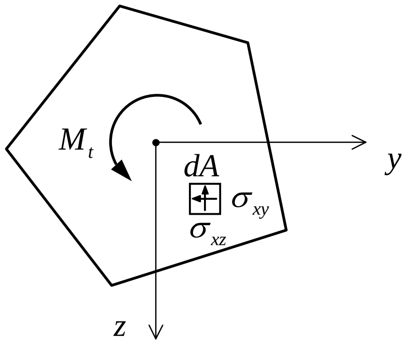


Fig. 6.18: Calculation of the resulting torsional moment.

It is advantageous to solve integral (6.24) in two parts. During the solution process integration by parts takes place. It is recalled to mind how this is done. Two functions $u(x)$ and $v(x)$ are considered on the interval $x_1 \le x \le x_2$. For x_1 the function values are u_1 and v_1 and for x_2 they are u_2 and v_2. For the product of the two functions it holds:

$$\int_{x_1}^{x_2} d(u\,v) = u_2 v_2 - u_1 v_1 \quad \text{or} \quad \int_{x_1}^{x_2} u\,dv + \int_{x_1}^{x_2} v\,du = u_2 v_2 - u_1 v_1$$

As rule for integration by parts, the last expression is used in the form:

$$\int_{x_1}^{x_2} u\,dv = -\int_{x_1}^{x_2} v\,du + \left(u_2 v_2 - u_1 v_1\right) \tag{6.25}$$

After this short intermezzo, it is continued with the determination of the surface integral of the vertical stresses. They deliver a share in the torsional moment given by:

$$M_{vertical} = \iint_A y\,\sigma_{xz}\,dA = \iint_A -y\frac{\partial\phi}{\partial y}\,dy\,dz$$

First the integration in y-direction is performed, see Fig. 6.19:

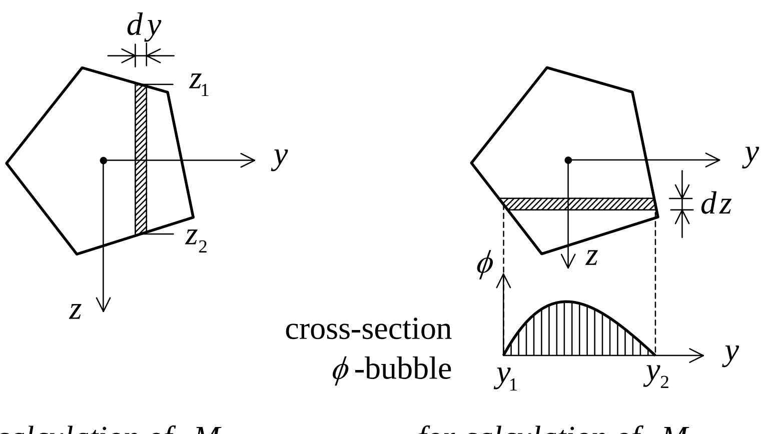


Fig. 6.19: Integration paths for the calculation of $M_{vertical}$ and $M_{horizontal}$.

$$M_{vetical} = \int \left(\int_{y_1}^{y_2} -y \frac{\partial \phi}{\partial y} \, dy \right) dz = \int \left(\int_{y_1}^{y_2} -y \, d\phi \right) dz$$

Now, the integral in y-direction will be integrated by parts according to the rule (6.25). Then it is found:

$$-\int_{y_1}^{y_2} y \, d\phi = \int_{y_1}^{y_2} \phi \, dy - \left(y_2 \, \phi_2 - y_1 \, \phi_1 \right)$$

Because ϕ_1 and ϕ_2 are situated on the edge they are zero, so that the following remains:

$$\int_{y_1}^{y_2} \phi \, dy \qquad (\textit{area of the vertical section of the } \phi\textit{-bubble}) \tag{6.26}$$

This result is exactly the area of the considered vertical section of the ϕ-bubble. It then is clear that the moment of the vertical stresses is equal to the volume of the ϕ-bubble:

$$\boxed{M_{vertical} = \iint_A \phi \, dy dz} \qquad (\textit{volume of the } \phi\textit{-bubble}) \tag{6.27}$$

In an analogous manner the contribution of $M_{horizontal}$ is calculated. For that purpose integration in z-direction is carried out. Also for this case it is found:

$$\boxed{M_{horizontal} = \iint_A \phi \, dy dz} \qquad (\textit{volume of the } \phi\textit{-bubble}) \tag{6.28}$$

Therefore, the total result for the torsional moment becomes:

$$\boxed{M_t = 2 \iint_A \phi \, dA} \qquad (\textit{two times the volume of the } \phi\textit{-bubble}) \tag{6.29}$$

Recalling to mind that for this moment it also holds $M_t = GI_t \theta$, for the torsional stiffness it is found:

$$\boxed{GI_t = \frac{2}{\theta} \iint_A \phi \, dA} \tag{6.30}$$

Conclusions

Up to now, results have been derived which are worthwhile mentioning in a summary. Fig. 6.20 supports these conclusions.

– The stiffness GI_t and the torsional moment M_t are determined by the double volume of the ϕ-bubble (for $\theta = 1$).
– The shear stress is determined by the slope of the ϕ-bubble perpendicular to the direction of the stress. This property holds for every direction.
– It appears that the contribution of the vertical and horizontal stresses to GI_t and M_t is the same, namely one volume of the ϕ-bubble each. This generally holds irrespective the shape of the cross-section, for example for a strip-shaped as well as a square cross-section.

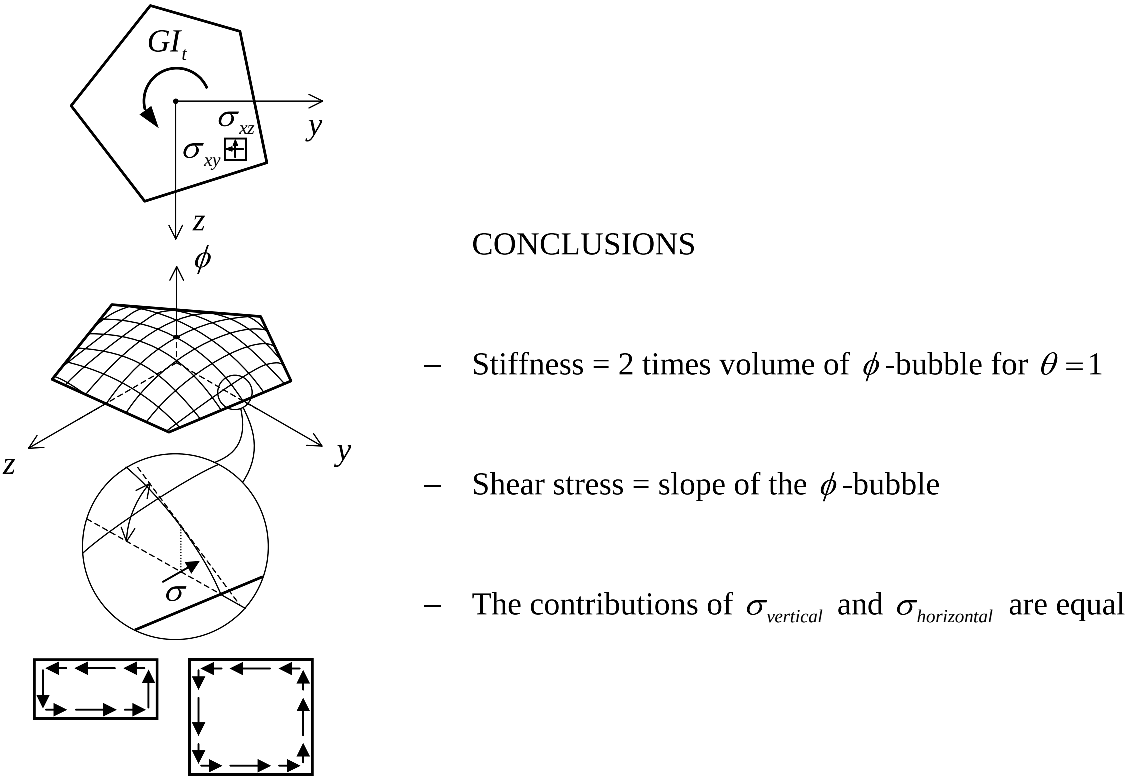


Fig. 6.20: Summary of the conclusions.

Remarks

1. The x-axis has been chosen arbitrarily parallel to the bar axis. The displacement field (6.4) contains a rotation about the x-axis. This creates the impression that this axis has certain special properties. However, this is not the case, because an extra rotation as a rigid body about the y- and z-axes can be added to the displacement field (6.4), such that any other line parallel to the x-axis start to act as the rotation axis. For the displacement field it then has to be chosen:

$$u_x = \psi(y,z)\theta - by\theta + az\theta$$

rotation about y-axis

$$u_y = x\left(z-b\right)\theta$$

rotation about z-axis

$$u_z = x\left(y-a\right)\theta$$

It simply can be established that the additional terms have no influence on the stresses, and that now the rotation takes place about the axis $y = a, z = b$. So, this indicates that the x-axis indeed can be chosen arbitrarily without loss of generality provided that it is parallel to the axis of the bar.

2. During the analysis it was made clear that σ_{xx} has to be zero, also at the ends of the bar and that for that reason the warping cannot be prevented. In section 6.12 the consequences of a prevented warping will be discussed. Theoretically, the shear stresses σ_{xy} and σ_{xz} at the ends have to be distributed exactly as the derivatives of the stress function ϕ prescribe. If this is not the case at the ends an interference length will occur in which the stress-state gradually evolves to the distribution according to the derivatives of ϕ.

3. The surface integral for M_t given by (6.29) could have been determined directly from (6.24) by replacement of σ_{xy} and σ_{xz} by the respective derivatives of ϕ, which is followed by the application of the proposition of Green for the transformation of a surface integral into a contour integral. However, the disadvantage of this approach is that it would not have revealed that the contributions of the horizontal and vertical stresses to the moment are identical.

6.5 Exact solution for an elliptic cross-section

For some cross-sections, it appears to be possible to derive an exact solution for the differential equation and boundary condition (6.22). An example is the elliptic cross-section (see Fig. 6.21). The equation of the edge in this case is:

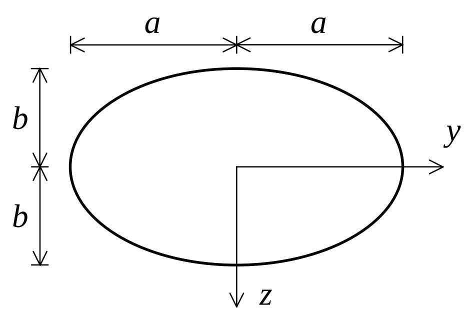


Fig. 6.21: Elliptic cross-section.

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Then the following function is zero all along the edge:

$$A\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)$$

It turns out that the differential equation can be satisfied if ϕ is made equal to this function:

$$\phi = A\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right)$$

Substitution of this relation into differential equation (6.22) yields the following result for A:

$$A = G\frac{a^2b^2}{a^2+b^2}\theta$$

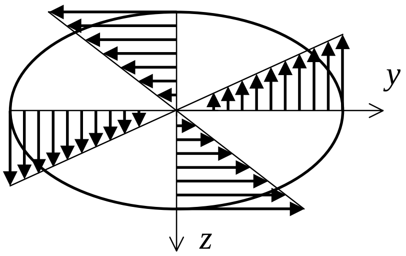
Therefore, the solution is:

$$\boxed{\phi = G\frac{a^2b^2}{a^2+b^2}\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right)\theta} \tag{6.31}$$

Using (6.15) the shear stresses become:

$$\sigma_{xy} = -2G\frac{a^2z}{a^2+b^2}\theta \quad ; \quad \sigma_{xz} = +2G\frac{b^2y}{a^2+b^2}\theta \tag{6.32}$$

The shear stresses are linearly distributed along straight lines through the origin, just as known for the circular cross-section (see Fig. 6.22). The largest shear stress occurs on the edge at the short axis (at $y=0$, $z=\pm b$, when $a>b$). The absolute value of this stress is:



In this figure the x-axis is pointing into the paper

Fig. 6.22: Stress distribution in an elliptic cross-section.

$$\sigma_{\max} = 2G\frac{a^2b}{a^2+b^2}\theta \quad \text{for} \quad a>b$$

The torsional moment is according to (6.29) two times the volume of the ϕ-bubble:

$$M_t = 2G\frac{a^2b^2}{a^2+b^2}\left\{\iint dy\,dz - \frac{1}{a^2}\iint y^2dy\,dz - \frac{1}{b^2}\iint z^2dy\,dz\right\}\theta$$

The first integral is the area of the ellipse, πab. The second and third integrals are the moments of inertia with respect to the z-axis and y-axis, respectively. The values are equal to $\pi a^3b/4$ and $\pi ab^3/4$, respectively. Thus the term between braces equals $\pi ab/2$ and the moment becomes:

$$M_t = G\pi \frac{a^3 b^3}{a^2 + b^2}\theta \tag{6.33}$$

Because it also holds:

$$M_t = G I_t \theta$$

For an ellipse it apparently is found:

$$I_t = \frac{\pi a^3 b^3}{a^2 + b^2} \tag{6.34}$$

By using (6.33), the relations (6.32) for the stresses can be expressed in the torsional moment M_t:

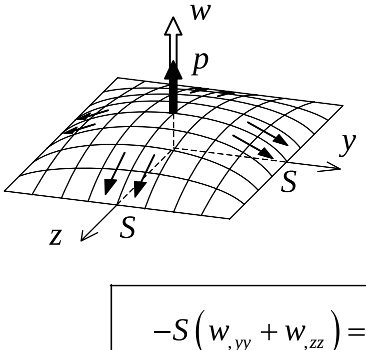
$$\sigma_{xy} = -\frac{z\,M_t}{\frac{1}{2}\pi a b^3} \quad ; \quad \sigma_{xz} = +\frac{y\,M_t}{\frac{1}{2}\pi a^3 b} \tag{6.35}$$

A designer mainly will be interested in the maximum shear stress. Expressed in the torsional moment this shear stress equals:

$$\sigma_{\max} = \frac{M_t}{\frac{1}{2}\pi a b^2} \quad \text{for} \quad a > b \tag{6.36}$$

6.6 Membrane analogy

Already during the discussion of the force method it was mentioned that analogies can be used. *Prandtl* introduced a well-known analogy. He recognised that the differential equation for the torsional problem was similar to the problem of a membrane under tension. This so-called *membrane analogy* is depicted in Fig. 6.23.



ANALOGY

– shape of membrane = shape of cross-section
– $p = 2\theta$
– $s = 1/G$

$$-S\left(w_{,yy} + w_{,zz}\right) = p$$
on edge: $w = 0$

$$-\frac{1}{G}\left(\phi_{,yy} + \phi_{,zz}\right) = 2\theta$$
on edge: $\phi = 0$

Fig. 6.23: The differential equations for a stressed membrane and for torsion have the same character.

The bending stiffness of the membrane is very low. In the plane of the membrane an omnidirectional tensile force per unit of width is present. When a part of the membrane is considered with unit width in z -direction and length dy, it can be modelled as a cable. For

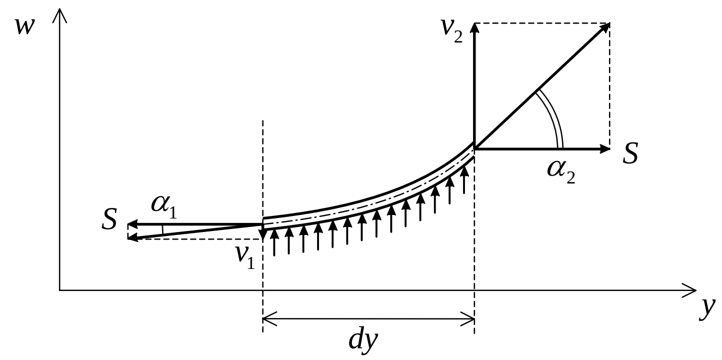


Fig. 6.24: Free-body diagram.

sufficiently small deflection w under a constant excess pressure p, the vertical equilibrium of this part of the membrane equals (see Fig. 6.24):

$$v_1 - v_2 = p\,dy$$

where v_1 and v_2 are the membrane shear forces, for which it can be written:

$$v_1 = S\,\alpha_1 \quad ; \quad v_2 = S\,\alpha_2$$

So:

$$S\left(\alpha_1 - \alpha_2\right) = p\,dy$$

Further, for the increase of the slope it holds:

$$\alpha_2 - \alpha_1 = w_{,yy}dy$$

This provides:

$$-S\,w_{,yy}\,dy = p\,dy \quad \rightarrow \quad -S\,w_{,yy} = p$$

When both the y -direction and z -direction are considered, the differential equation for a membrane is found:

$$\boxed{-S\left(w_{,yy} + w_{,zz}\right) = p} \tag{6.37}$$

Experimental membrane analogy
The membrane analogy can be utilised experimentally as discussed below. A box is made with vertical walls and horizontal bottom. The horizontal top of the box is open and is covered by a stretched rubber membrane. The plan view of the box has the shape of the cross-

section to be investigated. By pressurising the inside of the box, the membrane will bulge out. A shape will be created similar to the ϕ-bubble (Fig. 6.23 shows this phenomenon for a rectangular cross-section). By measurement of the slope of the membrane in different points, the stress distribution over the cross-section can be determined. In literature articles can be found of how this was done in the past with *soap films*. Therefore, this approach is also called the *soap-film analogy*.

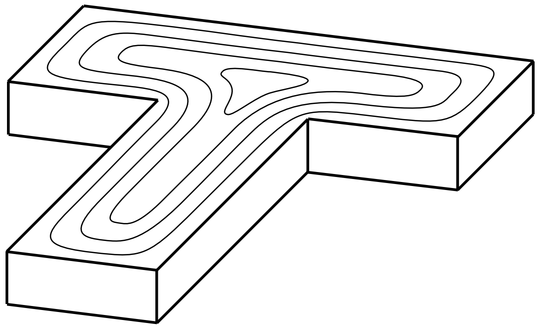


Fig. 6.25: Example of contour lines for a T-shaped cross-section.

A method to visualise the membrane surface is the drawing of contour lines of constant ϕ (see Fig. 6.25). When a s-n coordinate system is attached with s parallel and n perpendicular to the contour line, then along the contour line it holds $w_{,s} = 0$ and therefore $\phi_{,s} = 0$. This means that σ_{xn} is zero and only the shear stress σ_{xs} is present. So, the direction of the shear stress is tangent to the contour lines. The contour lines can be drawn at constant intervals of w (and thus ϕ). Then it can be concluded that the shear stress σ_{xs} is large where the density of he contour lines is high, because at those spots the gradient $\phi_{,n}$ is large. The contour lines can be visualised by the so-called “shadow moiré”. With optical means, the lines of constant displacement are visualised, with a constant difference in displacement between the lines.

Membrane analogy as mental experiment

The membrane analogy can be used too, without the actual execution of a real test with a membrane. The analogy is applied as a mental experiment. This can be done both qualitatively and quantitatively. Qualitatively, the analogy is useful, since it provides an indication where the largest shear stresses will occur and how the cross-section can be adapted to optimise it for torsion. Fig. 6.26 shows a triangular cross-section. When a soap film is imagined under pressure over this cross-section, the largest slope will occur halfway

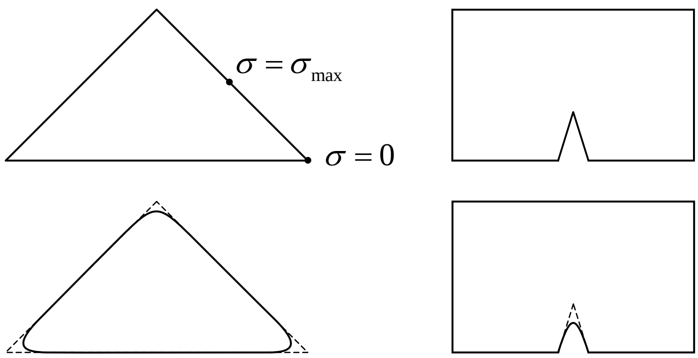


Fig. 6.26: membrane analogy as mental experiment to optimize cross-sections.

the sides of the triangle. At those positions the shear stress reaches its maximum. In the vertices, the soap film will be practically horizontal and no significant contribution to the moment and stiffness can be expected. Therefore, rounding off the vertices can save material. At the right side of Fig. 6.26, a notched rectangular cross-section is depicted. When the notch is sharp the contour lines will be concentrated around the tip of the notch and large stresses will occur. A blunt notch is much more favourable.

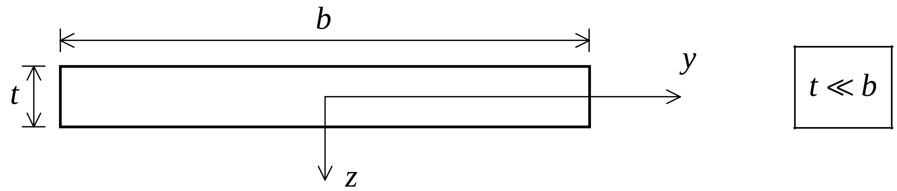


Fig. 6.27: Strip-shaped cross-section.

The mental experiment can be applied too for obtaining quantitative information. A well-known example is the torsion of a bar with a strip-shaped cross-section as shown in Fig. 6.27 (for this case, in previous lectures a solution was found already by a different method). When the experiment would be carried out with a membrane it can be expected that the deflection would be cylindrically shaped over practically the entire width b, independently of y. Only at the two ends $y = b/2$ and $y = -b/2$ a deviation from this shape would occur. For $t \ll b$ this will hardly affect the volume under the membrane, if it is assumed that the deflection w over the full width is only a function of z, see Fig. 6.28.

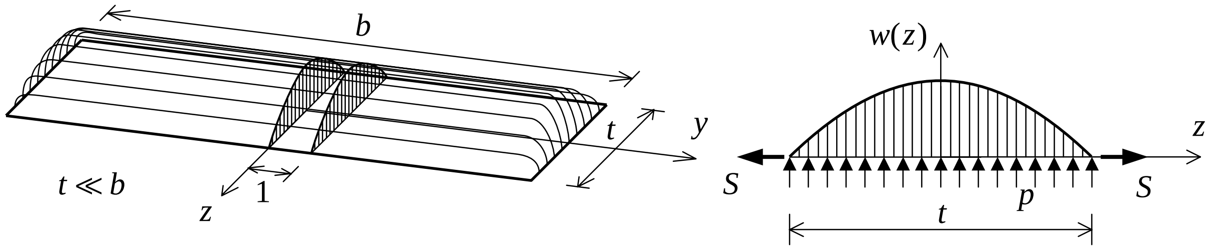


Fig. 6.28: Shape of the membrane for a strip-shaped cross-section.

The differential equation then simplifies to:

$$-\frac{1}{S}\frac{d^2 w}{d z^2} = p$$

with boundary condition that w is zero for $z = \pm t/2$.
The solution for this equation is:

$$w(z) = \frac{p}{8S} t^2 \left(1 - \frac{4z^2}{t^2}\right)$$

This is a parabola. In Fig. 6.23 it is shown that w is equivalent to ϕ, if at the same time it is substituted:

$$p = 2\theta \quad ; \quad s = \frac{1}{G}$$

Therefore, for the stress function ϕ it holds:

$$\phi(z) = \frac{Gt^2}{4}\left(1 - \frac{4z^2}{t^2}\right)\theta \tag{6.38}$$

This is a parabolic distribution with a maximum of:

$$\phi_{\max} = \frac{Gt^2}{4}\theta$$

The area between this parabola and the z-axis equals:

$$Area = \frac{2}{3} \times t \times \phi_{\max} = \frac{1}{6} G\, t^3 \theta \tag{6.39}$$

The torsional moment is twice the volume of the ϕ-bubble;

$$M_t = 2 \times b \times Area = \frac{1}{3} G\, b\, t^3 \theta$$

Since it also holds that:

$$M_t = G I_t\, \theta$$

For the torsional moment of inertia it is found:

$$I_t = \frac{1}{3} b\, t^3 \tag{6.40}$$

which already was indicated in Fig. 6.2.
Further, the stress distribution can be checked. In the direction of the long edge it holds:

$$\sigma_{xy} = \frac{\partial \phi}{\partial z} = -2Gz\theta \tag{6.41}$$

The maximum values in absolute sense occur for $z = \pm t/2$, see Fig. 6.29:

$$\sigma_{\max} = G\, t\theta$$

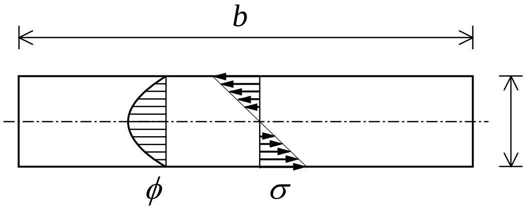


Fig. 6.29: Stress distribution in a strip-shaped cross-section.

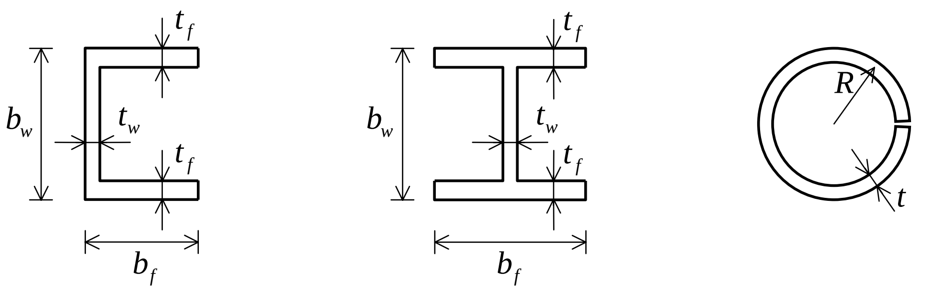
With the aid of (6.39) this maximum stress can be expressed in the moment, an interesting relation for design purposes:

$$\sigma_{\max} = \frac{M_t}{\frac{1}{3}bt^2} \tag{6.42}$$

This expression was mentioned in Fig. 6.2 too.

Remarks

1. From the mental experiment it follows that the formula $I_t = \frac{1}{3}bt^3$ also can be used (in a similar approach) to determine the torsional moment of inertia of cross-sections, which are built up out of strip-shaped parts as shown in Fig. 6.30.



$I_t = \frac{1}{3}b_w t_w^3 + \frac{2}{3}b_f t_f^3$ $\qquad$ $I_t = \frac{1}{3}b_w t_w^3 + \frac{2}{3}b_f t_f^3$ $\qquad$ $I_t = \frac{1}{3}bt^3$ where $b = 2\pi R$

Fig. 6.30: Torsional moments of inertia of thin-walled cross-sections.

2. The result for ϕ given by (6.38) leads to the stresses:

$$\sigma_{xy} = -\frac{M_t z}{\frac{1}{6}bt^3} \quad ; \quad \sigma_{xz} = 0$$

It was shown in general that the contribution to the moment of the stress σ_{xz} is the same as that of the stress σ_{xy}. However in this case this is not possible because σ_{xz} is zero. The share of the horizontal shear stresses σ_{xy} is correct:

$$-b\int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{xy}\, z\, dz = \frac{M_t b}{\frac{1}{6}bt^3}\int_{-\frac{1}{2}t}^{\frac{1}{2}t} z^2\, dz = \frac{1}{2}M_t$$

Nevertheless, in reality the missing part $M_t/2$ is delivered by vertical shear stresses σ_{xz}, which are present at the ends of the cross-section as shown in Fig. 6.31. At these ends the

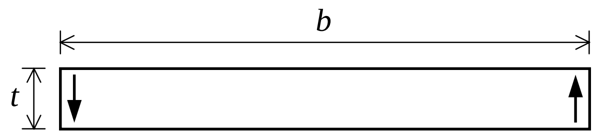


Fig. 6.31: Shear stresses at the ends contribute half of the torsional moment.

distribution of ϕ is not cylindrically in z but has to decrease to zero (see Fig. 6.28). These stresses are of the same order of magnitude as σ_{xy}, but because of the large distance between them (about b) they still produce half of the torsional moment M_t.

6.7 Numerical approach

The availability of fast computers makes it possible to generate numerical solutions. Suitable for this purpose is the Finite Element Method. A cross-section is divided into elements and in the nodes of the element mesh a value is determined for the displacement w of the membrane. It is a method of approximation, which produces more accurate results for finer meshes.

Strip-shaped cross-section
A very detailed discussion of this numerical method falls outside the scope of this course. Only the principle will be indicated and as an example the strip-shaped cross-section is taken, for which in section 6.6 already a solution has been determined. This solution will be labelled as the exact one. Since a cylindrically shaped displacement w is present, it is sufficient just to define an element distribution over the shortest edge t of the cross-section.

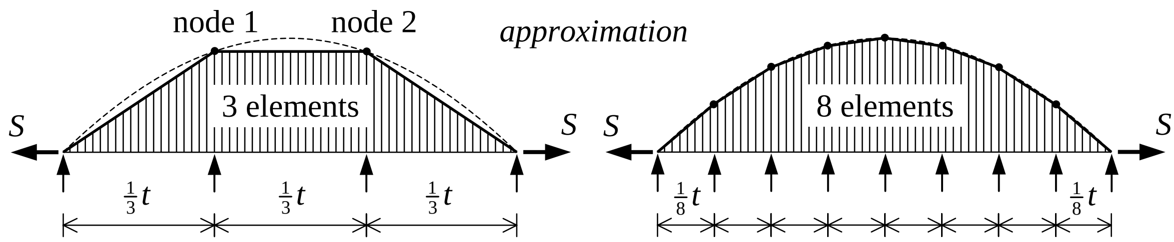
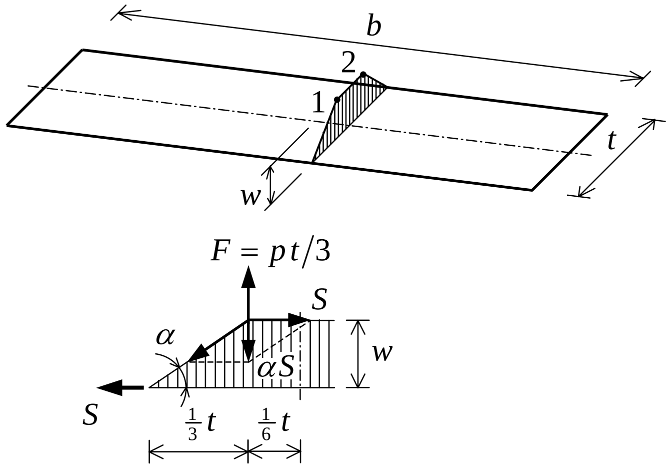


Fig. 6.32: Approximation of the shape of the membrane by three and eight elements.

The real distribution of the displacement $w(z)$ that is drawn by the dashed line in Fig. 6.32, is approximated by linear interpolation between two adjacent nodes. The figure shows the cases with three and eight elements. Now, the shape of the membrane is polygonal. With eight elements the approximation seems already to be quite good, but with three elements not yet. However for the coarse distribution with three elements the finite element method can be simulated by a calculation by hand. In the analysis, use will be made of the symmetry of the membrane surface. Fig. 6.33 shows the strip-shaped cross-section once more, including the section over the membrane. The uniformly distributed load p is concentrated as point loads F in the nodes at distances of $t/3$. The problem contains one degree of freedom w. This degree of freedom can be obtained from the equilibrium of node 1. After the determination of w, the torsional stiffness is calculated from the double volume beneath the membrane.
The calculation scheme of w is listed in Fig. 6.33. In node 1, the point load F has to be in equilibrium with the vertical component of the tension force S in the first element. Because w is small, $\tan\alpha$ can be replaced by the angle α itself and the displacement becomes:

$$w = \frac{t^2}{9S}\,p \tag{6.43}$$

The area of the section under the membrane over the full thickness t equals:



Concentrate : $F = \frac{1}{3} p t$ (1)

small w : $\alpha = \frac{w}{t/3}$ (2)

equilibrium of node 1 : $\alpha S = F$ (3)

substitution of (1) and (2) into (3):

$$\frac{3wS}{t} = \frac{1}{3} p t \quad \rightarrow \quad w = \frac{t^3}{9S} p$$

Fig. 6.33: Calculation of the displacement w for three elements.

$$Area = \frac{2}{3} w t$$

and the double volume:

$$2 \times Vol = 2 \times Area \times b = \frac{4}{3} w b t$$

Substitution of w from (6.43) yields:

$$2 \times Vol = \frac{4}{27} \frac{p}{S} b t^3$$

The ϕ-bubble is introduced by choosing:

$$p = 2\theta \quad ; \quad S = \frac{1}{G}$$

Then the double volume is equal to the torsional moment, and for $\theta = 1$ the result is the torsional stiffness GI_t:

$$GI_t = \frac{4}{27} \frac{2}{1/G} b t^3 = \frac{8}{27} G b t^3$$

The exact solution is:

$$GI_t = \frac{1}{3} G b t^3$$

The difference is in the order of 10 percent; for such a coarse element mesh this is quite a good result. The approximated solution appears to be exactly the inscribed polygon of the parabola.

In the computation with eight elements four different unknown displacements w occur. Then four equilibrium equations have to be set up and solved simultaneously. In that case, the error in GI_t already will be less than 1 percent.

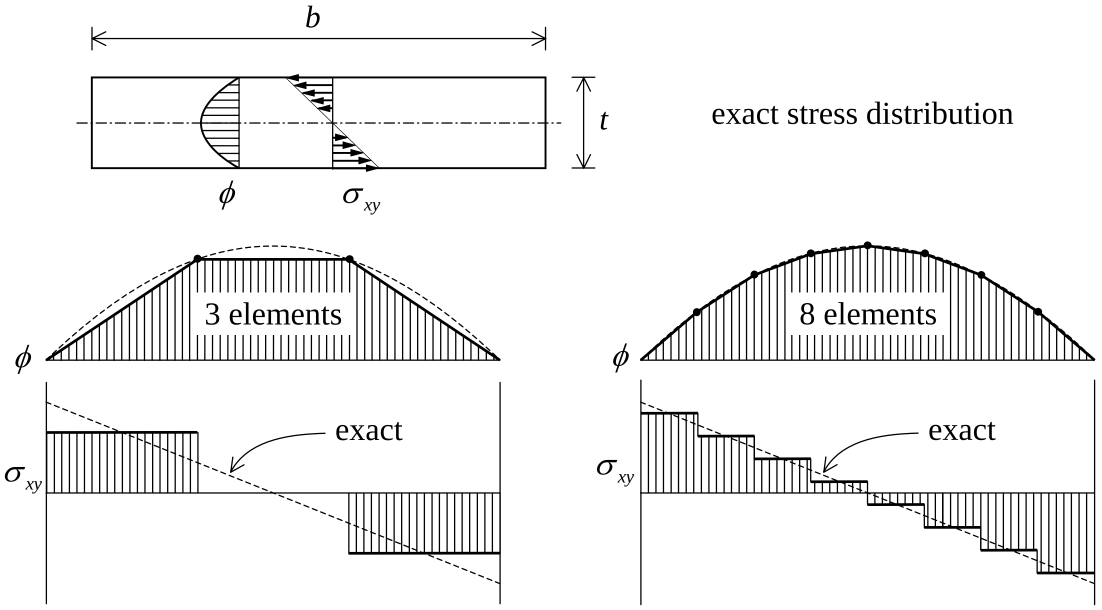


Fig. 6.34: Stress distribution obtained by finite element method.

The accuracy of the stress distribution is investigated as well. The exact solution is displayed in Fig. 6.34. Since ϕ is a parabolic function, its derivative the stress σ_{xy} will be linear over the thickness t of the cross-section. For a polygon description of ϕ with straight branches, the derivative will be constant per branch and will be discontinuous in the nodes. Fig. 6.34 shows the result that can be expected with the discussed finite element example. With three elements the largest error is not less than 33 percent, but for eight elements the error already reduces to 12 percent. In the middle of the elements always the correct value is found.

Arbitrarily shaped cross-sections

For arbitrarily shaped cross-sections a two-dimensional element mesh is applied. This can be done with triangular, rectangular and quadrilaterals of arbitrary shape (see Fig. 6.35). In the finite element approximation the load is again concentrated in the nodes. An unknown w (and therefore ϕ) is introduced in each node. Between the nodes, i.e. along the element edges the variation of w (and ϕ) is linear. Generally, the number of equations that can be formulated is equal to the number of nodes, thus equal to the number of unknown w's (and ϕ's). From this set the unknowns are solved.

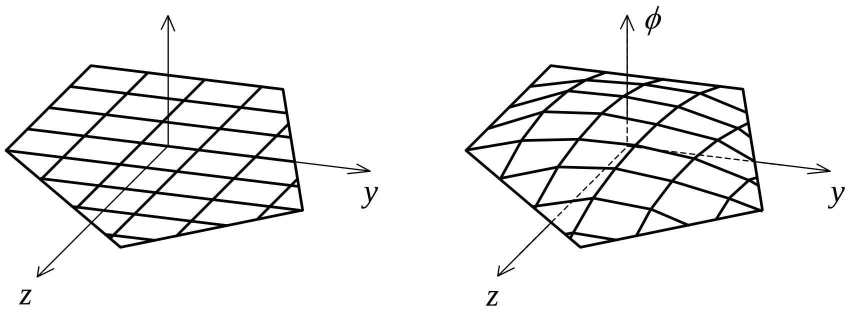
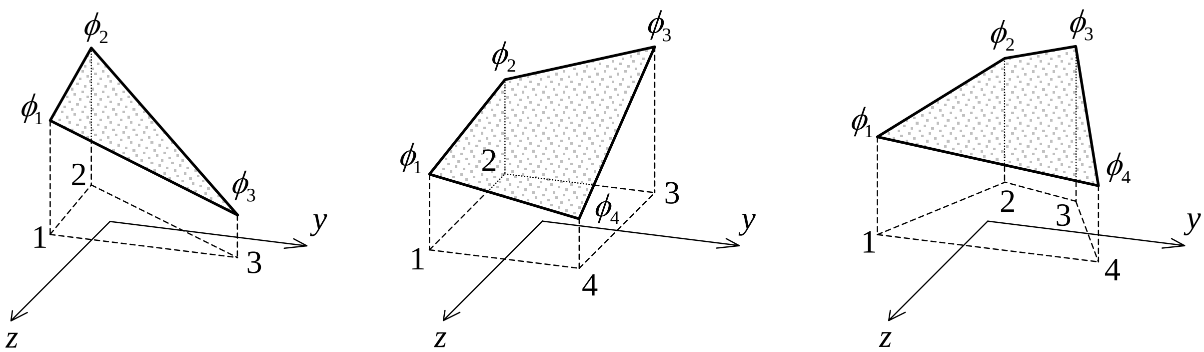


Fig. 6.35: Element mesh and ϕ-bubble for a cross-section of arbitrary shape.

Now, the ϕ-bubble is a collection of flat surfaces (above the triangular elements) and a collection of ruled surfaces (Dutch: “regelvlak”) (above the rectangular and quadrilateral elements). The formula read:

$$\phi(y,z) = a_1 + a_2 y + a_3 z \qquad (\textit{triangle})$$
$$\phi(y,z) = a_1 + a_2 y + a_3 z + a_4 yz \quad (\textit{rectangle})$$

The double content of the ϕ-bubble for $\theta = 1$ again provides the stiffness GI_t. The value of the stiffness is already quite good for relatively small numbers of elements. The stresses follow the slope of the ϕ-bubble. In a triangle both slopes $\phi_{,y}$ and $\phi_{,z}$ are constant over the entire element. The single value per element calculated for σ_{xy} and σ_{xz} is considered to be present in the centre of gravity of the element. In a rectangular element, each of the two slopes of ϕ is constant in one direction and linear in the other direction. This delivers two values for each of the stresses σ_{xy} and σ_{xz}. In an arbitrary quadrilateral the slopes vary in both directions and four values for σ_{xy} and σ_{xz} can be computed (see Fig. 6.36).



$\dfrac{\partial\phi}{\partial y}$ is constant	$\dfrac{\partial\phi}{\partial y}$ only varies in z-direction	$\dfrac{\partial\phi}{\partial y}$ varies in two directions
$\dfrac{\partial\phi}{\partial z}$ is constant	$\dfrac{\partial\phi}{\partial z}$ only varies in y-direction	$\dfrac{\partial\phi}{\partial z}$ varies in two directions
$\Downarrow$	$\Downarrow$	$\Downarrow$
one value for σ_{xy}	two values for σ_{xy}	four values for σ_{xy}
one value for σ_{xz}	two values for σ_{xz}	four values for σ_{xz}

Fig. 6.36: Stress distributions in the elements.

Increasing refinement of the mesh leads to better approximations approaching the exact solution. In Fig 6.37 this is demonstrated for a rectangular cross-section with a height-width ration of 2. When the number of elements $2N \times N$ increases, the ratio of the approximated and exact torsional stiffness GI_t approaches unity. This also holds for the maximum shear stress σ, provided it is evaluated halfway an elemental edge. When after a number of numerical tests it has become clear how fine the mesh should be for a particular accuracy, the calculation can be repeated for different height-width ratios. Then a table can be created as shown in Fig. 6.38. When $b \gg t$, the cross-section degenerates into a strip and for both GI_t and the maximum shear stress σ the coefficient $1/3$ is calculated, previously found in (6.40) and (6.42).

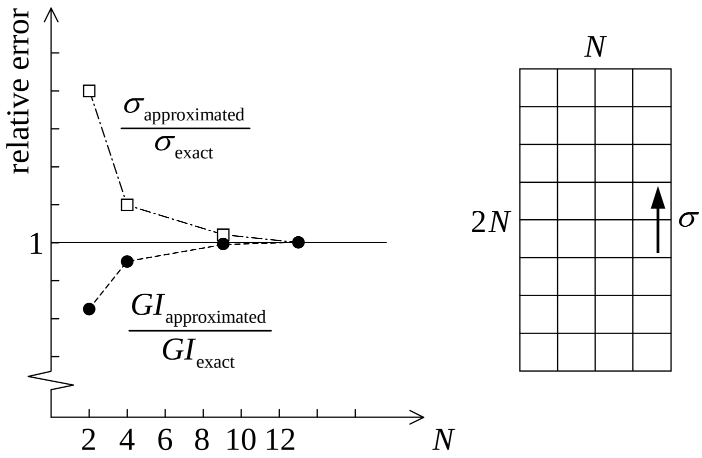
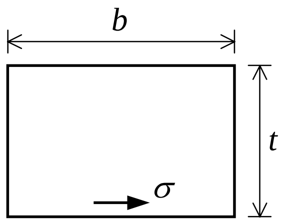


Fig. 6.37: Mesh refinement leads to convergence.



$\frac{b}{t}$	$\frac{GI_t}{bt^3G}$	$\frac{M_t}{\sigma b t^2}$
1.0	0.141	0.208
2.0	0.229	0.246
3.0	0.263	0.267
.	.	.
.	.	.
∞	0.333	0.333

Fig. 6.38: Stiffness and maximum stress for a rectangular cross-section.

6.8 Cross-section with holes

When the cross-section of the bar contains one or more cavities, the discussed theory requires some addition. First the case will be discussed with a single hole in the cross-section (see Fig. 6.39). Along the edge of the hole a n-s coordinate system is attached. The positive direction of n points into the hole. On the edge of the hole, no shear stress σ_{xn} different from zero can be present. Therefore it holds that:

$$\phi_{,s} = 0 \tag{6.44}$$

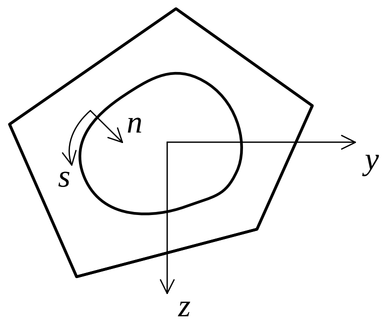


Fig. 6.39: Cross-section with one hole.

This means that ϕ is constant along the circumference of the hole. However in this case the value of ϕ cannot be set to zero, because this already has been done at the outer circumference. The unknown value is indicated by ϕ_h and is an undetermined degree of freedom of the problem.
The question arises which boundary condition for $\phi(y,z)$ has to be prescribed at the edge of the hole. The harmonic equation for ϕ is a second-order differential equation of the elliptic type, in which case generally only one boundary condition can be formulated on the edge. When this is the value of ϕ itself, one speaks about a Dirichlet problem as previously discussed. When the derivative $\phi_{,n}$ is prescribed the problem is of the Neumann type. Since at the outer circumference the value of ϕ was set to zero, at the edge of the hole ϕ is free and undetermined. Therefore, at that position the value of $\phi_{,n}$ has to be prescribed. Thus, the hole creates an extra unknown ϕ_h, which means that only one extra condition $\phi_{,n}$ has to be formulated along the edge of the hole, although $\phi_{,n}$ may vary itself along the circumference of the hole. It now will be investigated which condition this is. This will be done in two steps. First a special case is considered, for which the condition can be identified easily. After that it will be shown that this condition is generally valid.

Special case
Again the ϕ-bubble is considered occurring on a solid cross-section. In the left part of Fig. 6.40, the contour lines of the ϕ-bubble are indicated, i.e. the lines of constant ϕ. Along such a line the value of $\phi_{,s}$ is zero, which means that σ_{xn} is zero as well. This means that the part

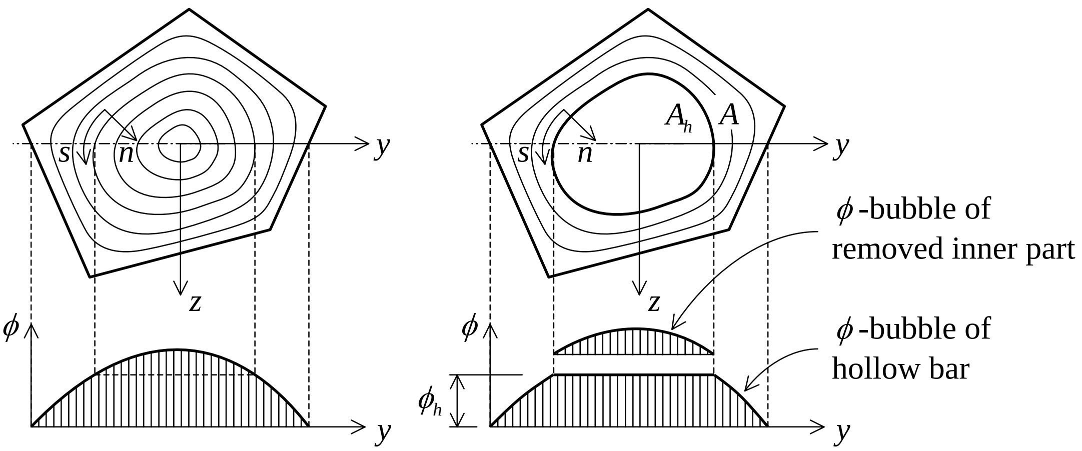


Fig. 6.40: Clarification on the ϕ-bubble of a cross-section with a hole.

of the bar inside the contour does not exert any force on the part outside the contour. Therefore, the inner part (with area A_h) can be removed without affecting the stress distribution outside the contour (with area A). This situation is depicted in the right part of Fig. 6.40. It also has been indicated how this affects the ϕ-bubble. For the solid cross-section a cut is made through the ϕ-bubble at the line $z = 0$. The contour line, inside which the hole will be created, intersects the curve twice with the same value for ϕ, namely ϕ_h.
In this case, the torsional stiffness GI_t for the hollow cross-section is equal to the difference of the torsional stiffness of the solid cross-section minus the torsional stiffness of the removed inner part. The cut-off cap of the ϕ-bubble represents the torsional stiffness of the removed part. It can be seen that for the ϕ-bubble of the hollow cross-section the hole continues to provide a contribution, but now with a constant value ϕ_h over the whole cavity. So, the

torsional stiffness is twice the volume of the truncated ϕ-bubble for $\theta = 1$, *including* the part above the hole. The formula reads:

$$\boxed{GI_t = 2\iint\limits_A \phi \, \mathrm{d}A + 2\phi_h A_h \quad \text{for} \quad \theta = 1} \tag{6.45}$$

For further interpretation it make sense to investigate what impact above explanation has on the membrane analogy (see Fig. 6.41). This analogy still can be used if a small adaptation is included. Again the membrane is fixed at the outer circumference and spans the cross-section.

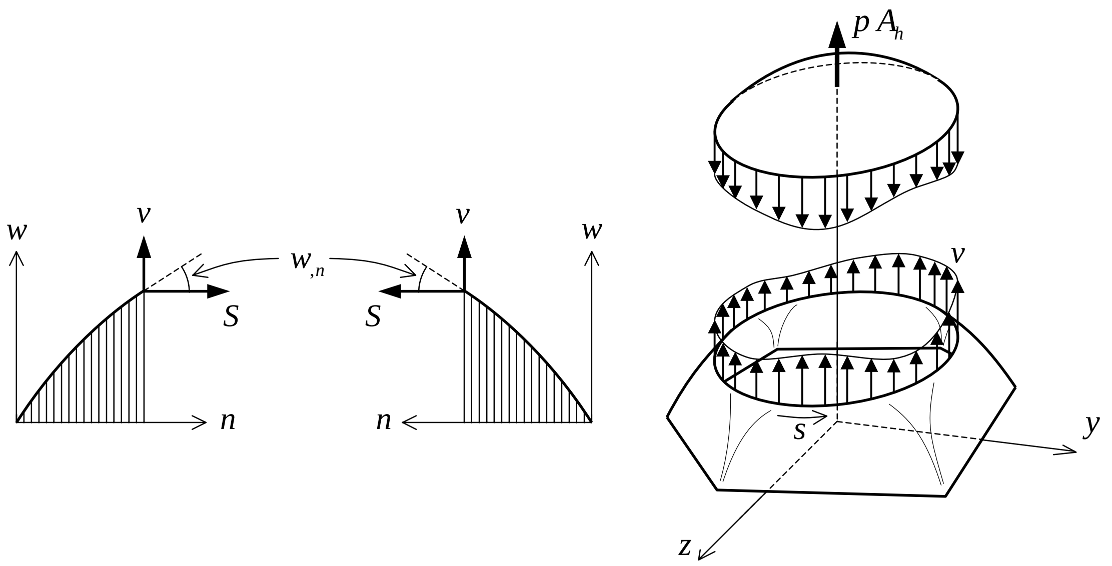


Fig. 6.41: The membrane analogy assists in finding the boundary condition for $\phi_{,n}$ along the hole.

On the edge of the hole the membrane is imaginarily fixed to a thin rigid weightless plate. This plate must be able to move freely. When a pressure p pressurises the membrane, it will load not only area A of the membrane with but also area A_h of the thin plate. Therefore, the weightless plate will be displaced parallel with respect to itself. The displacement of the membrane along the edge of the plate is the same and the slope with the plate is $w_{,n}$. The membrane analogy assists in finding the boundary condition for $\phi_{,n}$ along the circumference of the hole. For that purpose the equilibrium of the weightless plate is considered.
The weightless plate is subjected to a distributed load p over its surface and to the lateral membrane load v along the circumference. This lateral load has the value $S w_{,n}$ (n is positive if it points inside the hole). For the equilibrium of the weightless plate in w-direction it then can be written:

$$\oint v \, ds = pA_h \quad \rightarrow \quad \oint S \, w_{,n} \, ds = pA_h$$

When this result is reformulated in terms of the torsional problem ($p = 2\theta$, $S = 1/G$ and $w = \phi$), the required condition for $\phi_{,n}$ at the edge of the hole is found:

$$\oint \frac{1}{G}\phi_{,n}\, ds \quad = 2A_h\theta \tag{6.46}$$

General case

Now the idea is abandoned that a hole is created by removing material from a solid cross-section just inside a contour of the ϕ-bubble. In this case the hole is created arbitrarily and again a s-n coordinate system is attached along its circumference. Also the same boundary condition (6.44) holds and ϕ must have a constant value ϕ_h. In this case ϕ-contours are generated that generally do not correspond with those of the solid cross-section. This means that compared to the solid cross-section the stress distribution will be different. Therefore, it has to be proved separately that condition (6.46) holds for this case too. If so, the analogy of the weightless plate for the determination of the ϕ-bubble can be generalised. It also has to be shown that formula (6.45) for the determination of GI_t is generally valid.
Since $\phi_{,n}$ is equal to $-\sigma_{xs}$ and $\sigma_{xs} = G\gamma_{xs}$, condition (6.46) can be rewritten as:

$$\boxed{-\oint \gamma_{xs} ds \quad = 2A_h\theta} \tag{6.47}$$

In order to prove whether this contour integral is generally valid, it is investigated how γ_{xs} is expressed in the three-dimensional displacement field. This displacement field reads:

$$u_x(y,z) = \psi(y,z)\theta \quad ; \quad u_y(y,z) = -x\,z\,\theta \quad ; \quad u_z(y,z) = x\,y\,\theta \tag{6.48}$$

The shear angle γ_{xs} is defined by:

$$\gamma_{xs} = u_{x,s} + u_{s,x}$$

which means that the derivative in x-direction of the displacement $u_s(y,z)$ has to be determined. This displacement can be expressed in u_y and u_z as shown in Fig. 6.12:

$$u_s = -\sin\alpha\; u_y + \cos\alpha\; u_z$$

Then the required relation between γ_{xs} and the displacement field is found:

$$\gamma_{xs} = u_{x,s} - \sin\alpha\; u_{y,x} + \cos\alpha\; u_{z,x}$$

Substitution of (6.48) changes this result into:

$$\gamma_{xs} = \left\{\psi_{,s}(y,z) + z\sin\alpha + y\cos\alpha\right\}\theta$$

So, the contour integral under investigation becomes:

$$-\oint \gamma_{xs} ds \quad = \left\{-\oint \psi_{,s} ds - \oint z\sin\alpha\; ds - \oint y\cos\alpha\; dz\right\}\theta$$

Since $-\sin\alpha\, ds$ is just equal to dy and $\cos\alpha\, ds$ is just equal to dz this relation transforms into:

$$-\oint \gamma_{xs} ds = \left\{ -\oint \psi_{,s}\, ds + \oint z\, dy - \oint y\, dz \right\} \theta \qquad (6.49)$$

The three contour integrals in the right-hand side will be determined separately. The first one can be written as:

$$\oint \psi_{,s}\, ds = \oint d\psi = 0$$

The value has to be zero because of the uniqueness of the warping displacement ψ along the circumference. The second integral equals:

$$\oint z\, dy = A_h$$

This result can easily be understood by splitting the integral into two parts. In the left part of Fig. 6.42 the contour integral is split into a part from A to B through the region with positive z-values and in a part from B to A with negative z-values. For the first part dy is positive

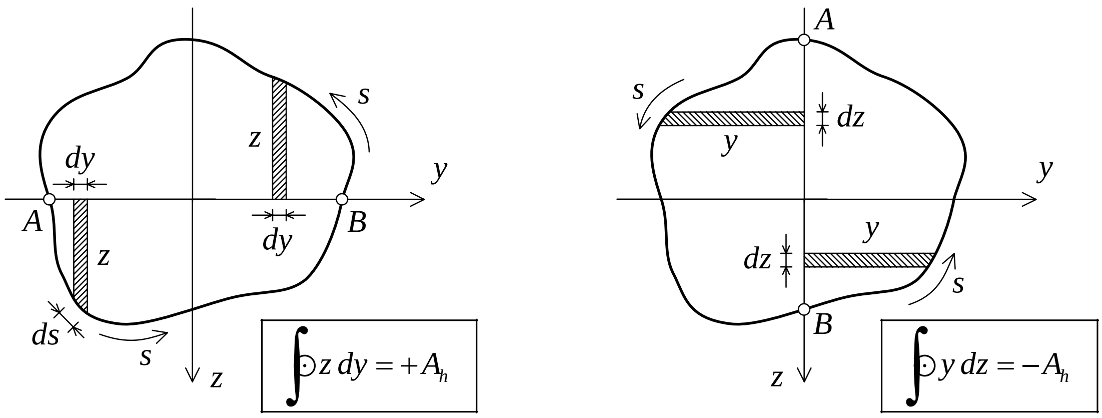


Fig. 6.42: Integrations of $\oint z\, dy$ and $\oint y\, dz$.

if s increases in positive direction and z is also positive in this area. Therefore, the integral over this part becomes:

$$\oint_{down} z\, dy = \textit{Area of the part of the hole for which } z \geq 0$$

In the second part of the contour integral dy is negative if s increases in positive direction, but z is negative as well, so that $z\, dy$ still is positive. Consequently, the integral over this part equals:

$$\oint_{up} z\,dy = Area\ of\ the\ part\ of\ the\ hole\ for\ which\ z < 0$$

The summation of both integrals just delivers the total area of the hole:

$$\oint z\,dy = \oint_{down} z\,dy + \oint_{up} z\,dy = A_h$$

Likewise, in the right part of Fig. 6.42 the area is split up into two parts for the third contour integral, a region where $y \geq 0$ and a region where $y < 0$. In a similar manner for the third contour integral it is found:

$$\oint y\,dz = \oint_{right} y\,dz + \oint_{left} y\,dz = -A_h$$

With these results for the three contour integrals, relation (6.49) transforms into:

$$-\oint \gamma_{xs}\,ds = 2A_h\,\theta$$

This is the same condition as found in (6.47) for the special case of a hole the edge of which coincides with a contour line of a solid cross-section. This means that it has been shown that this condition is valid too for an arbitrary position of the hole. So, the membrane analogy can also be applied for the determination of the ϕ-bubble. Only the plate will not automatically displace itself parallel to its original position and some sort of *guide* is required.

The only remaining aspect is to show that formula (6.45) retains its validity for the determination of the torsional stiffness GI_t from the ϕ-bubble. For the solid cross-section, the following relation was used:

$$GI_t = \iint_A \left(y\,\sigma_{xz} - z\,\sigma_{xy} \right) dA \qquad (\text{for } \theta = 1)$$

or by expressing the stresses in ϕ:

$$GI_t = \iint_A \left(-y\frac{\partial \phi}{\partial y} - z\,\frac{\partial \phi}{\partial z} \right) dA \qquad (\text{for } \theta = 1)$$

For the solid section, the integral over the area A was calculated in two parts. This will be done again, but now only that part of the area will be considered where material can be found (see Fig. 6.43).
The first integral can be worked out as follows:

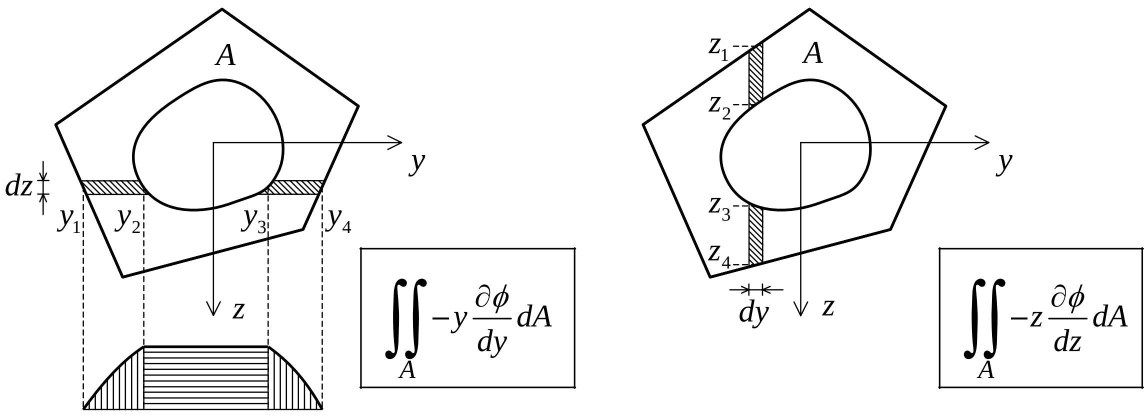


Fig. 6.43: Determination of the resulting moment for a cross-section with a hole.

$$\iint\limits_A -y\,\frac{\partial\phi}{\partial y}\,dA = \int\left(\int_{y_1}^{y_2} -y\;d\phi + \int_{y_3}^{y_4} -y\;d\phi\right)dz$$

Integration by parts with respect to y changes this relation into:

$$\int\left(\int_{y_1}^{y_2}\phi\;dy - \left(y_2\phi_2 - y_1\phi_1\right) + \int_{y_3}^{y_4}\phi\;dy - \left(y_4\phi_4 - y_3\phi_3\right)\right)dz$$

On the outer circumference, ϕ_1 and ϕ_4 are zero, while ϕ_2 and ϕ_3 have the same value ϕ_h at the circumference of the hole. This reduces the integral to:

$$\int\left(\int_{y_1}^{y_2}\phi\;dy - \phi_h\left(y_3 - y_2\right) + \int_{y_3}^{y_4}\phi\;dy\right)dz$$

The term between square brackets is just the area of the cross-section of the ϕ-bubble, including the part of the ϕ-bubble above the hole. Consequently it is found:

$$\iint\limits_A -y\,\frac{\partial\phi}{\partial y}\,dA = \iint\limits_A \phi\;dA + \phi_h A_h$$

Similarly it can be derived:

$$\iint\limits_A -z\,\frac{\partial\phi}{\partial z}\,dA = \iint\limits_A \phi\;dA + \phi_h A_h$$

For a cross-section with a hole it remains valid that the vertical and horizontal stresses contribute equally. Putting all results together it can be written:

$$GI_t = 2\iint\limits_{A} \phi \, dA + 2\,\phi_h A_h = 2 \iint\limits_{A+A_h} \phi \, dA \quad (\text{for } \theta = 1)$$

This completes the proof that for any arbitrary position of the hole the torsional stiffness is equal to twice the volume of the ϕ-bubble, for $\theta = 1$.

Cross-section with a number of holes

It simply can be indicated how the calculation should be performed for more than one hole in the cross-section. In that case, the amount of unknown ϕ_h's is the same as the number of holes, and all these ϕ_h's may have a different value.

For the membrane analogy this means that above each hole a weightless plate is present, and that for each plate an equilibrium equation has to be formulated. When the shape of the membrane has been determined in this manner and the conversion of the torsional problem is carried out, the torsional stiffness can be determined from:

$$\boxed{GI_t = 2\iint \phi \, dA + 2 \sum_{\text{all holes}} \phi_h A_h} \tag{6.50}$$

6.9 Thin-walled tubes with one cell

A special case of a cross-section with holes is a tube with a relatively small wall thickness. The circumference of the tube is C and it wall thickness t, such that $t \ll C$. Further it is assumed that the cross-sectional area inside the tube is equal to A_h.

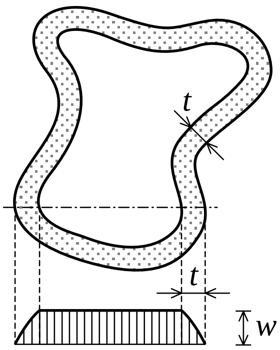


Fig. 6.44: Membrane for thin-walled tube.

According to the membrane analogy, the membrane and plate adjust themselves such that the plate elevates to a certain height w (see Fig. 6.44). Because t is very small, with a good approximation it can be assumed that the displacement of the membrane varies linearly from zero to w over the distance t. For the slope $w_{,n}$ it then holds:

$$w_{,n} = \frac{w}{t}$$

The equilibrium of the weightless plate is described by:

$$\oint S\, w_{,n}\, ds = p\, A_h$$

so that:

$$S\frac{w}{t}C = p\, A_h \quad \rightarrow \quad w = \frac{p}{S}\frac{t\,A_h}{C}$$

By substitution of $p = 2\theta$ and $S = 1/G$, the displacement w can be replaced by ϕ_h, i.e.:

$$\phi_h = 2G\frac{t\,A_h}{C}\theta$$

The torsional moment is twice the volume of the ϕ-bubble:

$$M_t = 2\iint_A \phi\, dA + 2\phi_h A_h$$

In this case, the area A of the material is equal to $t\,C$. For thin-walled tubes this area can be neglected with respect to the area of the hole A_h. The moment then becomes:

$$M_t = 2\phi_h A_h \quad \rightarrow \quad M_t = G\frac{4t\,A_h^2}{C}\theta$$

Therefore, the torsional moment of inertia becomes:

$$I_t = \frac{4t\,A_h^2}{C}$$

This formula is valid for a constant wall thickness t. When t varies along the circumference, the more general so-called *2nd formula of Bredt* holds:

$$\boxed{I_t = \frac{4A_h^2}{\oint \frac{ds}{t}}} \qquad (6.51)$$

This formula follows from the equilibrium of the weightless plate. When t is constant, application of the formula leads to the above-derived relation for I_t. The shear stresses σ_{xs} are approximately constant across the thickness. Apart from the sign, it holds:

$$\sigma_{xs} = \frac{\phi_h}{t} = 2G\frac{A_h}{C}\theta$$

or expressed in the torsional moment:

$$\boxed{\sigma_{xs} = \frac{M_t}{2\,t A_h}} \tag{6.52}$$

This relation is called the *1st formula of Bredt.*

Remarks

1. The assumption that the membrane varies linearly over the wall thickness is an approximation. In reality a weak parabolic variation has to be added to the linear profile as shown in Fig. 6.45. This parabolic contribution represents the torsional moment of inertia

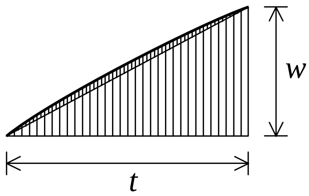


Fig. 6.45: Weakly curved membrane.

$\frac{1}{3}Ct^3$ of the wall itself, considered as a strip. However, compared to the torsional moment of the entire closed tube as a whole, the contribution of the wall can be neglected.

2. It is instructive to compare the results of a closed tube and an open tube (see Fig. 6.46).

closed tube | open tube

$$I_t = 2\pi R^3 t \qquad I_t = \frac{2\pi}{3} R t^3$$

$$\sigma_{\max} = \frac{M_t}{2\pi R^2 t} \qquad \sigma_{\max} = \frac{1}{3}\frac{M_t}{2\pi R t^2}$$

Fig. 6.46: Torsional stiffness of closed and open tubes.

The ratio of the stiffnesses is:

$$\frac{I_{closed\ tube}}{I_{open\ tube}} = 3\left(\frac{R}{t}\right)^2$$

The ratio of the stresses for the same moment equals:

$$\frac{\sigma_{closed\ tube}}{\sigma_{open\ tube}} = 3\left(\frac{t}{R}\right)$$

It can be seen that the stresses for the transmission of the same moment in the closed tube are an order t/R smaller than in the open tube, while the stiffness of the closed tube is much larger. This is caused by the fact that the “round-going” shear stresses in the closed tube have a large arm ($2R$), while this arm in the open tube is equal or smaller than t. This means that for the transmission of the same moment, in the last case the shear stresses are much larger.

6.10 Thin-walled tubes with multiple cells

In the building practice, it may be necessary to calculate the torsional stiffness of box-girders with more cells. An example with two cells is depicted in Fig. 6.47. The traffic arrangement on the upper deck of a bridge may be the cause that the vertical partitioning wall is applied

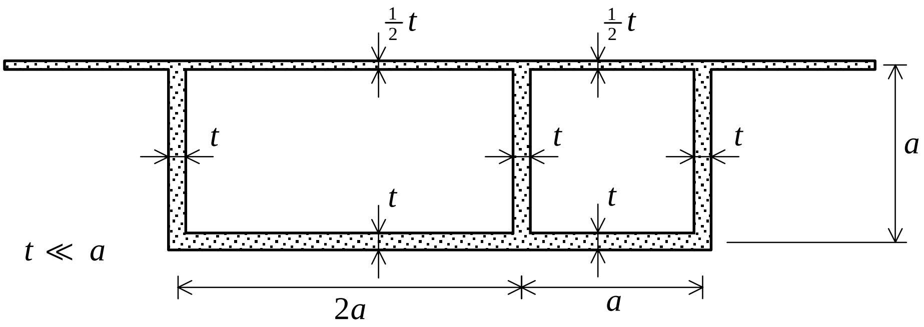


Fig. 6.47: Box-girder with two cells.

eccentrically. In this example, the thickness ($t/2$) of the upper deck is half the one (t) of the webs and the lower plate. Since $t \ll a$, the centre-to-centre distance (a and $2a$) of the box-girder walls can be used. This means that the contribution of the walls itself can be neglected. For the same reason, the flanges of the box-girder can be ignored as well.
Fig. 6.48 shows a cross-section of the membrane and the two weightless plates appearing in the membrane analogy. The left and right plate displace w_1 and w_2, respectively. In the drawing w_2 is chosen larger than w_1, which is consequently applied in the calculations as

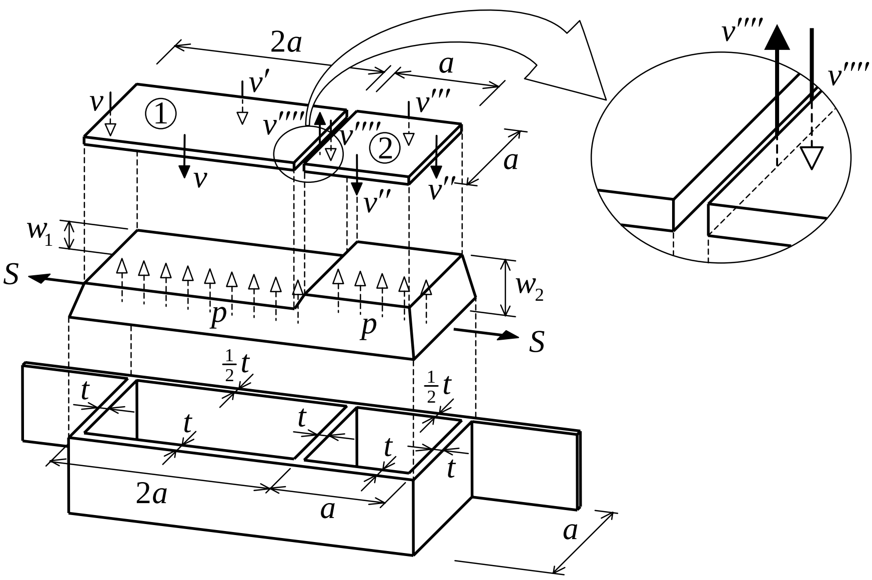


Fig. 6.48: Membrane and equilibrium of the plates.

well. The answers will reveal whether this assumption was correct. For the equilibrium of the two plates membrane shear forces of different magnitude play a role. When w_2 is larger than w_1, the shear forces as drawn in Fig. 6.48 will have a positive value. They are:

$$v = \frac{w_1}{t} S \quad ; \quad v' = \frac{w_1}{\frac{1}{2}t} S \quad ; \quad v'' = \frac{w_2}{t} S \quad ; \quad v''' = \frac{w_2}{\frac{1}{2}t} S \quad ; \quad v'''' = \frac{w_2 - w_1}{t} S$$

The first four shear forces v, v', v'' and v''' are applied in downward direction on the plates. The last force v'''' acts upward on plate 1 and downward on plate 2 (for the case of w_1 being larger than w_2 this is the other way round). The vertical equilibrium of the plates is:

$$\begin{aligned} &v * a \ + v * 2a \ + v' * 2a \ - v'''' * a \ = \ p * 2a * a \qquad (plate\ 1) \\ &v'' * a \ + v'' * a \ + v''' * a \ + v'''' * a \ = \ p * a * a \qquad (plate\ 2) \end{aligned}$$

Substitution of the forces provides:

$$\begin{aligned} &\frac{w_1}{t} S * a \ + \frac{w_1}{t} S * 2a \ + \frac{w_1}{\frac{1}{2}t} S * 2a \ - \frac{w_2 - w_1}{t} S * a \ = \ 2 p a^2 \qquad (plate\ 1) \\ &\frac{w_2}{t} S * a \ + \frac{w_2}{t} S * a \ + \frac{w_2}{\frac{1}{2}t} S * a \ + \frac{w_2 - w_1}{t} S * a \ = \ p a^2 \qquad (plate\ 2) \end{aligned}$$

Division by a changes this into:

$$\begin{aligned} &8 \frac{S}{t} w_1 \ - \ \frac{S}{t} w_2 \ = \ 2 p a \qquad (plate\ 1) \\ &-\frac{S}{t} w_1 \ + \ 5 \frac{S}{t} w_2 \ = \ p a \qquad (plate\ 2) \end{aligned}$$

Solution of the two equations provides:

$$w_1 = \frac{11}{39} \frac{p}{S} a t \quad ; \quad w_2 = \frac{10}{39} \frac{p}{S} a t$$

The displacement w_2 is smaller than w_1. Therefore, the shear force v'''' is pointing in the opposite direction than it is drawn.

Now the transition is made to the ϕ-bubble by the introduction of $p = 2\theta$ and $S = 1/G$. This delivers:

$$\phi_1 = \frac{22}{39} G a t \theta \quad ; \quad \phi_2 = \frac{20}{39} G a t \theta$$

The torsional moment is twice the volume of the ϕ-bubble:

$$M_t = 2\phi_1 * 2a * a + 2\phi_2 * a * a \quad \rightarrow \quad M_t = \frac{128}{39} G a^3 t \theta$$

Obviously the torsional moment of inertia equals:

$$I_t = \frac{128}{39} a^3 t$$

Expressing ϕ_1 and ϕ_2 in the moment provides:

$$\phi_1 = \frac{11}{64}\frac{M_t}{a^2} \quad ; \quad \phi_2 = \frac{10}{64}\frac{M_t}{a^2}$$

For the stresses it then can be derived (see Fig. 6.49):

$$\sigma = \frac{\phi_1}{t} = \frac{11}{64}\frac{M_t}{ta^2} \quad ; \quad \sigma' = \frac{\phi_1}{\frac{1}{2}t} = \frac{22}{64}\frac{M_t}{ta^2} \quad ; \quad \sigma'' = \frac{\phi_2}{t} = \frac{10}{64}\frac{M_t}{ta^2}$$

$$\sigma''' = \frac{\phi_2}{\frac{1}{2}t} = \frac{20}{64}\frac{M_t}{ta^2} \quad ; \quad \sigma'''' = \frac{\phi_1 - \phi_2}{t} = \frac{1}{64}\frac{M_t}{ta^2}$$

Fig. 6.49 indicates the proper directions of the shear stresses. The first one can be chosen, and considering the slope of the membrane the other ones can be indicated. The partitioning has the same slope as the right web, which means that σ'''' points in the same direction as σ''.

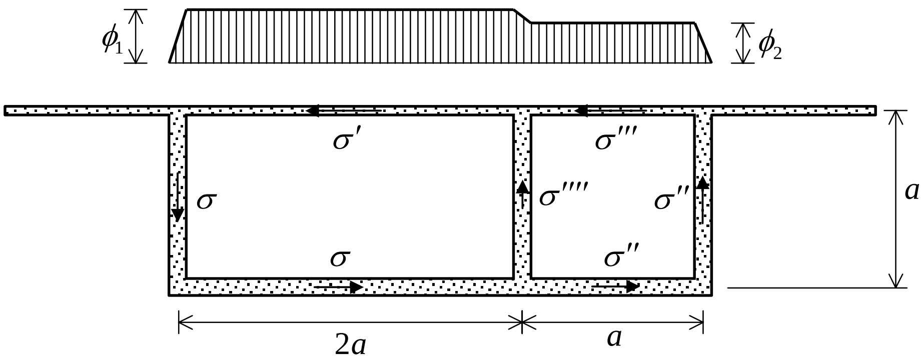


Fig. 6.49: ϕ-distribution and shear stresses in a section with two cells.

Exercises

1. Confirm that the resulting horizontal and vertical shear forces are zero in the discussed box-girder with two cells.
2. Check if the calculated vertical stresses deliver a torsional moment of $M_t/2$. Repeat the same exercise for the horizontal stresses.
3. Calculate with the formula of Bredt the torsional moment of inertia I_t and the stresses σ when the partitioning is left out of the structure. Compute the ratio of the I_t's and the maximum σ's for the situation with and without partitioning. What can be concluded?

6.11 Cross-section built up out of different materials

A cross-section composed out of two different materials A and B is considered as shown in Fig. 6.50. For these materials the respective shear moduli G_A and G_B are applicable. A n-s coordinate system is attached to the joining line, of which the s-coordinate is following the line and the n-coordinate is perpendicular to it. Now the boundary conditions along the

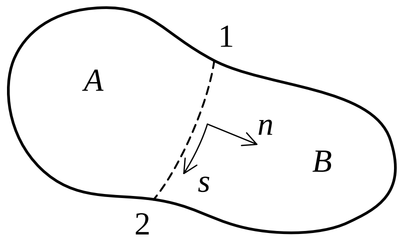
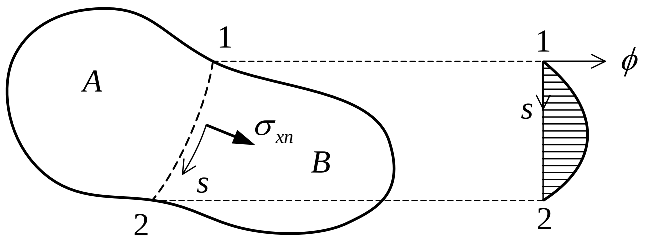


Fig. 6.50: Cross-section with two different materials A and B.

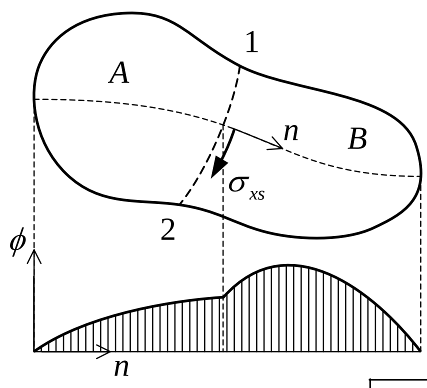
joining line are investigated. In Fig. 6.51 the n-direction is considered. The stress σ_{xn} has to be continuous across the joining line, because the stress σ_{xn} is transmitted from one material to the other. For the ϕ-bubble this means that the derivative $\partial\phi/\partial s$ has to be continuous. In both materials A and B, ϕ starts with zero value in edge point 1. Therefore, ϕ has to be continuous along the entire joining line.



$$\boxed{\sigma_{xn_A} = \sigma_{xn_B}} \quad \Rightarrow \quad \frac{\partial\phi}{\partial s} = \text{CONTINUOUS} \quad \Rightarrow \quad \boxed{\phi = \text{CONTINUOUS}}$$

Fig. 6.51: Stress condition perpendicular to a connection line of two materials.

In Fig. 6.52 the s-direction is considered. In that direction the deformation condition holds that the shear strain γ_{xs} on the joining line is the same for both materials. Consequently, the stress σ_{xs} is discontinuous across the joining line (because the shear moduli G_A and G_B are different), this means that the slope $\phi_{,n}$ of the ϕ-bubble is discontinuous too. Therefore the ϕ-bubble contains a kink.
In the membrane analogy, the same procedure as described before can be followed. Only the tensile force for both materials is different. So, distinction has to be made between S_A and



$$\boxed{\gamma_{xs_A} = \gamma_{xs_B}} \quad \Rightarrow \quad \sigma_{xs} = \text{DISCONTINUOUS} \quad \Rightarrow \quad \boxed{\frac{\partial\phi}{\partial n} = \text{DISCONTINUOUS}}$$

Fig. 6.52: Strain condition in the direction of a connection line of two materials.

S_B. This implicates again that a kink is created in the membrane. The shear force v_n must be continuous and because it holds that $v_n = S\, w_{,n}$ the slope $w_{,n}$ will be discontinuous for $S_A \neq S_B$.

Cross-section with hole as a special case

A cross-section with a hole can be regarded as a special case of a cross-section composed out of two materials. The hole is considered to be a special material with G equal to zero. Then

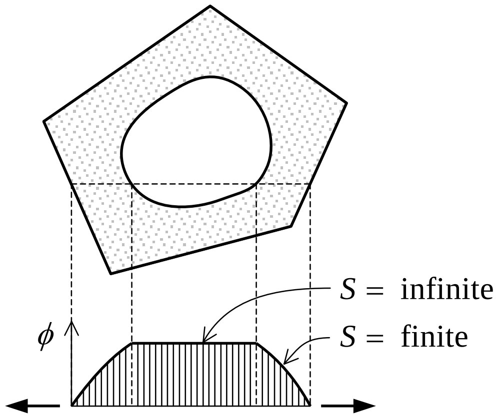


Fig. 6.53: Cross-section with a hole considered as a special case of two materials.

in the membrane analogy the tensile force S above this hole is infinitely large. Therefore, this part of the membrane remains flat (see Fig. 6.53). Making further use of the knowledge that:

$$\sigma_{xn_A} = \sigma_{xn_B} = 0 \quad \rightarrow \quad \phi_{,s} = 0 \quad \rightarrow \quad \phi_h = \text{constant}$$

Additionally it can be concluded that this flat part of the membrane remains horizontal. So, exactly the same goal is achieved as with the concept of a weightless plate!

6.12 Torsion with prevented warping

Up to now it has been assumed continuously that the ends of the bar were loaded in such a manner that no axial normal stresses σ_{xx} could be generated. An eventual warping of the cross-section could take place without any hindrance. However, if the warping at the end is prevented, for example by bonding the end to an undeformable body, a kinematic boundary condition is prescribed. Then it is not possible to prescribe zero stress values. So, generally normal stresses σ_{xx} will be generated. The influence of the prevented warping can be considerable. Especially the stiffness can be increased strongly. This can be demonstrated for example by the torsion of an I-section. When an I-section is subjected to torsion, the cross-

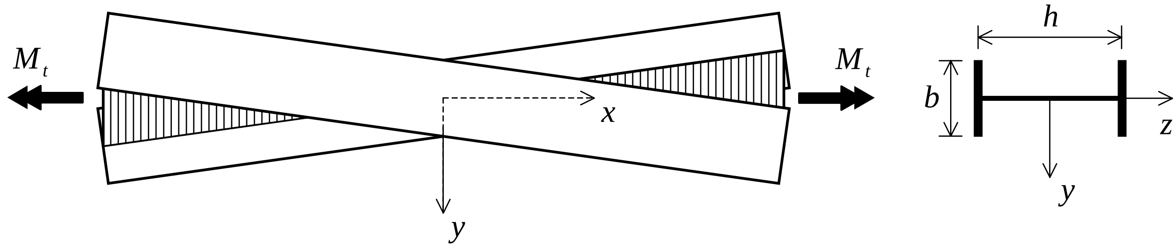


Fig. 6.54: Torsion of an I-section.

section warps. In each cross-section, the upper and lower flanges have opposite rotation directions as shown in Fig. 6.54. The right part of the section is drawn again in Fig. 6.55, but rotated over an angle of 90^0. At the left sketch the warping is free at the right sketch it is prevented. The prevention of the warping can be achieved by subjecting the top flange to a moment about the z-axis, while at the same time the bottom flange experiences a moment of the same magnitude but opposite direction. In both flanges this moment disappears gradually as the distance from the fixed end increases. These moments go together with shear forces V in y-direction in the flanges.

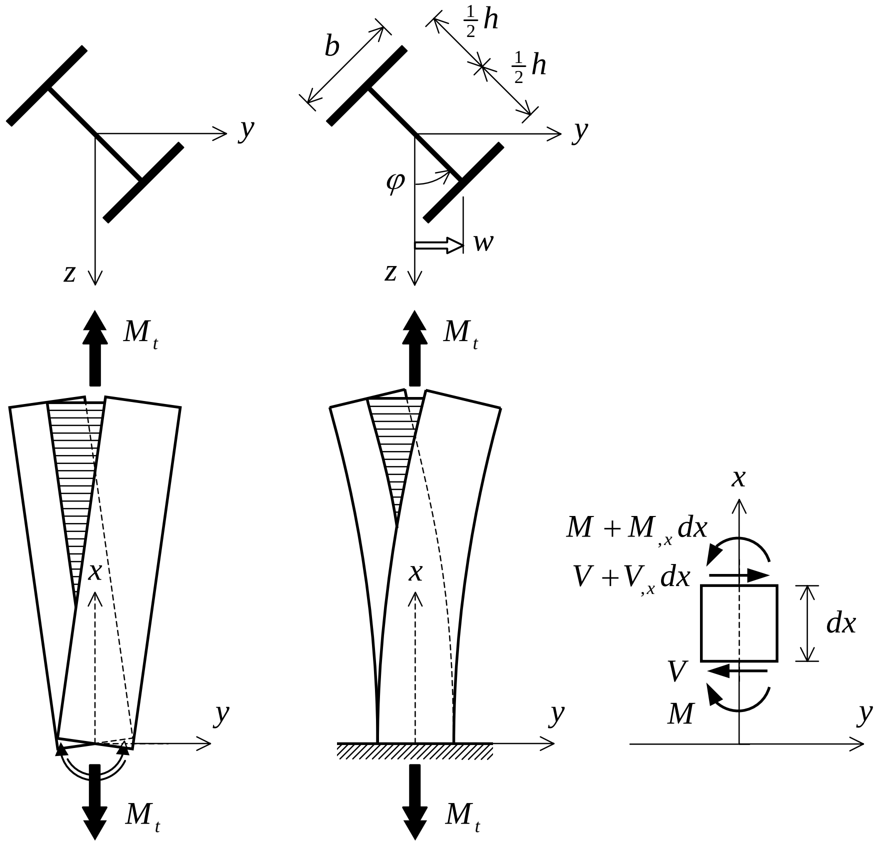
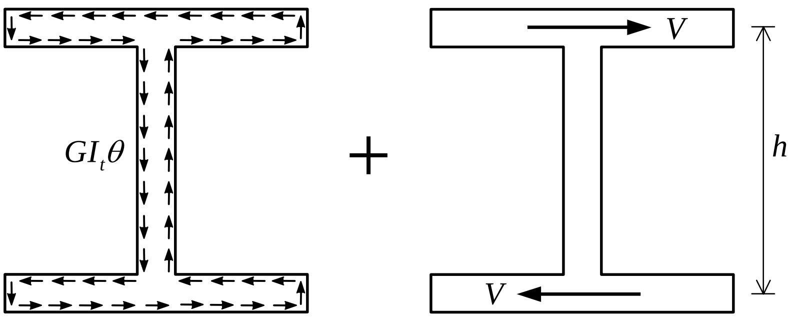


Fig. 6.55: I-section the warping of which is prevented.

The total torsional moment M_t is not only taken up by a "round-going" shear stream according to De Saint-Venant, but can partly be attributed to these shear forces V too (see Fig. 6.56). At the clamped end ($x = 0$), the warping is completely prevented. At that position, θ is zero and the moment is completely determined by Vh. For sufficiently large x it can be expected that V damps out to zero value and that θ has developed completely, so that at that position the torsional moment is resisted just as in the case of free warping ($G I_t \theta$).

The expected picture will now be worked out quantitatively. The rotation of the cross-section is equal to φ and the displacement of the top flange in y-direction is called w. The moment M and shear force V in this flange as drawn in Fig. 6.55 are considered positive. The following relations are valid:



$$\boxed{M_t = GI_t\theta + V\,h}$$

Fig. 6.56: Shear forces V due to prevented warping.

$$\left.\begin{aligned} V &= \frac{dM}{dx} \\ M &= -EI_f\,\frac{d^2w}{dx^2} \end{aligned}\right\} \rightarrow \quad V = -EI_f\,\frac{d^3w}{dx^3}$$

$$\left.\begin{aligned} w &= \frac{h}{2}\varphi \\ \theta &= \frac{d\varphi}{dx} \end{aligned}\right\} \rightarrow \quad \frac{d^3w}{dx^3} = \frac{h}{2}\frac{d^2\theta}{dx^2}$$

$$\rightarrow \quad V = -\frac{h}{2}EI_f\,\frac{d^2\theta}{dx^2}$$

where EI_f is the bending stiffness of a flange for bending in the plane of the flange. For the torsional moment it then follows:

$$M_t = GI_t\,\theta + V\,h \quad \rightarrow \quad M_t = GI_t\,\theta - \frac{1}{2}h^2 EI_f\,\frac{d^2\theta}{dx^2}$$

This is a differential equation in the unknown θ. After division by GI_t and the introduction of the characteristic length λ and the specific torsion angle θ_{sv} according to De Saint-Venant given by:

$$\lambda^2 = \frac{h^2 EI_f}{2GI_t} \quad ; \quad \theta_{sv} = \frac{M_t}{GI_t}$$

the differential equation becomes:

$$\theta - \lambda^2\,\frac{d^2\theta}{dx^2} = \theta_{sv}$$

The solution consists out of a particular and a homogeneous part:

$$\theta(x) = \theta_{sv} \qquad (\textit{particular part})$$
$$\theta(x) = C_1\,e^{x/\lambda} + C_2\,e^{-x/\lambda} \qquad (\textit{homogeneous part})$$

The total solution is:

$$\theta(x) = \theta_{sv} + C_1\, e^{x/\lambda} + C_2\, e^{-x/\lambda}$$

The coefficient C_1 has to be zero, because the specific torsional angle $\theta(x)$ has to approach the value θ_{sv} for $x \to \infty$. Then the influence of the bonded end should not be felt anymore. The constant C_2 can be found from the condition that θ is zero for $x = 0$, i.e.:

$$x \to \infty:\ C_1 = 0 \quad ; \quad x = 0:\ C_2 = -\theta_{sv}$$

Therefore, the solution becomes:

$$\theta(x) = \left(1 - e^{-x/\lambda}\right)\theta_{sv}$$

On basis of this solution all desired stresses can be calculated. In Fig. 6.57 it can be seen that the influence of the fixed end is practically damped out at a distance of 2λ to 3λ.

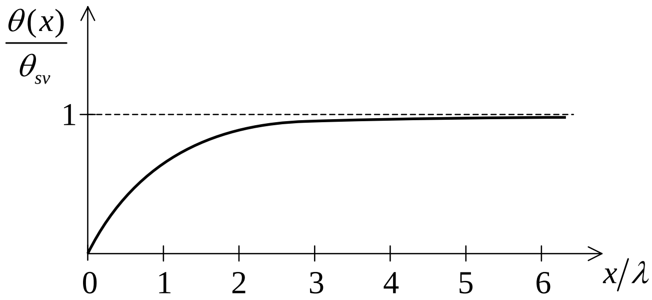


Fig. 6.57: Influence of the clamped end.

Example

For the sake of simplicity, Poisson's ratio υ is set to zero, so that $G = E/2$.
For an I-section with the same thickness for the web and the flanges it holds:

$$I_f = \frac{1}{12} t\, b^3 \quad ; \quad I_t = \frac{1}{3}\left(h + 2b\right)t^3$$

where b is the width of the flange. It then follows:

$$\frac{\lambda}{h} = \frac{b}{2t}\sqrt{\frac{b}{h+2b}}$$

For more or less equal b and h, this relation becomes:

$$\frac{\lambda}{h} \approx \frac{1}{2\sqrt{3}}\frac{b}{t}$$

When b is much larger than t, then λ is much larger than h. The influence of the disturbance by the bonded end is noticeable up to a distance (about 3λ), which is much larger than the size (h) of the disturbed cross-section. In this case, the principle of de Saint-Venant appears not to be applicable. This means that the principle has no general validity.

A practical example of disturbed warping occurs in the case of viaducts built up out of pre-stressed beams, on which a high-lying deck is cast (see Fig. 6.58). The disturbance occurs at the end cross-members and eventual intermediate cross-members. Both b and h are of the

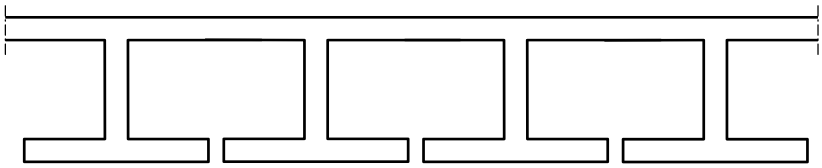


Fig. 6.58: Pre-stressed beam for viaduct.

order of magnitude of 1 m, while t is in the order of 0.20 m. With $b/t = 5$ the characteristic length becomes:

$$\lambda = \frac{5}{2\sqrt{3}} \approx 1.5 \text{ m}$$

a damping length of about 3λ provides more than 4 m. Compared to a span of 30 to 40 m this is a small part of the entire span. This means that the torsional stiffness of such a viaduct obtained through De Saint-Venant is sufficiently accurate, in case only end cross-members are applied. When intermediate cross-members are applied as well, the actual stiffness will be larger.