Delft University of Technology
Faculty of Civil Engineering and Geosciences

Theory of Elasticity Ct 5141

Energy Principles and
Variational Methods

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Ct 5141

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J. Blaauwendraad
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Introduction to part 1

Motivation
In the first years of university education, the mathematical background of structural mechanics was mainly based on three basic sets of equations, namely equilibrium, compatibility and material behaviour. These sets of equations were solved by so-called *direct methods*. During the basic training, the most important new aspect in the solution procedures was the introduction of the concept of virtual work and the virtual work equation. Application of this *indirect method* happened only occasionally.

In part 1 of the course CT5141, the lecture notes of which are presented here, the *indirect methods* or *variational methods* will be put in a wider context. It will be shown that the virtual work equation is a facet of a comprehensive and consistent theory. In the classical structural mechanics, two energy principles have always played an important role. They are the principle of *minimum potential energy* and the principle of *minimum complementary energy*. They are related to the displacement method and the force method, respectively. In a somewhat different formulation, these two energy principles are known as the two theorems of Castigliano. In the past, these methods were taught to provide the engineer with an analytical solution method for structural problems that were difficult to solve using direct solution procedures. Nowadays, computer programmes are available for many of these calculations, which means that analytical skills are less essential. However, indirect methods still are important to understand the theoretical background of the programmes.

There is another reason for the discussion of variational methods. The computer has not only adopted classical calculation procedures but also introduced new solution procedures such as the finite element method (FEM). The basis of this *approximation method* lies in the application of energy principles.

Course contents
This course is presented in such a manner that variational principles directly follow from the triplet: equilibrium equations, constitutive equations and kinematic equations for deformable continua. In this triplet, the principle of potential energy replaces the equilibrium equation and the principle of complementary energy replaces the kinematic equations. The first and second theorems of Castigliano will be derived from the principle of potential energy and from the principle of complementary energy, respectively. In this approach it also can be made very clear, when the variational principles are valid (valid or not for geometrical non-linear problems).

Actually, the derivations should be based on the equilibrium, constitutive and kinematic equations valid for three-dimensional continua. In that case the derivations require a lot of handwriting, unless the tensor notation is used. Therefore, in these lecture notes the derivations are performed for one-dimensional bodies. Then, the generalisation to three-dimensional bodies can directly be given.

Examples
The text of these lecture notes is quite extensive. The covered theory itself does not require such elaboration. A large part of the text consists of calculation examples applying the several variational methods.

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1 Work and some applications

1.1 Performed work

It is assumed that the material responds elastically. It is not necessary to adopt linear-elastic behaviour.
A cube of material is considered with a unit volume, which is loaded unaxially by a stress σ (see Fig. 1.1). The length 1 increases by an amount ε. The strain ε develops gradually, because the load σ is applied gently.

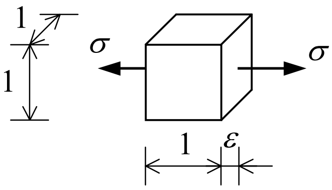


Fig. 1.1: Uniaxially loaded cube.

The work performed by this load, is determined as follows. Suppose that at a certain moment a load σ is present, then a small increment $d\sigma$ causes the strain to increase by $d\varepsilon$ (see Fig. 1.2a). The existing load σ performs an amount of work equal to $\sigma \bullet d\varepsilon$. Therefore, the total amount of work performed equals:

$$E_s' = \int_0^{\varepsilon} \sigma d\varepsilon \quad \text{with} \quad \sigma = \sigma(\varepsilon) \tag{1.1}$$

This is called the deformation energy per unit of volume. This amount of energy is indicated by the grey area in the σ-ε diagram (Fig. 1.2b). Deformation energy is potential energy, which is accumulated in the material.

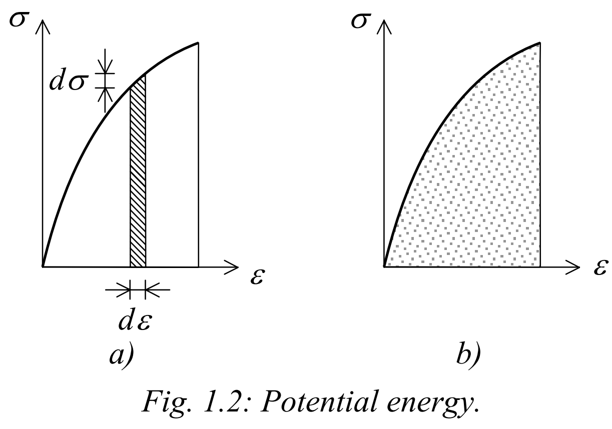


Fig. 1.2: Potential energy.

In structural mechanics another concept of energy plays an important role. It is called complementary energy indicated by E_c', it is defined by:

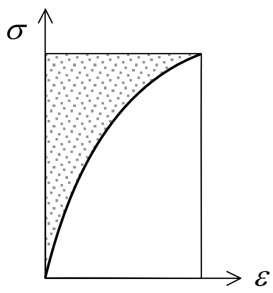


Fig. 1.3: Complementary energy.

$$E_c' = \int_0^{\sigma} \varepsilon d\sigma \quad \text{with} \quad \varepsilon = \varepsilon(\sigma) \tag{1.2}$$

In Fig. 1.3 the grey area represents the amount of complementary energy. In general, it is not easy to give a physical interpretation of the concept of complementary energy.

1.2 Linear-elasticity

From now on, only linear-elastic materials will be considered. In this case *the deformation energy* E_s' *is equal to the complementary energy* E_c' as shown in Fig. 1.4. Hooke's law then describes the constitutive property:

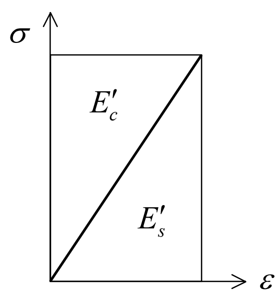


Fig. 1.4: Energy in a linear-elastic material.

$$\begin{aligned} \sigma &= E\varepsilon && (\textit{stiffness formulation}) \\ \varepsilon &= \frac{\sigma}{E} && (\textit{flexibility formulation}) \end{aligned} \tag{1.3}$$

where E is Young's modulus.
It holds:

$$E_s' = \frac{1}{2}\sigma\varepsilon \tag{1.4}$$

With the stiffness formulation of (1.3) this can be rewritten as:

$$\boxed{E_s' = \frac{1}{2} E \varepsilon^2} \tag{1.5}$$

It also holds:

$$E_c' = \frac{1}{2} \varepsilon \sigma \tag{1.6}$$

Substitution of the flexibility formulation of (1.3) provides:

$$\boxed{E_c' = \frac{1}{2} \frac{\sigma^2}{E}} \tag{1.7}$$

The deformation energy is expressed in *strains*, and is therefore also called *strain energy*. The complementary energy is expressed in *stresses*.

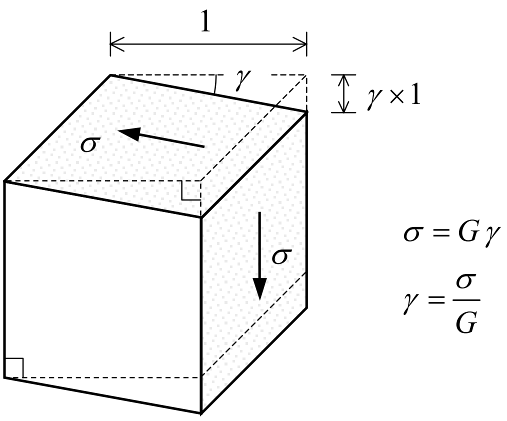


Fig. 1.5: Cube of unit volume subjected to a shear stress.

The same principle is applied to a shear stress σ. Again a cube of unit volume is considered (Fig. 1.5). A shear stress σ causes a shear deformation γ. The load acting on a surface of unit area (1×1), is then displaced over a distance $\gamma \times 1$. The work done, which is equal to the accumulated deformation energy, holds:

$$E_s' = \frac{1}{2} \sigma \gamma = \frac{1}{2} G \gamma^2 \tag{1.8}$$

where G is the shear modulus. The complementary energy equals:

$$E_c' = \frac{1}{2} \gamma \sigma = \frac{1}{2} \frac{\sigma^2}{G} \tag{1.9}$$

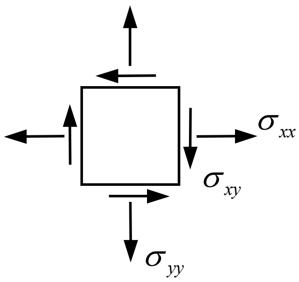


Fig. 1.6: Plane stress state.

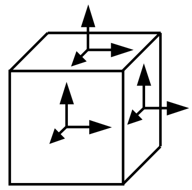


Fig. 1.7: Three-dimensional stress state.

For the case of an in-plane loaded plate (Fig. 1.6), a combination of shear stresses and normal stresses is present. It can be written:

$$E'_s = E'_c = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} + \frac{1}{2}\sigma_{yy}\varepsilon_{yy} + \frac{1}{2}\sigma_{xy}\gamma_{xy} \quad \rightarrow$$

$$E'_s = \frac{1}{2}\left\{\varepsilon_{xx} \;\; \varepsilon_{yy} \;\; \gamma_{xy}\right\}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} \tag{1.10}$$

$$E'_c = \frac{1}{2}\left\{\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{xy}\right\}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon} \tag{1.11}$$

where the superscript " T" indicates that the associated vector is transposed. Introduction of the more general matrix notation for the material properties given by:

$$\begin{aligned}&\boldsymbol{\sigma} = \boldsymbol{K}_\varepsilon\,\boldsymbol{\varepsilon} && (\textit{stiffness formulation})\\ &\boldsymbol{\varepsilon} = \boldsymbol{C}_\sigma\boldsymbol{\sigma} && (\textit{flexibility formulation})\end{aligned} \tag{1.12}$$

leads to:

$$E'_s = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{K}_\varepsilon\,\boldsymbol{\varepsilon} \qquad (\textit{an expression in strains}) \tag{1.13}$$

$$E'_c = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{C}_\sigma\,\boldsymbol{\sigma} \qquad (\textit{an expression in stresses}) \tag{1.14}$$

where $\boldsymbol{K}_\varepsilon$ is the *material stiffness matrix* and $\boldsymbol{C}_\sigma$ is the *material compliance or flexibility matrix*.
In three dimensions (Fig. 1.7) the more general expression holds:

$$E'_s = E'_c = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} + \frac{1}{2}\sigma_{yy}\varepsilon_{yy} + \frac{1}{2}\sigma_{zz}\varepsilon_{zz} + \frac{1}{2}\sigma_{xy}\gamma_{xy} + \frac{1}{2}\sigma_{yz}\gamma_{yz} + \frac{1}{2}\sigma_{zx}\gamma_{zx} \tag{1.15}$$

and again $E_s' = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{K}_\varepsilon \boldsymbol{\varepsilon}$ and $E_c' = \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{C}_\sigma \boldsymbol{\sigma}$. In this case $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are vectors with 6 components and $\boldsymbol{K}_\varepsilon$ and $\boldsymbol{C}_\sigma$ are square 6 by 6 matrices.

1.3 Bars

Analogously to the previously formulated energy relations per unit of volume, for bars definitions per unit of length will be introduced. This is done for the four basic cases: extension, bending, shear and torsion.

Extension

The governing equations are given by (see Fig. 1.8):

$$N = EA\varepsilon \quad (\textit{stiffness formulation}) \quad \Leftrightarrow \quad \varepsilon = \frac{N}{EA} \quad (\textit{flexibility formulation}) \tag{1.16}$$

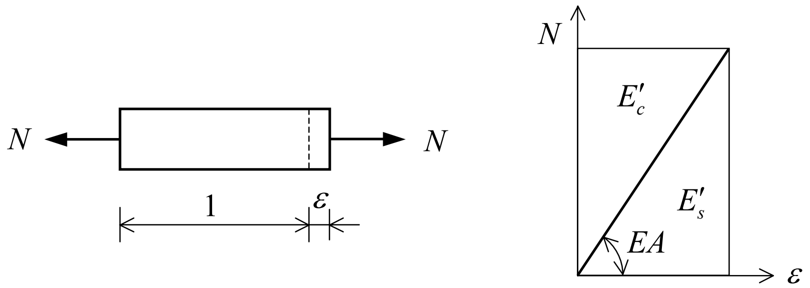


Fig. 1.8: Beam subjected to extension.

The potential and complementary energy become:

$$E_s' = \frac{1}{2}N\varepsilon = \frac{1}{2}EA\varepsilon^2 \qquad (\textit{per unit of length}) \tag{1.17}$$

$$E_c' = \frac{1}{2}\varepsilon N = \frac{1}{2}\frac{N^2}{EA} \qquad (\textit{per unit of length}) \tag{1.18}$$

Bending

The governing equations are given by (see Fig. 1.9):

$$M = EI\kappa \quad (\textit{stiffness formulation}) \quad \Leftrightarrow \quad \kappa = \frac{M}{EI} \quad (\textit{flexibility formulation}) \tag{1.19}$$

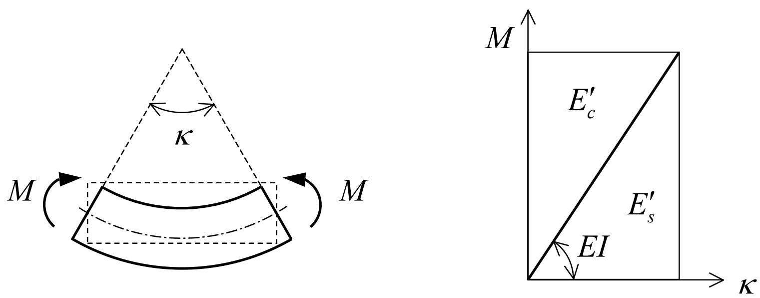


Fig. 1.9: Beam subjected to bending.

The potential and complementary energy become:

$$E_s' = \frac{1}{2} M \kappa = \frac{1}{2} EI \kappa^2 \qquad (\textit{per unit of length}) \tag{1.20}$$

$$E_c' = \frac{1}{2} \kappa M = \frac{1}{2} \frac{M^2}{EI} \qquad (\textit{per unit of length}) \tag{1.21}$$

Shear

The governing equations are given by (see Fig. 1.10):

$$V = GA_{sh} \gamma \quad (\textit{stiffness formulation}) \quad \Leftrightarrow \quad \gamma = \frac{V}{GA_{sh}} \quad (\textit{flexibility formulation}) \tag{1.22}$$

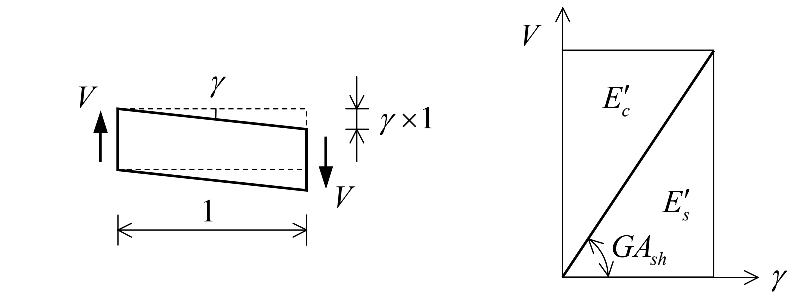


Fig. 1.10: Beam subjected to shear.

The potential and complementary energy become:

$$E_s' = \frac{1}{2} V \gamma = \frac{1}{2} GA_{sh} \gamma^2 \qquad (\textit{per unit of length}) \tag{1.23}$$

$$E_c' = \frac{1}{2} \gamma V = \frac{1}{2} \frac{V^2}{GA_{sh}} \qquad (\textit{per unit of length}) \tag{1.24}$$

Torsion

The governing equations are given by (see Fig. 1.11):

$$M_t = GI_t \vartheta \quad (\textit{stiffness formulation}) \quad \Leftrightarrow \quad \vartheta = \frac{M_t}{GI_t} \quad (\textit{flexibility formulation}) \tag{1.25}$$

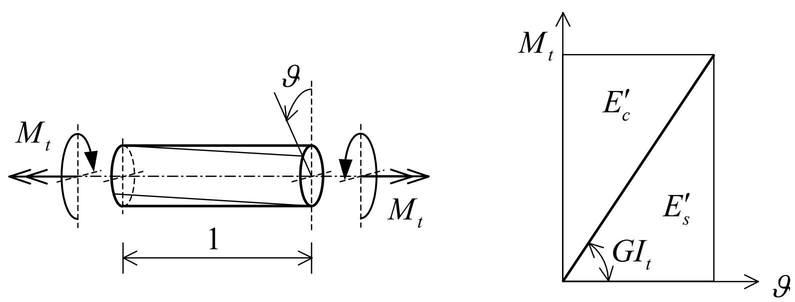


Fig. 1.11: Beam subjected to torsion.

The potential and complementary energy become:

$$E_s' = \frac{1}{2} M_t \vartheta = \frac{1}{2} GI_t \vartheta^2 \qquad (\textit{per unit of length}) \tag{1.26}$$

$$E_c' = \frac{1}{2} \vartheta M_t = \frac{1}{2} \frac{M_t^2}{GI_t} \qquad (\textit{per unit of length}) \tag{1.27}$$

1.4 Stiffnesses of bar cross-sections

For a bar the stiffnesses EA, EI, GA_{sh} and GI_t are important. One of the methods used for the determination of these stiffnesses is based on an energy approach. In this section this method will be discussed, in order to demonstrate that the work-concept sometimes simplifies the calculations.

Axial stiffness EA
A constant normal stress distribution across the cross-section is assumed (Fig. 1.12). The cross-section is square with height d and width b:

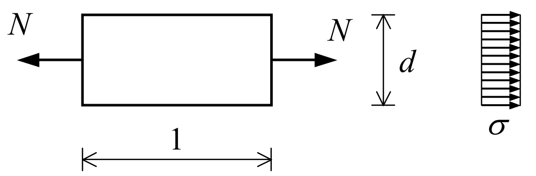


Fig. 1.12: Beam subjected to extension.

$$N = b\,d\,\sigma \quad \rightarrow \quad \sigma = \frac{N}{b\,d} \tag{1.28}$$

The complementary energy per unit of bar length equals:

$$E_c' = \iiint_V \frac{1}{2} \frac{\sigma^2}{E} dV = \frac{1}{2} \frac{\sigma^2}{E} \times b \times d \times 1 = \frac{1}{2} \left(\frac{N}{bd} \right)^2 \frac{bd}{E} = \frac{1}{2} \frac{N^2}{Ebd} \tag{1.29}$$

It also holds (see (1.18)):

$$E_c' = \frac{1}{2} \frac{N^2}{EA} \tag{1.30}$$

Therefore:

$$EA = E\,b\,d \tag{1.31}$$

Of course, this is a trivial result. For the cases of bending, shear and torsion it will be less obvious as shown below.

Bending stiffness EI

A linear distribution of stresses across the height *d* is assumed (Fig. 1.13). In the example a square cross-section is chosen (width *b* and height *d*). The stress distribution is:

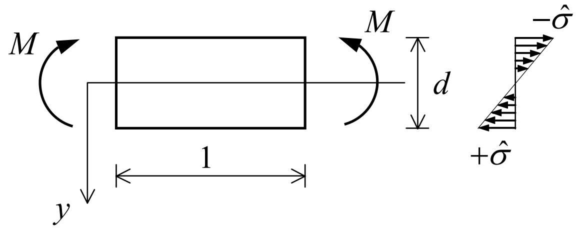


Fig. 1.13: Beam subjected to bending.

$$\sigma(y) = \frac{2y\hat{\sigma}}{d} \tag{1.32}$$

The moment *M* equals:

$$M = \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \sigma(y)\, by\, dy = \frac{2b\hat{\sigma}}{d}\int_{-\frac{1}{2}d}^{\frac{1}{2}d} y^2 dy = \frac{1}{6}bd^2\hat{\sigma} \quad \rightarrow \quad \hat{\sigma} = \frac{M}{\frac{1}{6}bd^2} \tag{1.33}$$

The complementary energy per unit of bar length is:

$$E_c' = \iiint_V \frac{1}{2}\frac{\sigma^2(y)}{E} dV = \frac{2b\hat{\sigma}^2}{Ed^2}\int_{-\frac{1}{2}d}^{\frac{1}{2}d} y^2 dy = \frac{1}{6}\frac{bd\hat{\sigma}^2}{E} = \frac{1}{2}\frac{M^2}{\frac{1}{12}Ebd^3} \tag{1.34}$$

It also holds (see (1.21)):

$$E_c' = \frac{1}{2}\frac{M^2}{EI} \tag{1.35}$$

From the last two equations it follows:

$$EI = \frac{1}{12}Ebd^3 \tag{1.36}$$

This is the well-known result for the rectangular cross-section.

Shear stiffness GA_{sh}

A shear-stress distribution is assumed, which is normally used in the beam theory (Fig. 1.14). In a rectangular cross-section (width *b*, height *d*) this is a parabolic profile across the height:

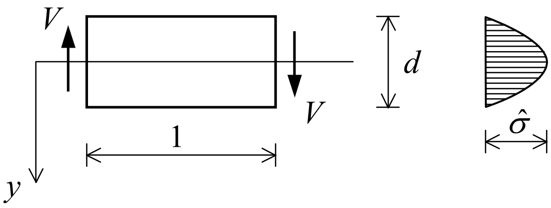


Fig. 1.14: Beam subjected to shear.

$$\sigma(y) = \left(1 - \frac{4y^2}{d^2}\right)\hat{\sigma} \tag{1.37}$$

The transverse force V equals:

$$V = \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \sigma(y)\, b\, dy = b\hat{\sigma} \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \left(1 - \frac{4y^2}{d^2}\right)^2 dy = \frac{2}{3} bd\hat{\sigma} \quad \rightarrow \quad \hat{\sigma} = \frac{3}{2}\frac{V}{bd} \tag{1.38}$$

The complementary energy per unit of bar length is:

$$E_c' = \iiint_V \frac{1}{2}\frac{\sigma^2(y)}{G}\, dV = \frac{b\hat{\sigma}^2}{2G} \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \left(1 - \frac{4y^2}{d^2}\right)^2 dy = \frac{4}{15}\frac{bd\hat{\sigma}^2}{G} = \frac{1}{2}\frac{V^2}{\frac{5}{6}Gbd} \tag{1.39}$$

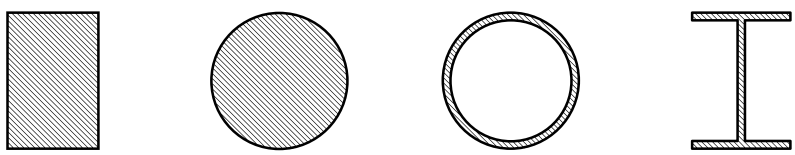
It also holds (see (1.24)):

$$E_c' = \frac{1}{2}\frac{V^2}{GA_{sh}} \tag{1.40}$$

From the last two equations it follows:

$$GA_{sh} = \frac{5}{6} Gbd \tag{1.41}$$

The area A_{sh} is often expressed in A by using:



$\eta = \frac{6}{5}$ $\qquad$ $\eta = \frac{37}{32}$ $\qquad$ $\eta = 2$ $\qquad$ $\eta \approx \frac{A}{A_{web}}$

Fig. 1.15: Shape factors for several cross-sections subjected to shear.

$$A_{sh} = \frac{A}{\eta} \tag{1.42}$$

For the rectangular cross section $\eta = \frac{6}{5} = 1.2$. The shape factor η depends on the shape of the cross-section. Fig. 1.15 shows some values of η for a number of common cross-sections.

Torsional stiffness GI_t

The energy concept can be used if the shear stress distribution is known. An example is a hollow thin-walled cross-section of arbitrary shape (see Fig. 1.16a). The thickness $t(s)$ may

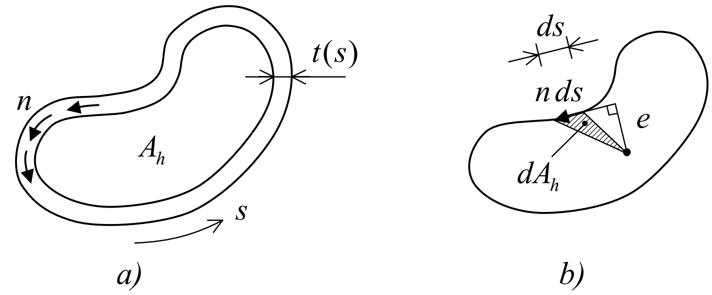


Fig. 1.16: Cross-section of a beam subjected to torsion.

vary along the circumference. The area of the hole in the cross-section is indicated by A_h. In the thin wall a shear-stream n is generated, which has the same value for all coordinates s. Therefore, along the whole circumference, it holds:

$$n = t(s)\sigma(s) \quad \rightarrow \quad \sigma(s) = \frac{n}{t(s)} \tag{1.43}$$

The torsional moment M_t in this hollow cross-section equals:

$$M_t = \oint (n\,ds) \bullet e \tag{1.44}$$

The dot product $e \bullet ds$ is exactly twice the hatched area dA_h of Fig. 1.16b. Therefore:

$$M_t = 2n \oint dA_h = 2n A_h \quad \rightarrow \quad n = \frac{M_t}{2A_h} \tag{1.45}$$

The complementary energy E_c' per unit of bar length now is:

$$E_c' = \int_{Vol} \frac{1}{2} \frac{\sigma^2(s)}{G} dV = \frac{1}{2} \frac{n^2}{G} \oint \left(\frac{1}{t(s)} \right)^2 t(s) \times 1 \times ds = \frac{1}{2} \frac{n^2}{G} \oint \frac{1}{t(s)} ds \tag{1.46}$$

With $n = M_t / 2A_h$ this becomes:

$$E_c' = \frac{1}{2}\frac{M_t^2}{G\dfrac{4A_h^2}{\oint \dfrac{1}{t(s)}ds}} \tag{1.47}$$

It also holds (see (1.27)):

$$E_c' = \frac{1}{2}\frac{M_t^2}{GI_t}$$

From the last two equations, it follows:

$$I_t = \frac{4A_h^2}{\oint \dfrac{1}{t(s)}ds} \qquad \text{(\textit{formula of Bredt})} \tag{1.48}$$

For other cross-sectional shapes, the same approach can be applied, as long as the shear stress distribution is known.

2 Virtual work equation and principle of minimum potential energy

2.1 Problem definition

For arbitrary three-dimensional bodies the following equations are valid for the volume V of the body:

Balance equations
The balance equations or equilibrium equations read:

$$\begin{aligned}
&\sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} + P_x = 0 \\
&\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z} + P_y = 0 \\
&\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + P_z = 0
\end{aligned} \tag{2.1}$$

Constitutive equations
The constitutive equations in matrix form are:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & \upsilon & \upsilon & 0 & 0 & 0 \\ & 1 & \upsilon & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & \text{symmetrical} & & 2(1+\upsilon) & 0 & 0 \\ & & & & 2(1+\upsilon) & 0 \\ & & & & & 2(1+\upsilon) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} \tag{2.2}$$

Written in the inverse form they read:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{(1-\upsilon)E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1 & \frac{\upsilon}{1-\upsilon} & \frac{\upsilon}{1-\upsilon} & 0 & 0 & 0 \\ & 1 & \frac{\upsilon}{1-\upsilon} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\upsilon}{2(1-\upsilon)} & 0 & 0 \\ & \text{symmetrical} & & & \frac{1-2\upsilon}{2(1-\upsilon)} & 0 \\ & & & & & \frac{1-2\upsilon}{2(1-\upsilon)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} \tag{2.3}$$

Kinematic equations
The kinematic or strain displacement relations are:

$$
\begin{aligned}
&\varepsilon_{xx} = u_{x,x} && 2\varepsilon_{xy} = u_{x,y} + u_{y,x} \\
&\varepsilon_{yy} = u_{y,y} && 2\varepsilon_{yz} = u_{y,z} + u_{z,y} \\
&\varepsilon_{zz} = u_{z,z} && 2\varepsilon_{zx} = u_{z,x} + u_{x,z}
\end{aligned}
\tag{2.4}
$$

Kinematic boundary conditions

The kinematic boundary conditions on the part S_u of the surface read:

$$
\left.
\begin{aligned}
u_x &= u_x^o \\
u_y &= u_y^o \\
u_z &= u_z^o
\end{aligned}
\right\} \quad \text{on } S_u
\tag{2.5}
$$

where u_x^o, u_y^o and u_z^o are prescribed values.

Dynamic boundary conditions

The dynamic boundary conditions on the part S_p of the surface equal:

$$
\left.
\begin{aligned}
\sigma_{xx}e_x + \sigma_{yx}e_y + \sigma_{zx}e_z &= p_x \\
\sigma_{xy}e_x + \sigma_{yy}e_y + \sigma_{zy}e_z &= p_y \\
\sigma_{xz}e_x + \sigma_{yz}e_y + \sigma_{zz}e_z &= p_z
\end{aligned}
\right\} \quad \text{on } S_p
\tag{2.6}
$$

where p_x, p_y and p_z are prescribed surface loads and e_x, e_y and e_z are the components of the unit outward-pointing normal on the surface.

Tensor notation

All equations above can be rewritten in tensor notation as:

$$
\left.
\begin{aligned}
&\sigma_{ij,i} + P_j = 0 \\
&\sigma_{ij} = K_{ijkl}\varepsilon_{kl} \\
&\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)
\end{aligned}
\right\} \quad \text{in } V
\tag{2.7}
$$

The boundary conditions expressed in tensor notation are:

$$
\begin{aligned}
&u_i = u_i^o && \text{on } S_u \\
&e_i\sigma_{ij} = p_j && \text{on } S_p
\end{aligned}
\tag{2.8}
$$

In above equations $i, j = 1,2,3$ and summation convention has to be applied.

From now on all derivations are restricted to the special one-dimensional continuum, where only u_x, P_x, p_x, ε_{xx} and σ_{xx} are different from zero. Since no confusion may arise anymore, the indices are left out of the variables and the notations u, P, p, ε and σ are used.
The cross-section of the one-dimensional body is chosen such that it has unit area. Integration over the volume of the body is then reduced to integration over its length. The surface S, being the sum of S_u and S_p, now consists out of both ends of the body (see Fig. 2.1). The

numbers 1 and 2 tag the left and right side, respectively. At one of these ends, the displacement may be prescribed. That cross-section then forms the part S_u of S, while the other end forms the part S_p. When at both ends a displacement is prescribed, S_u coincides

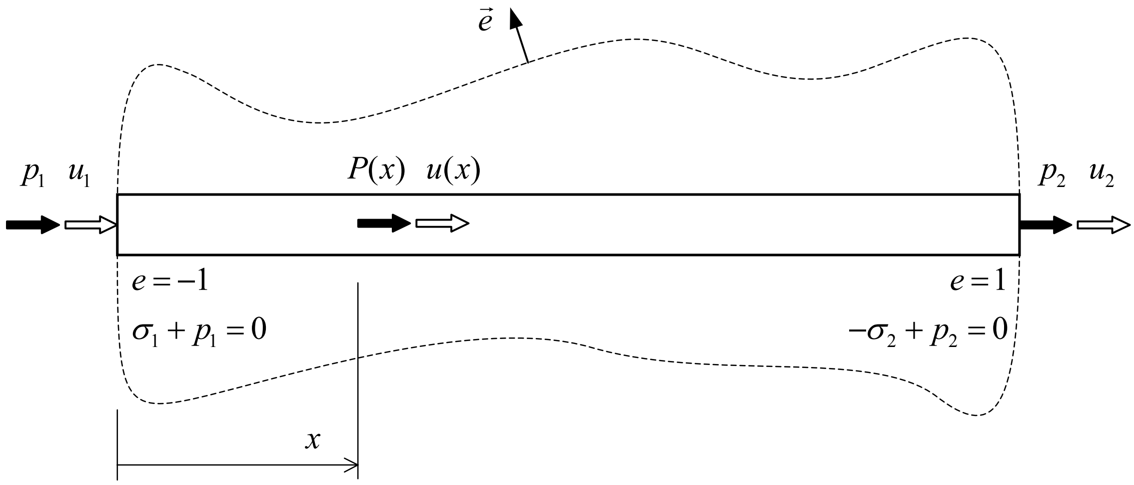


Fig. 2.1: One-dimensional body.

with S and there is no part S_p.
The relations (2.1) up to (2.6) can be simplified to:

$$\left.\begin{array}{ll} \sigma_{,x} + P = 0 & (\textit{balance equation}) \\ \sigma = E\varepsilon & (\textit{constitutive equation}) \\ \varepsilon = u_{,x} & (\textit{kinematic equation}) \end{array}\right\} \text{ in } V \qquad \begin{array}{r} (2.9) \\ (2.10) \\ (2.11) \end{array}$$

$$u - u^o = 0 \qquad (\textit{kinematic boundary condition}) \qquad \text{on } S_u \qquad (2.12)$$

$$-e\,\sigma + p = 0 \qquad (\textit{dynamic boundary condition}) \qquad \text{on } S_p \qquad (2.13)$$

2.2 The method of weighted residuals

It is assumed that the kinematic relation in V and the boundary condition on S_u are satisfied. In first instance, in V and on S_p stress fields are assumed that do not necessarily satisfy equilibrium. It then holds:

$$\begin{array}{ll} \sigma_{,x} + P = R & \text{in } V \\ -\sigma e + p = r & \text{on } S_p \end{array} \qquad (2.14)$$

where R and r are the so-called residuals. For the exact solution, these residuals have to be zero. This condition is fulfilled if for every kinematic admissible displacement field $\bar{u}$ the following relation is satisfied:

$$\iiint_V R\,\overline{u}\,dV + \iint_{S_p} r\,\overline{u}\,dS = 0 \tag{2.15}$$

Such a field is kinematic admissible if it is zero on S_u where the displacements have been prescribed and if it satisfies the kinematic relations in V.

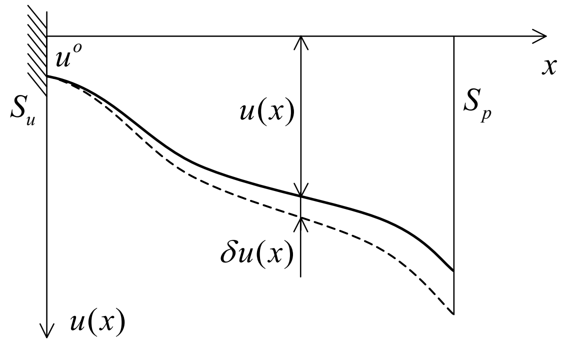


Fig. 2.2: Kinematically admissible displacement field.

For the displacement field $\overline{u}$ a variation δu on the real displacement field is chosen (also see Fig. 2.2). This idea was already used when the concept of virtual displacement was discussed. The condition (2.15) now becomes:

$$\iiint_V R\,\delta u\,dV + \iint_{S_p} r\,\delta u\,dS = 0 \tag{2.16}$$

The requirement that this condition has to hold for every arbitrarily admissible variation δu, can only be satisfied if the following relations are valid:

$$\begin{aligned} R &= 0 \qquad \text{in } V \\ r &= 0 \qquad \text{on } S_p \end{aligned} \tag{2.17}$$

Accordingly, condition (2.16) prescribes that exact equilibrium is required. The requirement that the "weighted residuals" have to be zero for all kinematically admissible variations of the displacement field is therefore a full replacement of the equilibrium conditions.

2.3 The equation of virtual work

Relation (2.16) will be reformulated. Firstly (2.14) is substituted, leading to:

$$\iiint_V \left(\sigma_{,x} + P\right)\delta u\,dV + \iint_{S_p} \left(-e\sigma + p\right)\delta u\,dS = 0 \tag{2.18}$$

The volume integral with integrand $\sigma_{,x}\delta u$ can be integrated by parts, i.e.:

$$\iiint\limits_V \sigma_{,x}\,\delta u\,dV = -\iiint\limits_V \sigma \underbrace{\delta u_{,x}}_{I}\,dV \underbrace{-\sigma_1\delta u_1 + \sigma_2\delta u_2}_{II} \tag{2.19}$$

The two terms indicated by I and II will be rephrased. Term II can be rewritten as:

$$-\sigma_1\delta u_1 + \sigma_2\delta u_2 = e_1\sigma_1\delta u_1 + e_2\sigma_2\delta u_2 = \iint\limits_{S_p} e\sigma\delta u\,dS \tag{2.20}$$

where S is the area of the entire surface, which is the sum of S_u and S_p. Because of the choice that δu is zero on S_u, this part of the surface integral vanishes. Therefore, (2.20) can be rewritten as:

$$-\sigma_1\delta u_1 + \sigma_2\delta u_2 = \iint\limits_{S_p} e\sigma\delta u\,dS \tag{2.21}$$

The virtual displacement $\delta u_{,x}$ (term I in (2.19)) can be replaced by $\delta\varepsilon$. After all, if δu is a variation of the displacement u, then $\delta u_{,x}$ is a variation of the strain ε, i.e.:

$$\delta\varepsilon = \delta u_{,x} \tag{2.22}$$

Combination of (2.18), (2.19), (2.21) and (2.22) finally leads to:

$$\boxed{\iiint\limits_V \sigma\delta\varepsilon\,dV - \iiint\limits_V P\delta u\,dV - \iint\limits_{S_p} p\delta u\,dS = 0} \tag{2.23}$$

This expression is the *virtual work equation* for a deformable body. The relation has been derived directly from the balance equation and the dynamic boundary condition. Since no constitutive conditions have been used, this relation is valid in the plastic range too.
It now also becomes clear why the variation of δu is set to zero on the part of the surface where u is prescribed, because otherwise in the virtual work equation also unknown support reactions p on S_u would appear.

Example
In spite of the fact, that the virtual work equation (2.23) is a preliminary result, already an application of the equation can be shown. The formulation in which it is presented is new, the application however is not. It can be used to calculate internal stresses or stress resultants in a statically determinate structure.
As an example, the overhanging truss as shown in Fig. 2.3 is considered. The normal N has to be determined in the bar of upper side adjacent to the support. On basis of the equilibrium of moments about point A it is known that $N = 2F$.
The principle of virtual work will be applied in two manners. In the first method the bar is imaginarily cut in half and the two cutting edges are displaced with regard to each other over a distance δe (see Fig. 2.4a). The formed mechanism causes the point load F to displace

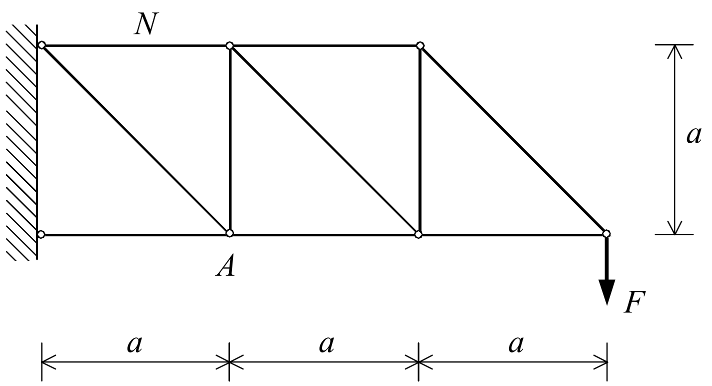


Fig. 2.3: Overhanging truss.

over a distance δu. From the geometry of the structure, it follows that $\delta u = 2\delta e$. The virtual work equation then becomes:

$$N \bullet \delta e = F \bullet \delta u \quad \rightarrow \quad N \bullet \delta e = F \bullet 2\delta e \quad \rightarrow \quad N = 2F$$

In the second method the following approach is adopted. The virtual work equation is used in the form given by (2.23). A virtual displacement field is chosen, which is uniquely defined in each point of the truss and still produces a mechanism. This can be achieved by a rigid body

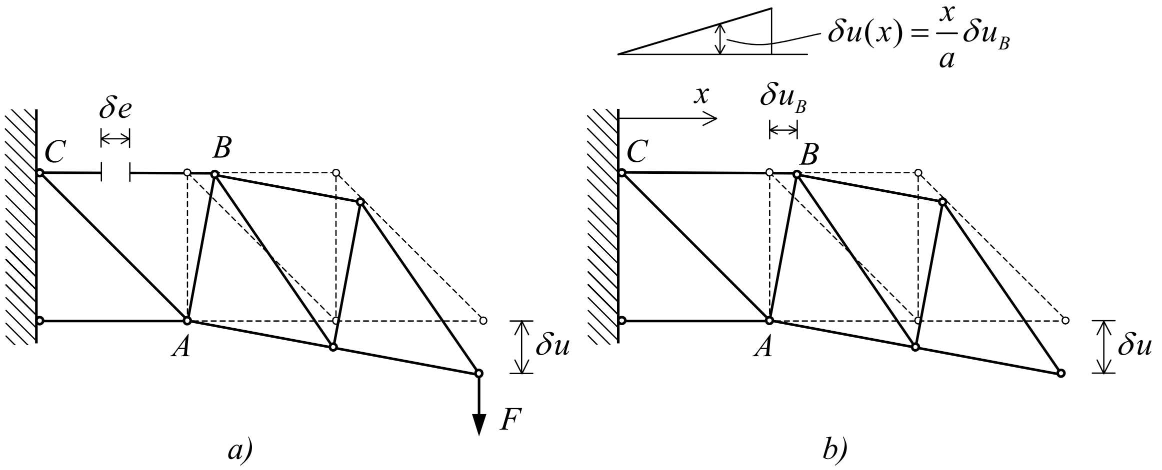


Fig. 2.4: Virtual displacements of the truss.

rotation about point A of the part of the truss situated at the right side of line-piece AB (see Fig. 2.4b). All displacements are expressed in the virtual displacement of load F. From geometrical considerations, the horizontal displacement δu_B of node B appears to be:

$$\delta u_B = \frac{1}{2}\delta u$$

Between the nodes C and B a linear displacement field is assumed, being equal to 0 in C and δu_B in B, i.e.:

$$\delta u(x) = \frac{x}{a}\delta u_B = \frac{x}{2a}\delta u$$

For the chosen mechanism, the strains in all bars with the exception of bar CB are zero. Therefore, the first integral of (2.23) becomes:

$$\iiint_V \sigma\,\delta\varepsilon\, dV = \int_0^a N\delta\varepsilon\, dx$$

For $\delta\varepsilon$ it follows:

$$\delta\varepsilon = \delta u_{,x} = \frac{1}{2a}\delta u$$

The virtual work equation can now be written as:

$$\int_0^a N\delta\varepsilon\, dx - F\delta u = 0 \quad \rightarrow \quad \int_0^a N\frac{1}{2a}\delta u\, dx - F\delta u = 0 \quad \rightarrow \quad \left(\frac{1}{2}N - F\right)\delta u = 0$$

This result should hold for every virtual displacement δu, this means that above equation only can be satisfied if the term between brackets is equal to zero. Therefore:

$$N = 2F$$

which is the required solution.

2.4 The principle of minimum potential energy

From now on only elastic materials will be considered. Further, it is assumed that the loads on the structures are conservative in nature. The effect of these conditions is that no energy is dissipated.
From now on, the constitutive equation plays a role too. For an elastic material the stress σ can uniquely be expressed in the strain ε:

$$\sigma = \sigma(\varepsilon) \tag{2.24}$$

The deformation energy per unit of volume E_s' is defined by:

$$E_s' = \int_0^\varepsilon \sigma\, d\varepsilon \tag{2.25}$$

and inversely the stress σ can be written as:

$$\sigma = \frac{dE_s'}{d\varepsilon} \tag{2.26}$$

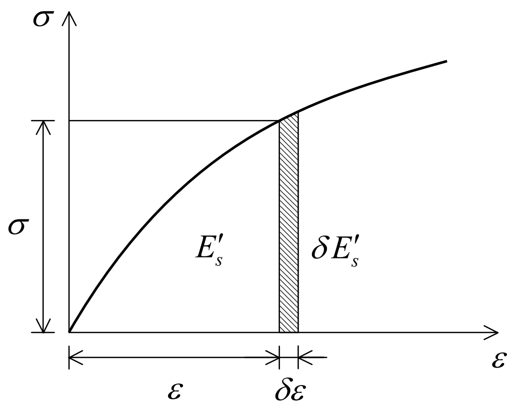


Fig. 2.5: Deformation energy.

The integral over the volume of E_s' is indicated by E_s. A variation δu is accompanied by a variation of the deformation energy:

$$\delta E_s' = \frac{dE_s'}{d\varepsilon}\delta\varepsilon \quad \rightarrow \quad \delta E_s' = \sigma\,\delta\varepsilon \tag{2.27}$$

In Fig. 2.5, this has been displayed graphically. The volume integral in the virtual work equation (2.23) is exactly equal to the variation δE_s of the potential energy accumulated in the material.
There is another form of potential energy, namely E_p of the position of the external loads. The potential of a distributed load per unit of volume P during a displacement u decreases by the amount P times u. For a variation δu of the displacement, per unit of volume it holds:

$$\delta E_p = -P\,\delta u$$

For the entire load P in the volume V and p on the surface S_p, it follows:

$$E_p = -\iiint\limits_V P\delta u\,dV - \iint\limits_{S_p} p\delta u\,dS \tag{2.28}$$

With the results (2.27) and (2.28) the virtual work equation can be interpreted as the condition that the potential energy E_{pot} has to be stationary with respect to variations of u:

$$\delta E_{pot} = 0 \tag{2.29}$$

or:

$$\boxed{E_{pot} = \underbrace{\iiint\limits_V E_s'\,dV}_{E_s} \underbrace{- \iiint\limits_V P\,u\,dV - \iint\limits_{S_p} p\,u\,dS}_{E_p}} \qquad (stationary) \tag{2.30}$$

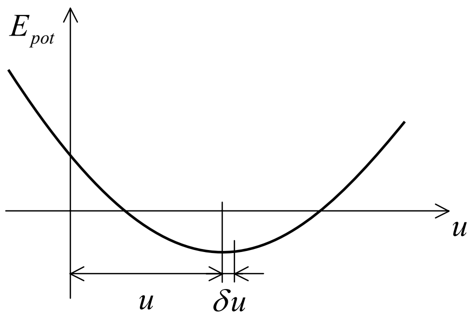


Fig. 2.6: Minimum of potential energy.

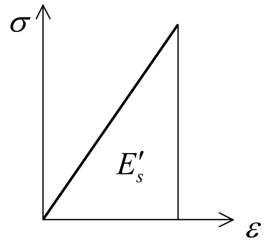


Fig. 2.7: Linear-elastic material.

It can be shown that the stationary value of E_{pot} is a minimum (see Fig. 2.6). Additionally the value of E_{pot} is negative.
Many problems can be solved by assuming that a linear-elastic material is applicable, for which Hooke's law is valid (Fig. 2.7):

$$\sigma = E\,\varepsilon \quad ; \quad E'_s = \frac{1}{2}E\varepsilon^2$$

The expression for the potential energy then becomes:

$$\boxed{E_{pot} = \iiint_V \frac{1}{2}E\varepsilon^2 dV - \iiint_V Pu\,dV - \iint_{S_p} p\,u\,dS} \qquad (stationary) \qquad (2.31)$$

Example
A bar with cross-section A, length l and modulus of elasticity E is loaded at one end by a compressive force F_1. The bar is restrained at the other end as shown in Fig. 2.8. The displacement of the free end is u_1. From the displacement u_1, the strain can be obtained which is uniformly distributed over the volume:

$$\varepsilon = -\frac{u_1}{l}$$

The deformation energy equals:

$$E_s = \iiint_V \frac{1}{2}E\left(-\frac{u_1}{l}\right)^2 dV = \frac{1}{2}\frac{EA}{l}u_1^2$$

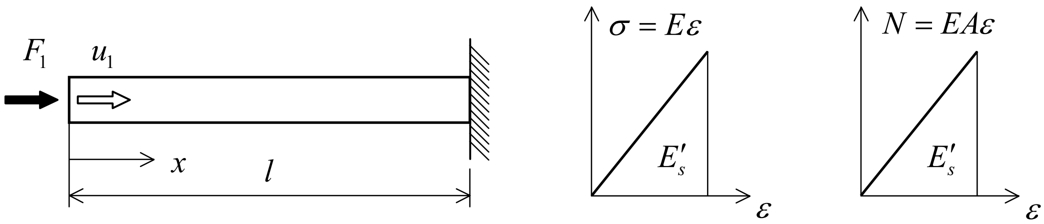


Fig. 2.8: Bar loaded by a compressive force.

Starting from the stress resultant $N = \sigma A = EA\varepsilon$ the same result is obtained:

$$E_s = \iiint_V \frac{1}{2} N\varepsilon \, dV = \int_0^l \frac{1}{2} N\varepsilon A \, dx = \int_0^l \frac{1}{2} EA\varepsilon^2 \, dx = \frac{1}{2} EA\left(-\frac{u_1}{l}\right)^2 l = \frac{1}{2}\frac{EA}{l} u_1^2$$

For the energy of position it holds:

$$E_p = -F_1 u_1$$

So that the potential energy is:

$$E_{pot} = \frac{1}{2}\frac{EA}{l} u_1^2 - F_1 u_1$$

For equilibrium it is required that:

$$\delta E_{pot} = \frac{dE_{pot}}{du_1}\delta u_1 = \left(\frac{EA}{l} u_1 - F_1\right)\delta u_1 = 0 \quad \rightarrow \quad F_1 = \frac{EA}{l} u_1$$

The value of E_{pot} in this state of equilibrium is:

$$E_{pot} = \frac{1}{2}\frac{EA}{l} u_1^2 - \left(\frac{EA}{l} u_1\right) u_1 = -\frac{1}{2}\frac{EA}{l} u_1^2$$

Indeed, the value of E_{pot} is negative. To demonstrate that this value is a minimum, the second derivative is investigated. It is found:

$$\frac{d^2 E_{pot}}{du_1^2} = \frac{d^2}{du_1^2}\left(\frac{1}{2}\frac{EA}{l} u_1^2 - F_1 u_1\right) = \frac{EA}{l}$$

So, the second derivative is positive for each arbitrary value of u_1. Therefore, the stationary value of E_{pot} (for which equilibrium occurs) is a minimum.

2.5 The principle of minimum potential energy in three dimensions

The potential energy expression (2.31) is derived for the one-dimensional case, only containing the stress σ_{xx}, the volume load P_x the surface load p_x, the displacement u_x and the strain ε_{xx}. Since no confusion could arise, the indices were removed.
The more general three-dimensional derivation provides a similar expression. Before this relation can be written down, firstly a number of vectors have to be introduced:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} \quad ; \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} \quad ; \quad \boldsymbol{P} = \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix} \quad ; \quad \boldsymbol{p} = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} \quad ; \quad \boldsymbol{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \tag{2.32}$$

The relation between stresses and strains is given by:

$$\boldsymbol{\sigma} = \boldsymbol{K}_{\varepsilon}\, \boldsymbol{\varepsilon} \tag{2.33}$$

where $\boldsymbol{K}_{\varepsilon}$ is the stiffness matrix already introduced in (2.3). The virtual work equation (2.23) for this case reads:

$$\boxed{\iiint_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \iiint_V \delta\boldsymbol{u}^T \boldsymbol{P}\, dV - \iint_{S_p} \delta\boldsymbol{u}^T \boldsymbol{p}\, dS = 0} \tag{2.34}$$

where the superscript "T" indicates that the associated vector is transposed. The expression (2.31) for the potential energy of a linear-elastic body becomes:

$$\boxed{E_{pot} = \iiint_V \frac{1}{2} \boldsymbol{\varepsilon}^T \boldsymbol{K}_{\varepsilon} \boldsymbol{\varepsilon}\, dV - \iiint_V \boldsymbol{u}^T \boldsymbol{P}\, dV - \iint_{S_p} \boldsymbol{u}^T \boldsymbol{p}\, dS} \qquad (\textit{stationary}) \tag{2.35}$$

2.6 The displacement method

The application of the principle of minimum potential energy actually boils down to the fact that in the triplet:

- kinematic equations
- constitutive equations (2.36)
- balance equations

the last one is replaced by the condition that the potential energy has to be stationary, changing the triplet into:

- kinematic equations
- constitutive equations (2.37)
- potential energy stationary

When the equations in the triplet are successively evaluated, the *strategy* of the displacement method is followed. When a compatible displacement field is assumed, then the potential energy is a function of that displacement field. Variation with respect to that displacement field provides the balance equations expressed in the displacements.

Example
As an example a truss is considered with two degrees of freedom u_x and u_y as shown in Fig. 2.9. The bars are loaded by normal forces only. The external load consists out of two point forces F_x and F_y. The axial stiffness and the length of the three bars are different. For

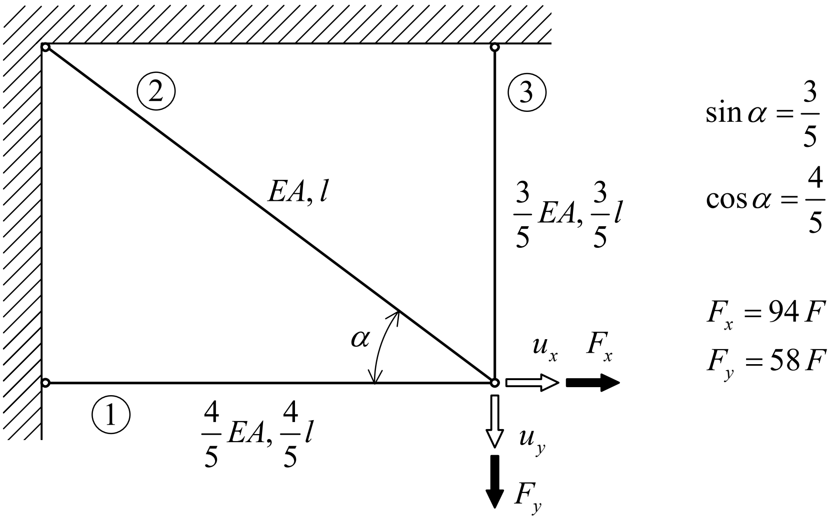


Fig. 2.9: Structure with two degrees of freedom.

comparison, the calculation is carried out in two manners. Firstly, a solution is obtained by a direct method and secondly the principle of minimum potential energy is applied.

Direct method
The kinematic, constitutive and balance equations respectively are:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{5}{4}\frac{u_x}{l} \\ \varepsilon_2 &= \frac{4}{5}\frac{u_x}{l} + \frac{3}{5}\frac{u_y}{l} \\ \varepsilon_3 &= \frac{5}{3}\frac{u_y}{l} \end{aligned}\right\} \qquad (\textit{kinematic equations})$$

$$\left.\begin{aligned} N_1 &= \frac{4}{5}EA\,\varepsilon_1 \\ N_2 &= \phantom{\frac{4}{5}}EA\,\varepsilon_2 \\ N_3 &= \frac{3}{5}EA\,\varepsilon_3 \end{aligned}\right\} \qquad (\textit{constitutive equations})$$

$$\left.\begin{aligned} N_1 + \frac{4}{5} N_2 &= F_x \\ \frac{3}{5} N_2 + \quad N_3 &= F_y \end{aligned}\right\} \qquad (\textit{balance equations})$$

By downward substitution the balance equations are transformed into:

$$\frac{EA}{25l}\left(41u_x + 12u_y\right) = F_x \quad ; \quad \frac{EA}{25l}\left(12u_x + 34u_y\right) = F_y$$

For the sake of simplicity a special load case is chosen:

$$F_x = 94F \quad ; \quad F_y = 58F$$

This provides the following solution:

$$u_x = 50\frac{Fl}{EA} \quad ; \quad u_y = 25\frac{Fl}{EA}$$

For the normal force it can be derived:

$$N_1 = 50F \quad ; \quad N_2 = 55F \quad ; \quad N_3 = 25F$$

Variational method

In the variational method, the same kinematic and constitutive equations are used. The balance equations are replaced by the minimisation of the potential energy equation:

$$E_{pot} = \int_0^{l_1} \frac{1}{2} EA_1\, \varepsilon_1^2\, dx + \int_0^{l_2} \frac{1}{2} EA_2\, \varepsilon_2^2\, dx + \int_0^{l_3} \frac{1}{2} EA_3\, \varepsilon_3^2\, dx - F_x u_x - F_y u_y \quad \rightarrow$$

$$E_{pot} = \frac{1}{2}\frac{4}{5} EA\left(\frac{5}{4}\frac{u_x}{l}\right)^2 \frac{4}{5} l + \frac{1}{2} EA\left(\frac{4}{5}\frac{u_x}{l} + \frac{3}{5}\frac{u_y}{l}\right)^2 l + \frac{1}{2}\frac{3}{5} EA\left(\frac{5}{3}\frac{u_y}{l}\right)^2 \frac{3}{5} l - F_x u_x - F_y u_y \quad \rightarrow$$

$$E_{pot} = \frac{EA}{50l}\left(41u_x^2 + 24u_x u_y + 34u_y^2\right) - F_x u_x - F_y u_y$$

Variation with respect to u_x and u_y delivers:

$$\frac{\partial E_{pot}}{\partial u_x} = \frac{EA}{25l}\left(41u_x + 12u_y\right) - F_x = 0 \quad ; \quad \frac{\partial E_{pot}}{\partial u_y} = \frac{EA}{25l}\left(12u_x + 34u_y\right) - F_y = 0$$

These equations are identical to the ones derived with the direct method. Therefore, the same solution is found.

2.7 Validity of the virtual work equation

In the form given by (2.23), the virtual work equation is valid for both elastic and plastic materials. After all, the constitutive equation has not been used in the derivation! The virtual work equation is valid for geometrical non-linear problems as well, where for example in the expression for the strains also terms are present that are quadratic in the displacements:

$$\varepsilon_{xx} = u_{x,x} + \frac{1}{2}\left(u_{x,x}\right)^2 + \frac{1}{2}\left(u_{y,x}\right)^2 + \frac{1}{2}\left(u_{z,x}\right)^2 \tag{2.38}$$

A small variation of the displacement field leads to a variation of the strain equal to:

$$\delta\varepsilon_{xx} = \delta u_{x,x} + \frac{1}{2}\left(\delta u_{x,x}\right)^2 + \frac{1}{2}\left(\delta u_{y,x}\right)^2 + \frac{1}{2}\left(\delta u_{z,x}\right)^2 \tag{2.39}$$

If the variations are small, then the last three terms are an order of magnitude smaller than the first term. Therefore for this case, it also holds:

$$\delta\varepsilon_{xx} = \delta u_{x,x} \tag{2.40}$$

This relation also has been used in the derivation of the virtual work equation, which means that it keeps its validity for geometrical non-linear problems.

In summary, the principle of minimum potential energy was obtained by assuming that a unique relation exists between stresses and strains. Therefore, this principle is only valid for elastic media. However, for geometrical non-linear elastic problems the principle is valid too.

3 Complementary virtual work equation and principle of minimum complementary energy

3.1 Starting points

Again, a one-dimensional continuum is considered, as used in chapter 2. As defined before, only the variables u_x, P_x, p_x, ε_{xx} and σ_{xx} are different from zero. Also in this case, the indices x are left out of the variables (see Fig. 3.1).

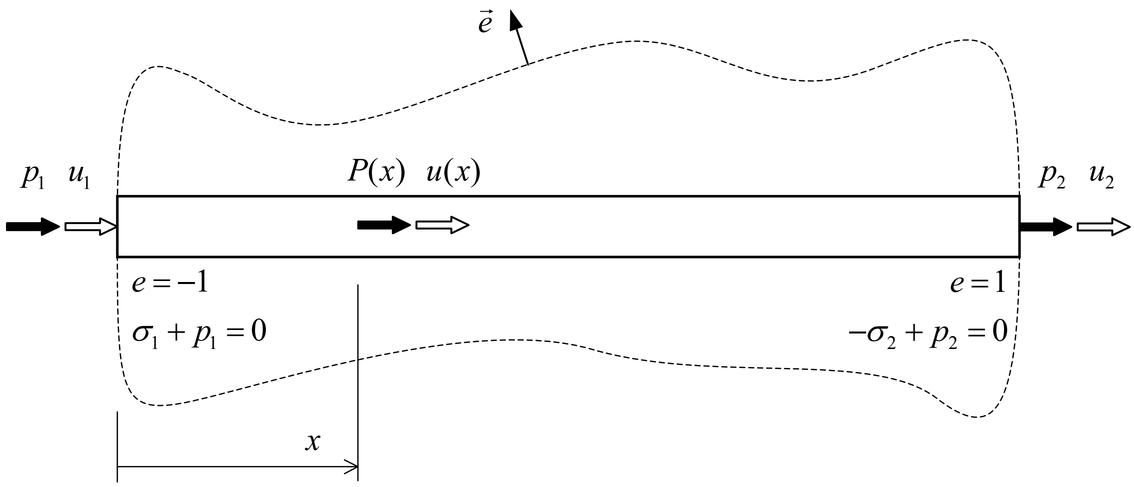


Fig. 3.1: One-dimensional body.

The cross-section of the one-dimensional body is chosen such that it has unit area. The relations for this body are (also see (2.9) up to (2.13)):

$$\left.\begin{aligned} &\sigma_{,x} + P = 0 && (\textit{balance equation}) && (3.1)\\ &\varepsilon = \frac{1}{E}\sigma && (\textit{constitutive equation}) && (3.2)\\ &\varepsilon = u_{,x} && (\textit{kinematic equation}) && (3.3)\end{aligned}\right\} \text{ in } V$$

$$u - u^o = 0 \qquad (\textit{kinematic boundary condition}) \qquad \text{on } S_u \qquad (3.4)$$

$$-e\,\sigma + p = 0 \qquad (\textit{dynamic boundary condition}) \qquad \text{on } S_p \qquad (3.5)$$

Note that the constitutive equation is presented in flexibility formulation, while in chapter 2 this was done in stiffness formulation.

3.2 The method of weighted residuals

It is assumed that the balance equation in V and the dynamic boundary conditions on S_p are satisfied. In first instance, in V and on S_p displacement fields are assumed, which not necessarily do satisfy the kinematic relation and the kinematic boundary condition. It then holds:

$$\begin{aligned} \varepsilon - u_{,x} &= R && \text{in } V \\ u - u^o &= r && \text{on } S_u \end{aligned} \tag{3.6}$$

where R and r are the residuals. For the exact solution, these residuals have to be zero. This condition is fulfilled if for every statically admissible stress field $\bar{\sigma}$ the following relation is satisfied:

$$\iiint_V R\,\bar{\sigma}\,dV + \iint_{S_u} r\,\bar{p}\,dS = 0 \tag{3.7}$$

Such a field is statically admissible if it is zero on S_p where the stresses have been prescribed and if it satisfies the homogeneous balance equation in V.

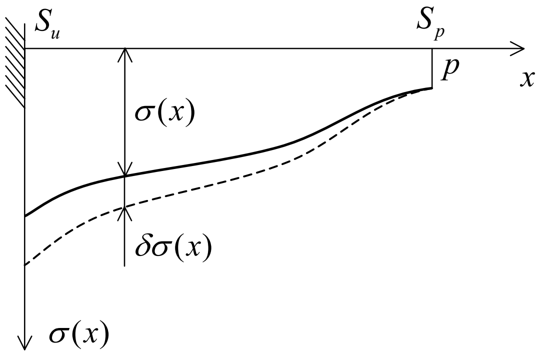


Fig. 3.2: Dynamically admissible stress field.

For the stress field $\bar{\sigma}$ a variation $\delta\sigma$ on the real stress field is chosen (also see Fig. 3.2). That these stresses satisfy the requirements of a statically admissible stress field can easily be shown. Since the balance equation has to be satisfied all the time, for the volume V the following variation has to be satisfied:

$$\left.\begin{aligned} \sigma_{,x} + P = 0 \\ (\sigma + \delta\sigma)_{,x} + P = 0 \end{aligned}\right\} \;\rightarrow\; \boxed{\delta\sigma_{,x} = 0} \quad \text{in } V \tag{3.8}$$

On the surface S_p it holds:

$$\left.\begin{aligned} -e\sigma + p = 0 \\ -e(\sigma + \delta\sigma) + p = 0 \end{aligned}\right\} \;\rightarrow\; e\,\delta\sigma = 0 \;\rightarrow\; \boxed{\delta p = 0} \quad \text{on } S_p \tag{3.9}$$

The concept $\delta\sigma$ is called a virtual stress. The condition (3.7) now becomes:

$$\iiint_V R\,\delta\sigma\,dV + \iint_{S_u} r\,\delta p\,dS = 0 \tag{3.10}$$

The requirement that this condition has to hold for <u>every</u> arbitrarily admissible variation $\delta\sigma$, can only be satisfied if the following relations are valid:

$$
\begin{aligned}
&R = 0 && \text{in } V \\
&r = 0 && \text{on } S_u
\end{aligned} \tag{3.11}
$$

Therefore, the condition (3.10) prescribes that exact compatibility is required. The requirement that the "weighted residuals" have to be zero for all statically admissible variations of the stress field is therefore a full replacement of the compatibility conditions.

3.3 The equation of complementary virtual work

Relation (3.10) will be reformulated. Firstly (3.6) is substituted, leading to:

$$\iiint_V \left(\varepsilon - u_{,x}\right)\delta\sigma\, dV + \iint_{S_u} \left(u - u^\circ\right)\delta p\, dS = 0 \tag{3.12}$$

The volume integral with integrand $-u_{,x}\delta\sigma$ can be integrated by parts, i.e.:

$$-\iiint_V u_{,x}\,\delta\sigma\, dV = \iiint_V u\delta\sigma_{,x}\, dV - u_2\delta\sigma_2 + u_1\delta\sigma_1 \tag{3.13}$$

The last two terms can be rewritten as:

$$-u_2\delta\sigma_2 + u_1\delta\sigma_1 = -u_2\delta p_2 - u_1\delta p_1 = -\iint_{S_u} u\,\delta p\, dS \tag{3.14}$$

where S is the area of the entire surface, which is the sum of S_u and S_p. Because of the choice that δp is zero on S_p, this part of the surface integral vanishes. Therefore, (3.14) can be rewritten as:

$$-u_2\delta\sigma_2 + u_1\delta\sigma_1 = -\iint_{S_u} u\delta p\ dS \tag{3.15}$$

Further, the volume integral of the right-hand side of (3.13) is equal to zero, because of the choice that $\delta\sigma_{,x}$ is equal to zero in V. With (3.13) and (3.15) the principle (3.12) becomes:

$$\boxed{\iiint_V \varepsilon\delta\sigma\, dV - \iint_{S_u} u^\circ\,\delta p\, dS = 0} \tag{3.16}$$

This expression is the *complementary virtual work equation* for a deformable body. The relation has been derived directly from the compatibility conditions. Now, it also becomes clear why a variation of the stress field is required that satisfies the conditions (3.8) and (3.9), because otherwise unknown displacements u in V and on S_p would persist in the equation. Comparison with (2.23) makes it clear why (3.16) is called the complementary energy equation. The roles of σ and ε are exchanged in the volume integral, and the contributions of P and p are replaced by the known displacement u^o.

Example

The complementary virtual work equation (3.16) is just like the virtual work equation (2.23) a preliminary result. However, in this form it already can be applied. It can be used for the calculation of displacements when the internal stresses or stress resultants are known. The same overhanging truss will be considered, as used in the example with the application of the virtual work equation (2.23). All bars have the same axial stiffness EA.

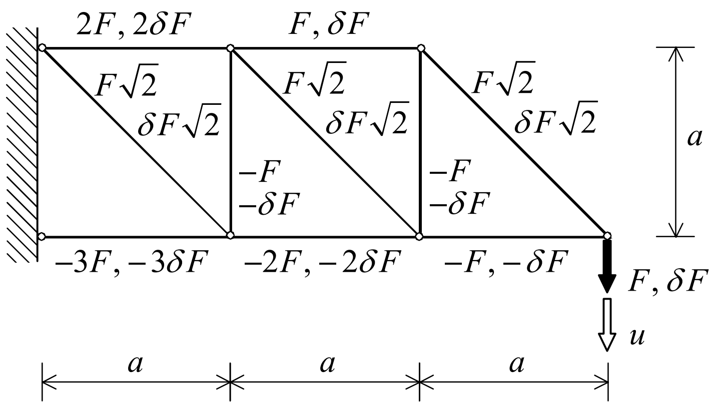


Fig. 3.3: Overhanging truss.

The given load F generates normal forces in the bars as displayed in Fig. 3.3. The displacement u is the result of the load F. But as a line of thought it also can be stated that a prescribed displacement u actually causes the support reaction F and all indicated normal forces in the bars. The virtual stresses are chosen to be the increase δF of this support reaction and the resulting increased normal forces. This delivers a statically admissible stress field. In all freely movable nodal point directions equilibrium is satisfied, except where the displacement is prescribed. Application of (3.16) now provides:

$$\sum_{\text{all bars}} \frac{l_i N_i}{EA} \delta N_i - u\,\delta F = 0 \quad \rightarrow$$

$$\frac{a\,F\,\delta F}{EA}\left(\underbrace{4+1}_{\substack{top\\edge}} + \underbrace{2\sqrt{2}+2\sqrt{2}+2\sqrt{2}}_{\substack{diagonal\\bars}} + \underbrace{1+1}_{\substack{vertical\\bars}} + \underbrace{9+4+1}_{\substack{bottom\\edge}} \right) - u\,\delta F = 0 \quad \rightarrow$$

$$\left\{ \frac{aF}{EA}\left(21+6\sqrt{2}\right) - u \right\} \delta F = 0$$

This has to hold for every variation $\delta F \neq 0$, i.e.:

$$u = \frac{aF}{EA}\left(21+6\sqrt{2}\right)$$

The same value would have been found from a Williot diagram.

3.4 The principle of minimum complementary energy

Again, only elastic materials and applied conservative loads will be considered. Also the constitutive equation will be introduced. In chapter 2 the deformation energy per unit of volume E_s' was introduced as a function of ε:

$$E_s' = \int_0^{\varepsilon} \sigma \, d\varepsilon \tag{3.17}$$

where the stress σ is a function of the strain ε:

$$\sigma = \sigma(\varepsilon) \tag{3.18}$$

Now it is meaningful to define a complementary energy per unit of volume E_c' according to:

$$E_c' = \sigma\varepsilon - E_s'(\varepsilon) \tag{3.19}$$

where in this case the strain ε is a function of the stress σ:

$$\varepsilon = \varepsilon(\sigma) \tag{3.20}$$

The variation of this complementary energy caused by a variation of the stress field equals:

$$\delta E_c' = \frac{dE_c'}{d\sigma}\delta\sigma \tag{3.21}$$

During the determination of $dE_c'/d\sigma$ from (3.19) one should realise that ε is a function of σ and that E_s' is a function of ε (and through (3.20) also a function of σ):

$$\frac{dE_c'}{d\sigma} = \varepsilon + \sigma\frac{d\varepsilon}{d\sigma} - \frac{dE_s'}{d\varepsilon}\frac{d\varepsilon}{d\sigma} = \varepsilon + \left(\sigma - \frac{dE_s'}{d\varepsilon}\right)\frac{d\varepsilon}{d\sigma}$$

Introduction of (2.26) given by:

$$\sigma = \frac{dE_s'}{d\varepsilon}$$

provides:

$$\frac{dE_c'}{d\sigma} = \varepsilon \tag{3.22}$$

From (3.21) it now follows:

$$\delta E_c' = \varepsilon\,\delta\sigma \tag{3.23}$$

For E_c' per unit of volume it also can be written:

$$E_c' = \int_0^{\sigma} \varepsilon \, d\sigma \tag{3.24}$$

Then, for the total volume it holds:

$$E_c = \iiint_V E_c' \, dV \tag{3.25}$$

This has been displayed in Fig. 3.4.
The volume integral in the complementary virtual work equation (3.16) is in view of (3.23) just equal to the variation δE_c of the complementary energy E_c accumulated in the material. For the total structure the complementary energy E_{compl} can now be defined as follows:

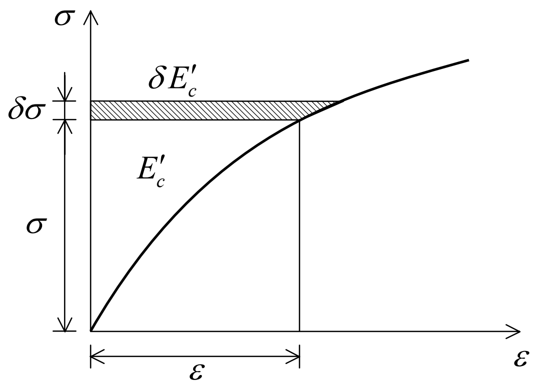


Fig. 3.4: Complementary energy.

$$\boxed{E_{compl} = \underbrace{\iiint_V E_c' \, dV}_{E_c} - \underbrace{\iint_{S_u} p u^o \, dS}_{E_c^o}} \tag{3.26}$$

The complementary virtual work equation can be interpreted as the condition that the complementary energy E_{compl} is stationary for variations of σ:

$$\delta E_{compl} = 0 \tag{3.27}$$

It can be shown that the stationary value of E_{compl} is a minimum (see Fig. 3.5). Additionally, the value of E_{compl} is negative.
For the special case involving linear-elastic media the expression (3.26) can be developed a bit further. Hooke's law is applied in its flexibility formulation (Fig. 3.6):

$$\varepsilon = \frac{1}{E}\sigma \quad ; \quad E_c' = \frac{1}{2}\frac{1}{E}\sigma^2$$

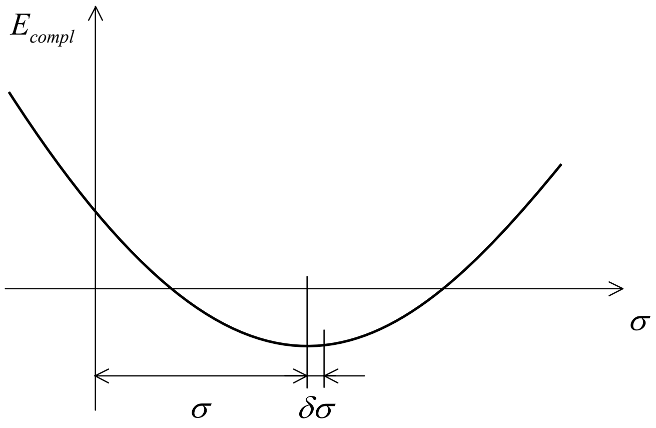


Fig. 3.5: Minimum of complementary energy.

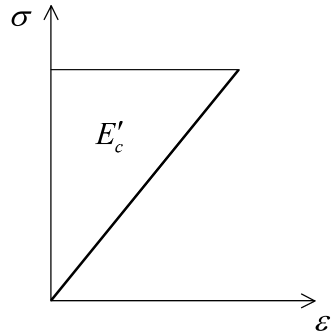


Fig. 3.6: Linear-elastic material.

The expression for the complementary energy then becomes:

$$\boxed{E_{compl} = \iiint_V \frac{1}{2}\frac{1}{E}\sigma^2 dV - \iint_{S_u} p u^o \, dS} \qquad (stationary) \qquad (3.28)$$

Example

The same bar of the example in section 2.4 is considered, which was analysed with the principle of minimum potential energy. The bar has cross-section A, length l and modulus of elasticity E and is loaded at the free end by a compressive force F_1, which causes a displacement u_1 (see Fig. 3.7). Requested is the relation between F_1 and u_1.

In order to be able to apply the principle of minimum complementary energy the next line of thought is followed. The left end of the bar is not considered as a free end with a given load F_1, but as the end where the displacement u_1 is prescribed. For this given u_1 the support

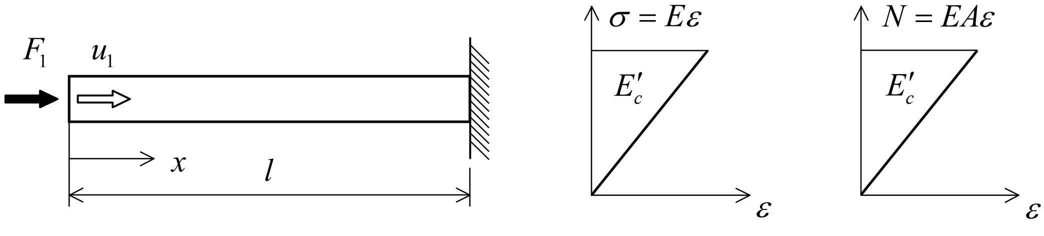


Fig. 3.7: Bar loaded by a compressive force.

reaction F_1 is calculated. Although S_u only consists out of the right end of the bar, this surface is artificially extended with the left end. The calculation is then as follows.

No volume load P is present so that σ has to be constant in order to satisfy the balance equation in V. So, there is only one stress parameter σ. The force F_1 and the stress resultant N can be expressed in this σ on basis of equilibrium:

$$F_1 = -\sigma A \quad ; \quad N = \sigma A$$

One of the variables σ, N or F_1 can be chosen as the fundamental unknown. In this case the choice is F_1.
The complementary energy E_c is:

$$E_c = \frac{1}{2E}\sigma^2 A l \quad \rightarrow \quad E_c = \frac{1}{2}\frac{l}{EA}F_1^2$$

Consequently, the total complementary energy is:

$$E_{compl} = E_c - F_1 u_1 = \frac{1}{2}\frac{l}{EA}F_1^2 - F_1 u_1$$

Variation with respect to F_1 delivers:

$$\delta E_{compl} = \frac{dE_{compl}}{dF_1}\delta F_1 = \left(\frac{l}{EA}F_1 - u_1\right)\delta F_1 = 0 \quad \rightarrow \quad u_1 = \frac{l}{EA}F$$

This is the same result as found in section 2.4.
Substitution of the found relation for u_1 in:

$$E_{compl} = \frac{1}{2}\frac{l}{EA}F_1^2 - F_1 u_1$$

delivers:

$$E_{compl} = \frac{1}{2}\frac{l}{EA}F_1^2 - F_1\left(\frac{l}{EA}F_1\right) = -\frac{1}{2}\frac{l}{EA}F_1^2$$

which is negative and a minimum. The minimum can be confirmed by determination of the second derivative of E_{compl} with respect to F_1, followed by substitution of the value of F_1 that makes E_{compl} stationary:

$$\frac{\partial^2 E_{compl}}{\partial F_1^2} = \frac{\partial^2}{\partial F_1^2}\left(\frac{1}{2}\frac{l}{EA}F_1^2 - u_1 F_1\right) = \frac{l}{EA} > 0$$

3.5 The principle of minimum complementary energy in three dimensions

The complementary energy expression (3.28) is derived for the one-dimensional case. The more general three-dimensional derivation provides a similar expression. Before this relation can be written down, the following vectors are introduced:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} \quad ; \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} \quad ; \quad \boldsymbol{p} = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} \quad ; \quad \boldsymbol{u}^o = \begin{Bmatrix} u_x^o \\ u_y^o \\ u_z^o \end{Bmatrix} \tag{3.29}$$

For the constitutive relations between stresses and strains the flexibility formulation is used:

$$\boldsymbol{\varepsilon} = \boldsymbol{C}_\sigma \, \boldsymbol{\sigma} \tag{3.30}$$

where $\boldsymbol{C}_\sigma$ is the compliance matrix, which is the inverse of the stiffness matrix $\boldsymbol{K}_\varepsilon$. The complementary work equation can now be written down as:

$$\boxed{\iiint\limits_V \delta\boldsymbol{\sigma}^T \boldsymbol{\varepsilon} \, dV - \iint\limits_{S_u} \delta\boldsymbol{p}^T \boldsymbol{u}^o \, dS = 0} \tag{3.31}$$

The expression (3.28) for the complementary energy of a linear-elastic body becomes:

$$\boxed{E_{compl} = \iiint\limits_V \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{C}_\sigma \, \boldsymbol{\sigma} \, dV - \iint\limits_{S_u} \boldsymbol{p}^T \boldsymbol{u}^o \, dS} \qquad (\textit{stationary}) \tag{3.32}$$

The volume integral is the deformation work. In many cases the surface integral will be zero, because in practical problems for the supports it holds $\boldsymbol{u}^o = \boldsymbol{0}$. In literature this specific case is referred to as the *principle of minimum deformation work*.

3.6 The force method

The application of the principle of minimum complementary energy actually amounts to it, that in the triplet:

- balance equations
- constitutive equations (3.33)
- kinematic equations

the last one is replaced by the condition that the complementary energy has to be stationary, changing the triplet into:

- balance equations
- constitutive equations (3.34)
- complementary energy stationary

When the equations in the triplet are successively evaluated the *strategy* of the force method is followed. A stress field is assumed that still contains one or more redundants (statisch onbepaalden), which is in equilibrium with the external load. Then the complementary energy is a function of this stress field. By variation with respect to the redundants of this stress field, the compatibility conditions are obtained, which are expressed in the stresses.

Example
The same example will be used as analysed by the displacement method in section 2.6. In that example the truss was treated as a structure with two degrees of freedom. Here the truss will be considered as a structure being statically indeterminate to first degree (see Fig. 3.8).

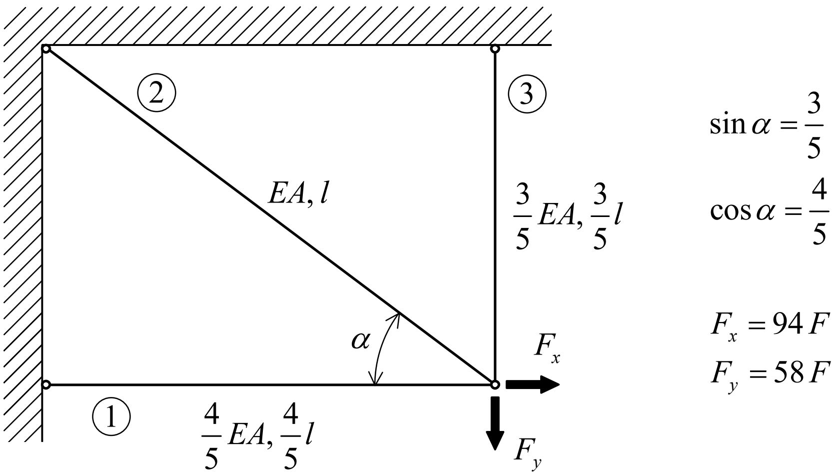


Fig. 3.8: Structure with two degrees of freedom.

For comparison, the calculation will firstly be carried out directly without the use of variational methods.

Direct method
The same kinematic, constitutive and balance equations are used:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{5}{4}\frac{u_x}{l} \\ \varepsilon_2 &= \frac{4}{5}\frac{u_x}{l} + \frac{3}{5}\frac{u_y}{l} \\ \varepsilon_3 &= \frac{5}{3}\frac{u_y}{l} \end{aligned}\right\} \qquad (\textit{kinematic equations})$$

$$\left.\begin{aligned} \varepsilon_1 &= \frac{5}{4}\frac{1}{EA}N_1 \\ \varepsilon_2 &= \frac{1}{EA}N_2 \\ \varepsilon_3 &= \frac{5}{3}\frac{1}{EA}N_3 \end{aligned}\right\} \qquad (\textit{constitutive equations})$$

$$\left.\begin{aligned} N_1 + \frac{4}{5}N_2 &= F_x \\ \frac{3}{5}N_2 + \quad N_3 &= F_y \end{aligned}\right\} \qquad (\textit{balance equations})$$

The force method begins by setting up an internal force distribution that satisfies the balance equations. The bar force N_2 is introduced as the redundant φ. Then the equilibrium equations can be solved for N_1 and N_3:

$$N_1 = F_x - \frac{4}{5}\varphi \quad ; \quad N_2 = \varphi \quad ; \quad N_3 = F_y - \frac{3}{5}\varphi$$

Thus, the bars 1 and 3 form the statically determinate primary system. The deformations follow from the constitutive equations:

$$\varepsilon_1 = \frac{1}{EA}\left(\frac{5}{4}F_x - \varphi\right) \quad ; \quad \varepsilon_2 = \frac{1}{EA}\varphi \quad ; \quad \varepsilon_3 = \frac{1}{EA}\left(\frac{5}{3}F_y - \varphi\right)$$

Using a compatibility condition the redundant can be solved. The values of the three deformations ε_1, ε_2 and ε_3 cannot be changed independently. There is a dependency the relation of which can be obtained from the kinematic equations, by elimination of the degrees of freedom u_x and u_y. This can be done by addition of the three equations after they have been multiplied by respectively the following factors:

$$\left(\frac{4}{5}\right)^2 \quad ; \quad -1 \quad ; \quad \left(\frac{3}{5}\right)^2$$

This delivers:

$$\frac{16}{25}\varepsilon_1 - \varepsilon_2 + \frac{9}{25}\varepsilon_3 = 0$$

In this compatibility condition the previously found relation between $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and F, φ is substituted:

$$\frac{1}{EA}\left\{\frac{16}{25}\left(\frac{5}{4}F_x - \varphi\right) - \varphi + \frac{9}{25}\left(\frac{5}{3}F_y - \varphi\right)\right\} = 0 \quad \rightarrow \quad \frac{1}{EA}\left\{\frac{4}{5}F + \frac{3}{5}F_y - 2\varphi\right\} = 0$$

So, for the redundant φ it follows:

$$\varphi = \frac{2}{5}F_x + \frac{3}{10}F_y$$

In this example the force components are:

$$F_x = 94F \quad ; \quad F_y = 58F$$

Therefore:

$$\varphi = 55F$$

The normal forces then become:

$$N_1 = 94F - \frac{4}{5}\varphi = 50F \quad ; \quad N_2 = 55F \quad ; \quad N_3 = 58F - \frac{3}{5}\varphi = 25F$$

This is the same solution as found by the displacement method. In this example the values for u_x and u_y directly follow from the kinematic relations:

$$u_x = \frac{4}{5} l\,\varepsilon_1 = \frac{4}{5} l \frac{1}{EA}\left(\frac{235}{2}F - 55F\right) = 50\frac{Fl}{EA}$$
$$u_y = \frac{3}{5} l\,\varepsilon_3 = \frac{3}{5} l \frac{1}{EA}\left(\frac{290}{3}F - 55F\right) = 25\frac{Fl}{EA}$$

Variational method
In the variational method the compatibility condition is replaced by the minimisation of the complementary energy. Because no prescribed displacements different from zero are present, it holds:

$$E_{compl} = \int_0^{l_1} \frac{1}{2EA_1} N_1^2 dx + \int_0^{l_2} \frac{1}{2EA_2} N_2^2 dx + \int_0^{l_3} \frac{1}{2EA_3} N_3^2 dx$$

In this case the expression for the deformation work becomes:

$$E_{compl} = \frac{1}{2}\frac{5}{4}\frac{1}{EA}\left(F_x - \frac{4}{5}\varphi\right)^2 \frac{4}{5} l + \frac{1}{2}\frac{1}{EA}\varphi^2 l + \frac{1}{2}\frac{5}{3}\frac{1}{EA}\left(F_y - \frac{3}{5}\varphi\right)^2 \frac{3}{5} l \quad \rightarrow$$
$$E_{compl} = \frac{1}{2}\frac{1}{EA}\left\{F_x^2 + F_y^2 - \left(\frac{8}{5}F_x + \frac{6}{5}F_y\right)\varphi + 2\varphi^2\right\}$$

Variation with respect to φ delivers:

$$\frac{\partial E_{compl}}{\partial \varphi} = \frac{1}{2}\frac{l}{EA}\left(-\frac{8}{5}F_x - \frac{6}{5}F_y + 4\varphi\right) = 0$$

where $F_x = 95F$ and $F_y = 58F$, leading to:

$$\varphi = 55F$$

This result was found by the direct method too. Therefore, the normal forces are the same too. When the displacements are required as well, u_x and u_y can be considered as prescribed displacements and F_x and F_y as support reactions. Then two extra redundants are introduced and it can be written:

$$E_{compl} = \frac{1}{2}\frac{l}{EA}\left\{\left(F_x - \frac{4}{5}\varphi\right)^2 + \varphi^2 + \left(F_y - \frac{3}{5}\varphi\right)^2\right\} - u_x F_x - u_y F_y \quad \rightarrow$$

$$E_{compl} = \frac{1}{2}\frac{l}{EA}\left\{\left(F_x^2 + F_y^2\right) - \left(\frac{8}{5}F_x + \frac{6}{5}F_y\right)\varphi + 2\varphi^2\right\} - u_x F_x - u_y F_y$$

where φ, F_x and F_y are the redundants.
Variation with respect to φ, F_x and F_y respectively gives:

$$-\left(\frac{8}{5}F_x + \frac{6}{5}F_y\right) + 4\varphi = 0 \quad ; \quad \frac{l}{EA}\left(F_x - \frac{4}{5}\varphi\right) - u_x = 0 \quad ; \quad \frac{l}{EA}\left(F_y - \frac{3}{5}\varphi\right) - u_y = 0$$

From the first relation, with $F_x = 94F$ and $F_y = 58F$ it follows again:

$$\varphi = 55F$$

and from the second and third equation:

$$u_x = 50\frac{Fl}{EA} \quad ; \quad u_y = 25\frac{Fl}{EA}$$

3.7 Validity of the virtual complementary work equation

In the form given by (3.16), the complementary work equation is valid for both elastic and plastic materials. After all, the constitutive equation has not been used in the derivation! However, the complementary work equation is not valid for geometrical non-linear problems, because in the derivation a linear kinematic equation has been assumed, see (3.12). So, this is a restriction compared to the virtual work equation of chapter 2.

In summary, the principle of minimum complementary energy was obtained by assuming that a unique relation exists between stresses and strains. Therefore, this principle is only valid for elastic media. In addition, the principle is restricted to geometrical linear problems.

4 The two theorems of Castigliano and the law of Maxwell

When an elastic body is loaded by a number of discrete forces and an identical number of discrete displacements are associated with those forces, the minimum principles can be formulated differently. The principles discussed in this chapter are called the *first and second theorem of Castigliano*. Then from these theorems directly *Maxwell's law of reciprocal deflections* can be derived.

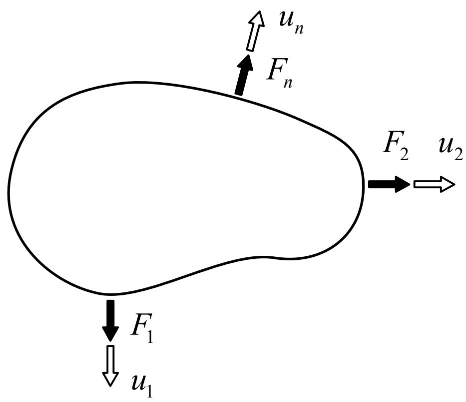


Fig. 4.1: Body subjected to external forces and/or displacements.

In the derivation of the theorems, the distributed volume load P and the distributed surface load p are not considered, only concentrated forces and displacements are taken into account (see Fig. 4.1). Likewise, concentrated moments can be considered together with the corresponding rotations. Generally one may speak about a discrete number of generalised forces with a matching number of generalised displacements.

4.1 The first theorem of Castigliano

The elastic construction is loaded by n forces $F_1, F_2, \cdots, F_n$ in different arbitrary directions. The displacements in those directions are respectively $u_1, u_2, \cdots, u_n$. The potential energy for the structure consists out of a contribution E_s of the internal deformation energy and a contribution E_p of the position of the external load, i.e.:

$$E_{pot} = E_s + E_p \tag{4.1}$$

The internal energy E_s is a function of the strains ε of the structure:

$$E_s = \iiint_V E_s'(\varepsilon)\, dV \tag{4.2}$$

The kinematic relations relate the strains ε to the displacements $u_1, u_2, \cdots, u_n$, transforming E_s into an expression of the displacements:

$$E_s = E_s(u_1, u_2, \cdots, u_n) \tag{4.3}$$

In the example of section 2.4 this was already demonstrated.

For a discrete number of forces, the external energy E_p holds:

$$E_p = -F_1 u_1 - F_2 u_2 - \cdots - F_n u_n \tag{4.4}$$

The total potential energy is the sum of the contributions (4.3) and (4.4):

$$E_{pot} = E_s(u_1, u_2, \cdots, u_n) - F_1 u_1 - F_2 u_2 - \cdots - F_n u_n \tag{4.5}$$

The potential energy has to become stationary for variations of each of the discrete displacements u_i:

$$\delta E_{pot} = \sum_{i=1}^{n} \frac{\partial E_{pot}}{\partial u_i} \delta u_i = 0 \quad \rightarrow \quad \delta E_{pot} = \sum_{i=1}^{n} \left(\frac{\partial E_s}{\partial u_i} - F_i \right) \delta u_i = 0 \tag{4.6}$$

From this relation, directly the *first theorem of Castigliano* can be obtained:

$$\boxed{F_i = \frac{\partial E_s}{\partial u_i}} \tag{4.7}$$

Actually, the example in section 2.4 was already an application of this theorem.

4.2 The second theorem of Castigliano

Again a set of forces $F_1, F_2, \cdots, F_n$ and the associated displacements $u_1, u_2, \cdots, u_n$ are considered. In the previous section, the concept was used that forces are applied (cause) and the displacements follow (effect). In this case the displacements will be prescribed (cause) and the forces are resulting support reactions (effect). The complementary energy of the structure consists out of an internal volume part E_c and an external surface part E_c^o on the surface S_u:

$$E_{compl} = E_c + E_c^o \tag{4.8}$$

The volume part of E_{compl} is a function of the stresses σ in the structure:

$$E_c = \iiint_V E_c'(\sigma)\, dV \tag{4.9}$$

Since these stresses satisfy the balance equations a relation is established between the stresses σ and the forces $F_1, F_2, \cdots, F_n$. This means that E_c is a function of those forces:

$$E_c = E_c(F_1, F_2, \cdots, F_n) \tag{4.10}$$

This already was shown in the example of section 3.4.
The complementary energy E_c^o on the surface can now directly be written as:

$$E_c^o = -u_1 F_1 - u_2 F_2 - \cdots - u_n F_n \tag{4.11}$$

The total complementary energy is the sum of the contributions (4.10) and (4.11):

$$E_{compl} = E_c(F_1, F_2, \cdots, F_n) - u_1 F_1 - u_2 F_2 - \cdots - u_n F_n \tag{4.12}$$

The complementary energy has to become stationary for variations of each of the discrete forces F_i:

$$\delta E_{compl} = \sum_{i=1}^{n} \frac{\partial E_{compl}}{\partial F_i} \delta F_i = 0 \quad \rightarrow \quad \delta E_{compl} = \sum_{i=1}^{n} \left(\frac{\partial E_c}{\partial F_i} - u_i \right) \delta F_i = 0 \tag{4.13}$$

From this relation, directly the *second theorem of Castigliano* can be obtained:

$$\boxed{u_i = \frac{\partial E_c}{\partial F_i}} \tag{4.14}$$

Actually, the example in section 3.4 was already an application of this theorem.
For linear-elastic materials E_c is equal to E_s. Therefore, E_s is often used but than as a function of the stresses.

4.3 Maxwell's law of reciprocal deflections

The restriction is made that only linear-elastic materials are considered and that geometrical non-linear effects are excluded. Then the principle of superposition is applicable, and the several resulting effects can be summed up. Every force can be written as a sum of the effects of the (applied) displacements and every displacement can be written as a sum of the effects of the (applied) forces. For the two forces F_i and F_j it is found:

$$\begin{aligned} F_i &= k_{i1} u_1 + k_{i2} u_2 + \cdots + k_{ii} u_i + \cdots k_{ij} u_j + \cdots k_{in} u_n \\ F_j &= k_{j1} u_1 + k_{j2} u_2 + \cdots + k_{ji} u_i + \cdots k_{jj} u_j + \cdots k_{jn} u_n \end{aligned} \tag{4.15}$$

For the two displacements u_i and u_j it follows:

$$\begin{aligned} u_i &= c_{i1} F_1 + c_{i2} F_2 + \cdots + c_{ii} F_i + \cdots c_{ij} F_j + \cdots c_{in} F_n \\ u_j &= c_{j1} F_1 + c_{j2} F_2 + \cdots + c_{ji} F_i + \cdots c_{jj} F_j + \cdots c_{jn} F_n \end{aligned} \tag{4.16}$$

The quantities k_{ij} are stiffness terms and are the coefficients of the stiffness matrix $\boldsymbol{K}$, which relates the vector $\boldsymbol{f}$ containing all forces to the vector $\boldsymbol{u}$ containing all displacements:

$$\boldsymbol{f} = \boldsymbol{K}\,\boldsymbol{u} \tag{4.17}$$

On the other hand, the flexibility terms c_{ij} are the coefficients of the flexibility or compliance matrix $\boldsymbol{C}$:

$$\boldsymbol{u} = \boldsymbol{C}\,\boldsymbol{f} \tag{4.18}$$

The matrix $\boldsymbol{C}$ is the inverse of the matrix $\boldsymbol{K}$.
Now it will be shown that $\boldsymbol{K}$ is symmetrical, i.e. it will be shown that k_{ij} is equal to k_{ji}. On basis of (4.15), for this two coefficients it respectively holds:

$$k_{ij} = \frac{\partial F_i}{\partial u_j} \quad ; \quad k_{ji} = \frac{\partial F_j}{\partial u_i} \tag{4.19}$$

From the fist theorem of Castigliano for F_i and F_j it follows:

$$F_i = \frac{\partial E_s}{\partial u_i} \quad ; \quad F_j = \frac{\partial E_s}{\partial u_j} \tag{4.20}$$

Substitution of these relations in to (4.19) leads to:

$$\boxed{k_{ij} = \frac{\partial^2 E_s}{\partial u_j \partial u_i} \quad ; \quad k_{ji} = \frac{\partial^2 E_s}{\partial u_i \partial u_j}} \tag{4.21}$$

Since the two right-hand sides are identical, it has been proved that:

$$\boxed{k_{ij} = k_{ji}} \tag{4.22}$$

Analogously it can be shown that the matrix $\boldsymbol{C}$ is symmetrical too, i.e. that c_{ij} is equal to c_{ji}. On basis of (4.16) for these two coefficients it holds:

$$c_{ij} = \frac{\partial u_i}{\partial F_j} \quad ; \quad c_{ji} = \frac{\partial u_j}{\partial F_i} \tag{4.23}$$

The second theorem of Castigliano provides:

$$u_i = \frac{\partial E_c}{\partial F_i} \quad ; \quad u_j = \frac{\partial E_c}{\partial F_j} \tag{4.24}$$

Substitution of this result into (4.23) leads to:

$$\boxed{c_{ij} = \frac{\partial^2 E_c}{\partial F_j \partial F_i} \quad ; \quad c_{ji} = \frac{\partial^2 E_c}{\partial F_i \partial F_j}} \tag{4.25}$$

Since the two right-hand sides are identical, it has been proved that:

$$\boxed{c_{ij} = c_{ji}} \tag{4.26}$$

By convention this is called the *law of Maxwell of reciprocal deflections*.

4.4 Remarks

From a historical point of view it is not correct to attribute the second theorem of Castigliano to Castigliano himself. He has derived the theorem only for linear-elastic systems (in 1873). About the same time, Fransesco Grotti who was a friend of Castigliano provided the general proposition (also) valid for non-linear systems. Independent of Grotti, Engesser has derived the proposition in 1889. In both cases this theorem drew little attention, until Westergaard rediscovered it in 1942.

In literature Maxwell's law of reciprocal deflections is called the Law of Betti as well. Betti has derived a similar relation in terms of energy.

4.5 Summary

From chapter 2 up to here a considerable number of equations, principles and laws have been derived. In order to indicate the coherence and field of application, in the table below a schematic overview is given.

field of application	*Displacement method*	*force method*
generally valid in spite of the constitutive property	virtual work equation	complementary virtual work equation
for elastic materials	principle of minimum potential energy	principle of minimum complementary energy
restriction to generalised forces	1st theorem of Castigliano $F_i = \dfrac{\partial E_s}{\partial u_i}$	2nd theorem of Castigliano $u_i = \dfrac{\partial E_c}{\partial F_i}$
further restriction to linear-elastic systems	stiffness matrix $k_{ij} = \dfrac{\partial^2 E_s}{\partial u_i \partial u_j}$ law of Maxwell $k_{ij} = k_{ji}$	flexibility matrix $c_{ij} = \dfrac{\partial^2 E_c}{\partial F_i \partial F_j}$ law of Maxwell $c_{ij} = c_{ji}$

4.6 Application of stiffness and flexibility matrices

The two “recipes” for the determination of the quantities c_{ij} and k_{ij} will be applied to a prismatic beam element, which is loaded at the ends by the moments M_1 and M_2. These moments are associated with the rotations φ_1 and φ_2, respectively (see Fig. 4.2).

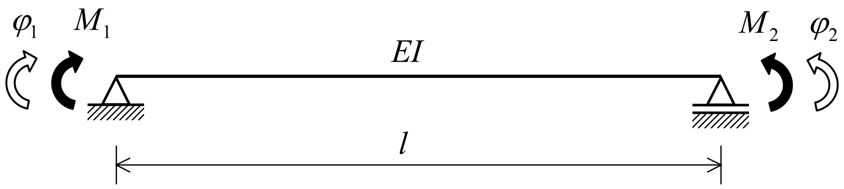


Fig. 4.2: Beam subjected to bending.

Flexibility matrix

The moment distribution is linearly interpolated between M_1 and M_2 (see Fig. 4.3):

$$M(x) = \left(1 - \frac{x}{l}\right) M_1 + \frac{x}{l} M_2$$

The internal complementary energy equals:

$$E_c = \frac{1}{2EI} \int_0^l M^2(x)\,dx \quad \rightarrow$$

$$E_c = \frac{1}{2EI} \left\{ \int_0^l \left(1 - \frac{x}{l}\right)^2 M_1^2\,dx + 2 \int_0^l \frac{x}{l}\left(1 - \frac{x}{l}\right) M_1 M_2\,dx + \int_0^l \left(\frac{x}{l}\right)^2 M_2^2\,dx \right\} \quad \rightarrow$$

$$E_c = \frac{l}{EI} \left(\frac{1}{6} M_1^2 + \frac{1}{6} M_1 M_2 + \frac{1}{6} M_2^2 \right)$$

From this it follows:

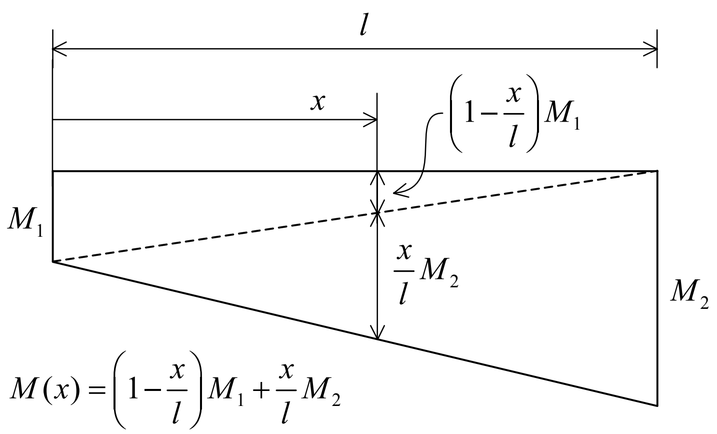


Fig. 4.3: Moment distribution along the beam.

$$c_{11} = \frac{\partial^2 E_c}{\partial M_1^2} = \frac{1}{3}\frac{l}{EI} \quad ; \quad c_{12} = \frac{\partial^2 E_c}{\partial M_1 \partial M_2} = \frac{1}{6}\frac{l}{EI}$$

$$c_{21} = \frac{\partial^2 E_c}{\partial M_2 \partial M_1} = \frac{1}{6}\frac{l}{EI} \quad ; \quad c_{22} = \frac{\partial^2 E_c}{\partial M_2^2} = \frac{1}{3}\frac{l}{EI}$$

The relation between rotations and moments is:

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \frac{l}{EI}\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

Stiffness matrix

Now the stiffness matrix will be derived by prescribing the displacement field $w(x)$. For the distribution $w(x)$ the following function is chosen (see Fig. 4.4):

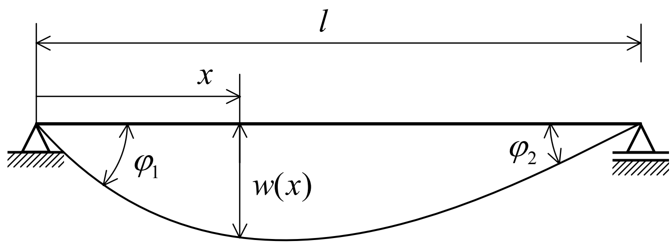


Fig. 4.4: Deflection of beam.

$$w(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

The coefficients a_1 up to a_4 can be determined from the four conditions:

$$x = 0 \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dx} = \varphi_1 \end{cases}$$

$$x = l \quad \rightarrow \quad \begin{cases} w = 0 \\ \dfrac{dw}{dx} = -\varphi_2 \end{cases}$$

For $w(x)$ it then is found:

$$w(x) = x\left(1 - \frac{x}{l}\right)^2 \varphi_1 + \frac{x^2}{l}\left(1 - \frac{x}{l}\right) \varphi_2$$

The curvature becomes:

$$\kappa(x) = -w_{,xx} = \left(\frac{4}{l} - \frac{6x}{l^2}\right)\varphi_1 - \left(\frac{2}{l} - \frac{6x}{l^2}\right)\varphi_2$$

The internal energy equals:

$$E_s = \frac{1}{2EI}\int_0^l \kappa^2(x)\,dx \quad \rightarrow$$

$$E_s = \frac{1}{2EI}\left\{ \int_0^l \left(\frac{4}{l}-\frac{6x}{l^2}\right)^2 \varphi_1^2\,dx - 2\int_0^l \left(\frac{4}{l}-\frac{6x}{l^2}\right)\left(\frac{2}{l}-\frac{6x}{l^2}\right)\varphi_1\,\varphi_2\,dx + \int_0^l \left(\frac{2}{l}-\frac{6x}{l^2}\right)^2 \varphi_2^2\,dx \right\} \quad \rightarrow$$

$$E_s = \frac{EI}{l}\left(2\varphi_1^2 - 2\varphi_1\varphi_2 + 2\varphi_2^2\right)$$

From this it follows:

$$k_{11} = \frac{\partial^2 E_s}{\partial \varphi_1^2} = 4\frac{EI}{l} \quad ; \quad k_{12} = \frac{\partial^2 E_s}{\partial \varphi_1 \partial \varphi_2} = -2\frac{EI}{l}$$

$$k_{21} = \frac{\partial^2 E_s}{\partial \varphi_2 \partial \varphi_1} = -2\frac{EI}{l} \quad ; \quad k_{22} = \frac{\partial^2 E_s}{\partial \varphi_2^2} = 4\frac{EI}{l}$$

The relation between moments and rotations is:

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \frac{EI}{l}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}$$

As expected, the product of the found stiffness and flexibility matrices appears to be equal to the unit matrix, which means that they are each other's inverse.

In this example a choice has been made for the distributions of $M(x)$ and $w(x)$. The assumptions made were correct, because for a beam subjected to bending in the absence of a distributed load along the beam, it holds:

$$\frac{d^2M}{dx^2} = 0 \quad ; \quad EI\frac{d^4w}{dx^4} = 0$$

The chosen interpolations satisfy these conditions.

4.7 Applications of the theorems of Castigliano

First theorem: example 1
The first theorem of Castigliano is just as the principle of minimum potential energy also valid for elastic *geometrical non-linear* systems. For a change, such a problem is addressed in this example.
Two horizontal bars AC and CB are considered, which are connected by a hinge (see Fig. 4.5). The points A and B are connected to static hinged supports. At point C a vertical force F is applied, causing this point to be displaced by w.

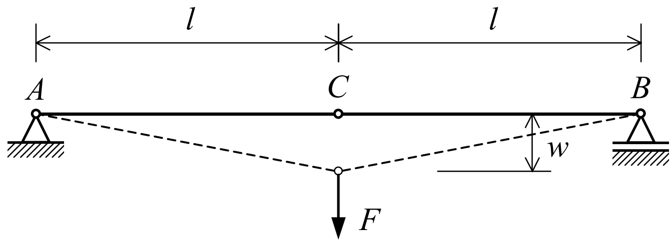


Fig. 4.5: Simply supported hinged bars.

The aim is to determine the relation between F and w.
In the deformed state the elongation e of each bar equals:

$$e = \sqrt{l^2 + w^2} - l = l\left(\sqrt{1 + \left(\frac{w}{l}\right)^2} - 1\right)$$

The ratio w/l is small compared to unity, this means that $(w/l)^2 \ll 1$. Therefore, the square root can be replaced by the following binomial series:

$$\sqrt{1 + \left(\frac{w}{l}\right)^2} = 1 + \frac{1}{2}\left(\frac{w}{l}\right)^2 + \cdots$$

The elongation and the normal force then become:

$$e = \frac{1}{2}\frac{w^2}{l} \quad ; \quad N = \frac{EA}{l}\,e$$

Subsequently, the deformation energy for both bars follows from:

$$E_s = 2\left(\frac{1}{2}\frac{EA}{l}\,e^2\right) = \frac{1}{4}\frac{EA}{l^3}\,w^4$$

Application of the first theorem of Castigliano provides:

$$F = \frac{dE_s}{dw} = \frac{EA}{l^3}\,w^3$$

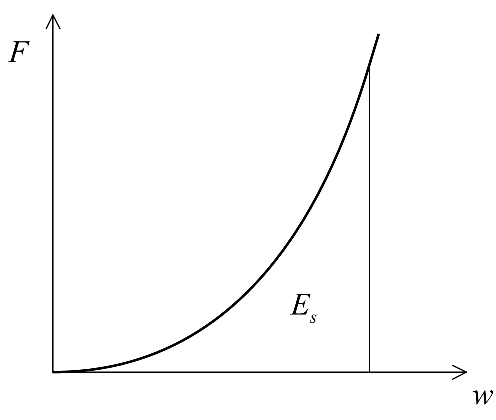


Fig. 4.6: Relation between force and deflection.

Which is the required relation between force F and displacement w. The load F can only be carried after the generation of a displacement w (see Fig. 4.6). In the initial state no force can be carried since the curve is tangent to the w-axis. Further because of the nature of the third-order curve, initially large displacements only provide small forces. For increasing deformation the structure becomes stiffer.

Second theorem: Example 2

A uniformly distributed load f, as shown in Fig. 4.7, loads a prismatic beam on three supports of equal span l. For this statically indeterminate problem to the first degree the stress distribution will be obtained.

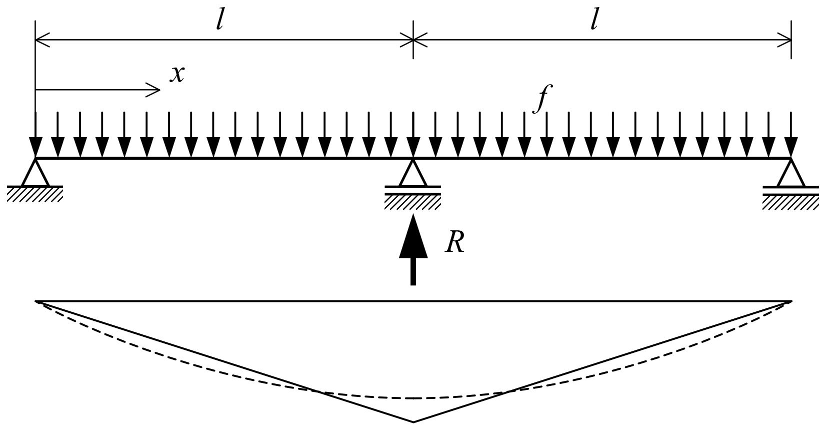


Fig. 4.7: Uniformly loaded beam on three simple supports with redundant R.

The reaction R in the central support is introduced as the redundant. The moment line is the superposition of a parabolic distribution caused by the distributed load f and a triangular distribution caused by the reaction R.
At a distance x from the end, the moment in the left field equals:

$$M(x)=\frac{1}{2}f\,x\left(2l-x\right)-\frac{1}{2}R\,x$$

The complementary energy becomes (for two fields):

$$E_c=2\left\{\int_0^l \frac{1}{2}\frac{M^2(x)}{EI}dx\right\}=\frac{1}{EI}\int_0^l\left(\frac{1}{2}f\,x\left(2l-x\right)-\frac{1}{2}Rx\right)^2 dx \quad\rightarrow$$

$$E_c=\frac{f^2}{4EI}\int_0^l x^2\left(2l-x\right)^2 dx-\frac{fR}{2EI}\int_0^l x^2\left(2l-x\right)dx+\frac{R^2}{4EI}\int_0^l x^2 dx \quad\rightarrow$$

$$E_c=\frac{2l^5}{15EI}f^2-\frac{5l^4}{24EI}fR+\frac{l^3}{12EI}R^2$$

where no deformation energy caused by the transverse force has been taken into account. Application of the second theorem of Castigliano delivers the displacement of the middle support. Because this displacement is equal to zero it holds:

$$\frac{dE_c}{dR} = 0$$

So, in this case the second theorem of Castigliano is identical to the principle of minimum deformation work. Differentiation of the complementary energy yields:

$$-\frac{5l^4}{24EI} f + \frac{l^3}{6EI} R = 0 \quad \rightarrow \quad R = \frac{5}{4} fl$$

Now the force distribution is known. The moment of the middle support becomes:

$$M(x = l) = -\frac{1}{8} fl^2$$

One integration could have been avoided, if the following approach had been chosen:

$$E_c = \frac{1}{EI} \int_0^l M^2(x)\, dx$$

$$\frac{dE_c}{dR} = \frac{1}{EI} \int_0^l 2M(x) \frac{dM(x)}{dR} dx = \frac{1}{EI} \int_0^l \left\{ fx(2l - x) - Rx \right\} \left(-\frac{1}{2} x \right) dx = 0 \quad \rightarrow$$

$$-\frac{1}{2} \frac{f}{EI} \int_0^l x^2 (2l - x)\, dx + \frac{R}{2EI} \int_0^l x^2 dx = 0 \quad \rightarrow \quad -\frac{5}{24} \frac{fl^4}{EI} + \frac{1}{6} \frac{Rl^3}{EI} \quad \rightarrow \quad R = \frac{5}{4} fl$$

Instead of R another choice for the redundant can be made, namely the moment M in the beam at the middle support (see Fig. 4.8: $M > 0$). Now the gap between the two beam parts above the support has to be zero, i.e.:

$$\frac{dE_c}{dM} = \frac{d}{dM} \left(\frac{1}{EI} \int_0^l M^2(x)\, dx \right) = \frac{2}{EI} \int_0^l M(x) \frac{dM(x)}{dM} dx = 0$$

At distance x from the end it holds:

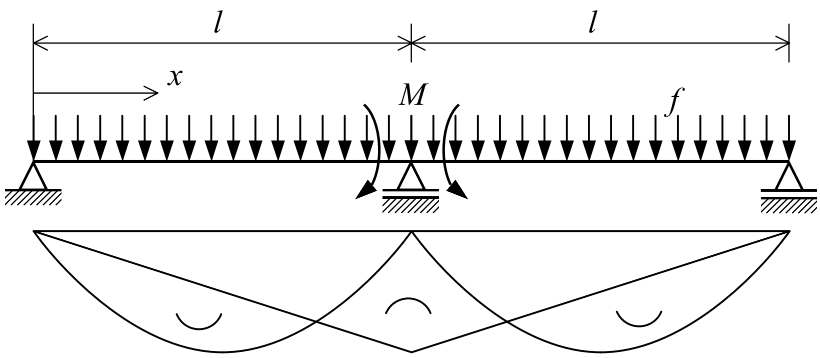


Fig. 4.8: Uniformly loaded beam on three simple supports with redundant M.

$$M(x) = \frac{1}{2} f x (l - x) - \frac{x}{l} M$$

Substitution of this relation provides:

$$\frac{2}{EI}\left(-\frac{f}{l} \int_0^l \frac{1}{2} x^2 (l - x)\, dx + \frac{M}{l^2} \int_0^l x^2 dx \right) = 0 \quad \rightarrow \quad \frac{2}{EI}\left(\frac{fl^3}{24} + \frac{Ml}{3} \right) = 0$$

Which shows that the gap caused by f is exactly compensated by the gap caused by M, so compatibility is satisfied. From above relation it finally follows:

$$M = \frac{1}{8} f l^2$$

Second theorem: Example 3

The second theorem of Castigliano is often used to calculate the stress distribution in arches. In this example a circular ring is chosen, which is loaded by two opposite forces $2F$ as shown in Fig. 4.9. This problem is statically indeterminate to the first degree. As the

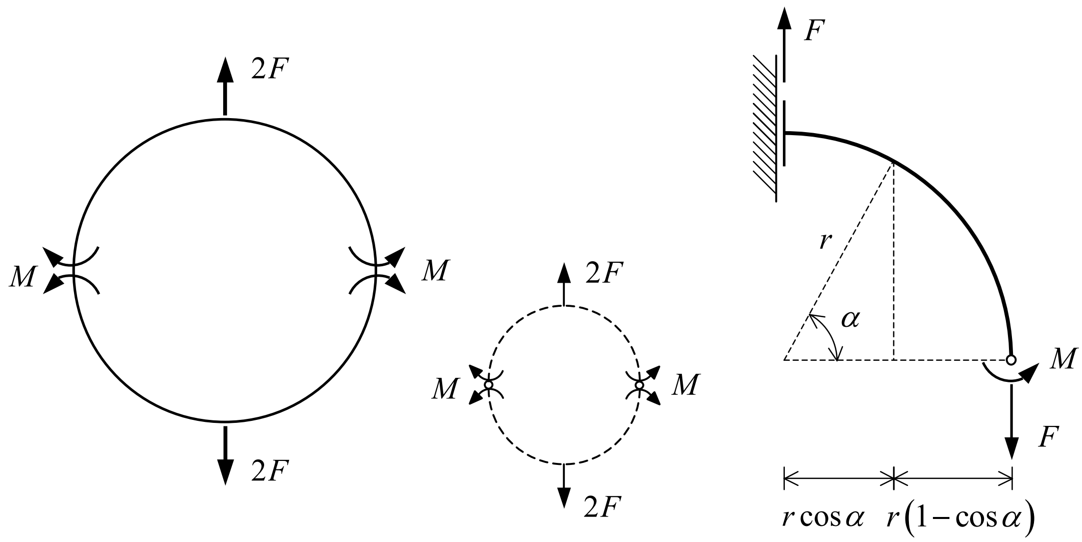


Fig. 4.9: Calculation of moment distribution in test ring.

redundant, the moment is introduced halfway the height of the ring. For reasons of symmetry only a quarter of the ring needs to be considered. In the quadrant as drawn, the moment is called positive if it causes tensile stresses at the inner side of the ring. It holds:

$$M(\alpha) = M - (1 - \cos\alpha) rF \quad ; \quad E_c = \frac{1}{2} \int_0^{\pi/2} \frac{1}{EI} M^2(\alpha)\, r\, d\alpha$$

The derivative with respect to M has to be zero, because the slope is zero at the end where the moment M is applied:

$$\frac{dE_c}{dM} = \frac{r}{EI} \int_0^{\pi/2} M(\alpha) \frac{dM(\alpha)}{dM} d\alpha = 0$$

The integral equals:

$$\int_0^{\pi/2} \left\{ M - \left(1 - \cos\alpha\right) rF \right\} d\alpha = 0 \quad \rightarrow \quad M \int_0^{\pi/2} d\alpha - rF \int_0^{\pi/2} \left(1 - \cos\alpha\right) d\alpha = 0$$

$$\frac{\pi}{2} M - \left(\frac{\pi}{2} - 1\right) rF = 0 \quad \rightarrow \quad M = \left(1 - \frac{2}{\pi}\right) rF$$

Assignment

Prove with the second theorem of Castigliano that the displacement of the end in the direction of F is equal to $\left(\pi^2 - 8\right) Fr^3 \big/ \left(4\pi EI\right)$.

5 Variational methods and differential equations

The variational methods for potential and complementary energy can be used to derive differential equations with corresponding dynamic boundary conditions. The principle of minimum potential energy provides differential equations with respect to unknown displacements and from the principle of minimum complementary energy differential equations with respect to unknown stress functions can be obtained. Variational methods can be used if the derivation of the differential equation by a direct method is not very transparent. This chapter is confined to examples using the principle of minimum potential energy. In addition only simple linear structures are considered, for which a direct solution is known as well. This offers the advantage that the validity of the found results can be tested.

5.1 Beam subjected to extension (displacement method)

A prismatic beam is considered with a continuous axial stiffness *EA* as shown in Fig. 5.1. At end 1 the beam is restrained. Along the beam a distributed load $f(x)$ per unit of length is acting. At end 2 an external concentrated force F_2 is applied.

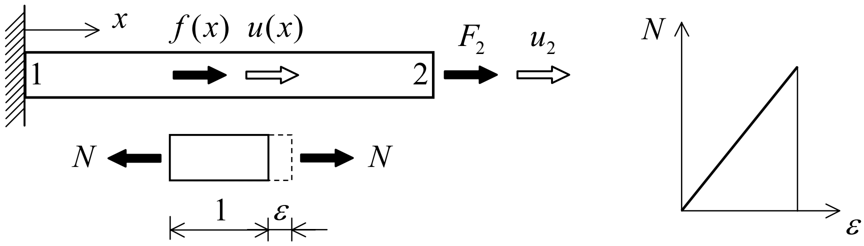


Fig. 5.1: Bar loaded by a tensile force.

The question now is to set up the balance equation together with the dynamic boundary condition at end 2 and to express these in the displacement field $u(x)$.

Direct method
The triplet of equations reads:

$$
\begin{aligned}
&\varepsilon = u_{,x} && \text{(\textit{kinematic equation})}\\
&N = EA\,\varepsilon && \text{(\textit{constitutive equation})}\\
&N_{,x} + f = 0 && \text{(\textit{equilibrium equation})}
\end{aligned}
\qquad (5.1)
$$

Successive substitution in the direction of the arrow transforms the equilibrium equation into:

$$\boxed{EA\,u_{,xx} + f = 0} \qquad (5.2)$$

The dynamic boundary condition at end 2 between the internal normal force N_2 and the external force F_2 reads:

$$-N_2 + F_2 = 0$$

With $N = EA\varepsilon$ and $\varepsilon = u_{,x}$ this boundary condition becomes:

$$\boxed{-EAu_{,x_2} + F_2 = 0} \tag{5.3}$$

Variational method

The expression for the potential energy equals:

$$E_{pot} = \int_1^2 \frac{1}{2} N\varepsilon \, dx - \int_1^2 fu \, dx - F_2 u_2$$

Together with $N = EA\varepsilon$ and $\varepsilon = u_{,x}$ this relation becomes an expression of the unknown displacement field and the known external load:

$$E_{pot} = \frac{1}{2} \int_1^2 EA\left(u_{,x}\right)^2 dx - \int_1^2 fu \, dx - F_2 u_2 \tag{5.4}$$

More precisely formulated, the potential energy is a functional in which $u_{,x}$ and u appear as variables, which in their turn are functions of x. In short it can be written:

$$E_{pot} = E_{pot}(u_{,x}\,;\, u\,;\, x) \tag{5.5}$$

This quantity has to be stationary with respect to variations of the displacement field. The variational process involves both δu and its derivative $\delta u_{,x}$. The variation δu is chosen such that it is zero at the position where the displacement u is prescribed. In this case, this is end 1, so it holds $\delta u_1 = 0$ (see Fig. 5.2).

From the potential energy expression (5.5), the following variation can be derived:

$$\delta E_{pot} = \frac{\partial E_{pot}}{\partial u_{,x}} \delta u_{,x} + \frac{\partial E_{pot}}{\partial u} \delta u = 0 \tag{5.6}$$

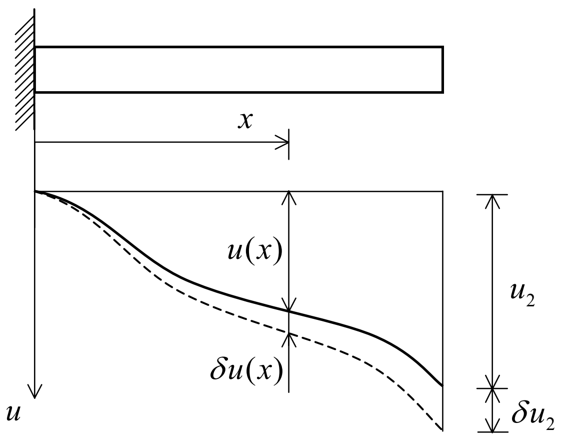


Fig. 5.2: Kinematically admissible displacement field.

From (5.4) it can be found:

$$\frac{\partial E_{pot}}{\partial u_{,x}}\delta u_{,x} = \frac{1}{2}\int_1^2 2EA\,u_{,x}\,\delta u_{,x}\,dx \quad ; \quad \frac{\partial E_{pot}}{\partial u}\delta u = -\int_1^2 f\,\delta u\,dx - F_2\,\delta u_2 = 0$$

Which transforms (5.6) into:

$$\delta E_{pot} = \int_1^2 EA\,u_{,x}\,\delta u_{,x}\,dx - \int_1^2 f\,\delta u\,dx - F_2\,\delta u_2 = 0 \qquad (5.7)$$

This expression will be rephrased such that in the right-hand side only δu appears and no variation of the derivative $\delta u_{,x}$. This can be achieved by partial integration of the first integral:

$$\int_1^2 \left(EA\,u_{,x}\right)\delta u_{,x}\,dx = -\int_1^2 EA\,u_{,xx}\,\delta u\,dx - EA\,u_{,x_1}\,\delta u_1 + EA\,u_{,x_2}\,\delta u_2$$

Substitution of this result into (5.7) together with the condition that δu_1 is zero yields:

$$\delta E_{pot} = -\int_1^2 \left(EA\,u_{,xx} + f\right)\delta u\,dx - \left(-EA\,u_{,x_2} + F_2\right)\delta u_2 = 0 \qquad (5.8)$$

If this relation has to be satisfied for <u>every arbitrary</u> kinematically allowable variation δu, then for every position x along the beam it should hold:

$$\boxed{EA\,u_{,xx} + f = 0} \qquad (5.9)$$

and also at end 2:

$$\boxed{-EA\,u_{,x_2} + F_2 = 0} \qquad (5.10)$$

These are exactly the same results as derived by the direct method in this section.

Solution

For the sake of completeness, a solution of the differential equation is given too. It is assumed that the distributed load f is uniform and that the force F_2 is equal to zero. A particular solution then is:

$$u(x)_{\text{part}} = -\frac{f}{2EA}x^2$$

The homogeneous solution equals:

$$u(x)_{\text{hom}} = a_1 + a_2 x$$

From the total solution given by:

$$u(x) = a_1 + a_2 x - \frac{f}{2EA} x^2$$

The constants a_1 and a_2 are solved from the two boundary conditions:

$$x = 0 \quad \rightarrow \quad u_1 = 0 \qquad ; \qquad x = l \quad \rightarrow \quad EA\, u_{,x_2} = 0$$

This delivers:

$$a_1 = 0 \quad ; \quad a_2 = \frac{fl}{EA}$$

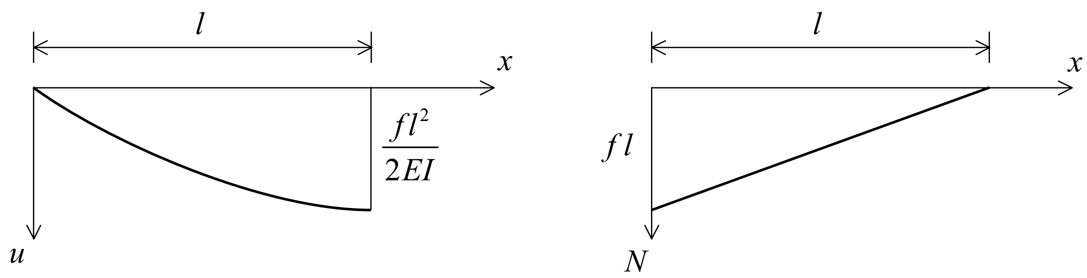


Fig. 5.3: Deflection and normal force along the bar.

This means that as shown in Fig. 5.3 the functional relations of the displacement and normal force are:

$$u(x) = \frac{fl^2}{EA}\left(\frac{x}{l} - \frac{1}{2}\frac{x^2}{l^2}\right) \quad ; \quad N(x) = EA\, u_{,x} = fl\left(1 - \frac{x}{l}\right) \tag{5.11}$$

5.2 Beam subjected to bending (displacement method)

As shown in Fig. 5.4, again a prismatic beam is considered, now with a constant bending stiffness EI. The beam is fixed at end 1. At that position, the kinematic boundary conditions are that both the displacement and the slope are equal to zero. The external load consists out of a distributed load $f(x)$ per unit of length and a force F_2 and moment T_2 at end 2 of the beam.

The question now is to set up the balance equation together with the dynamic boundary condition at end 2 and to express these in the displacement field $u(x)$, which in this case is perpendicular to the beam axis. It is assumed that the deformation caused by the transverse force can be neglected.

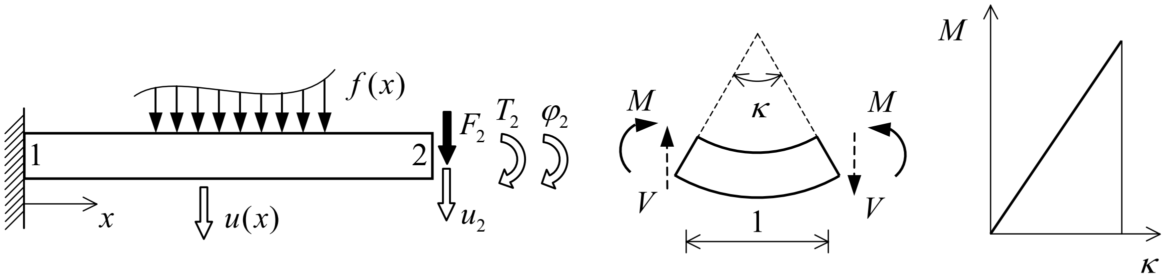


Fig. 5.4: Beam subjected to bending.

Direct method
The triplet of equations reads:

$$\begin{aligned} &\kappa = -u_{,xx} && (\textit{kinematic equation}) \\ &M = EI\,\kappa && (\textit{constitutive equation}) \\ &M_{,xx} + f = 0 && (\textit{equilibrium equation}) \end{aligned} \tag{5.12}$$

Successive substitution in the direction of the arrow transforms the equilibrium equation into:

$$\boxed{-EI\,u_{,xxxx} + f = 0} \tag{5.13}$$

The two dynamic boundary conditions at end 2 are:

$$-V_2 + F_2 = 0 \quad ; \quad M_2 + T_2 = 0$$

With $M = EI\,\kappa$, $\kappa = -u_{,xx}$ and $V = M_{,x}$ these boundary conditions become:

$$\boxed{EI\,u_{,xxx_2} + F_2 = 0 \quad ; \quad -EI\,u_{,xx_2} + T_2 = 0} \tag{5.14}$$

Variational method
The expression for the potential energy equals:

$$E_{pot} = \int_1^2 \frac{1}{2} M \kappa\, dx - \int_1^2 f u\, dx - F_2 u_2 - T_2 \varphi_2$$

Together with $M = EI\,\kappa$ and $\kappa = -u_{,xx}$ this becomes an expression of the displacement field:

$$E_{pot} = \frac{1}{2} \int_1^2 EI \left(u_{,xx}\right)^2 dx - \int_1^2 f u\, dx - F_2 u_2 - T_2 \varphi_2 \tag{5.15}$$

In short:

$$E_{pot} = E_{pot}(u_{,xx}\,;\,u\,;\,x) \tag{5.16}$$

The variation of this expression is:

$$\delta E_{pot} = \frac{\partial E_{pot}}{\partial u_{,xx}}\delta u_{,xx} + \frac{\partial E_{pot}}{\partial u}\delta u = 0 \tag{5.17}$$

Together with (5.15) it can be derived:

$$\delta E_{pot} = \int_1^2 EI\,u_{,xx}\delta u_{,xx}\,dx - \int_1^2 f\,\delta u\,dx - F_2\delta u_2 - T_2\delta\varphi_2 = 0 \tag{5.18}$$

In order to arrive at an expression only containing variations δu and no variation of $u_{,xx}$, the first integral of the right-hand side has to be partially integrated twice. The first partial integration provides:

$$\int_1^2 EI\,u_{,xx}\delta u_{,xx}\,dx = -\int_1^2 EI\,u_{,xxx}\delta u_{,x}\,dx - EI\,u_{,xx_1}\delta u_{,x_1} + EI\,u_{,xx_2}\delta u_{,x_2}$$

Since the variation of u has to satisfy the kinematic boundary condition at end 1 (at that position it holds $\varphi_1 = u_{,xl} = 0$), the variation $\delta u_{,x_1}$ has to be zero and above expression becomes:

$$\int_1^2 EI\,u_{,xx}\delta u_{,xx}\,dx = -\int_1^2 EI\,u_{,xxx}\delta u_{,x}\,dx + EI\,u_{,xx_2}\delta\varphi_2$$

where it has been used that $\delta u_{,x_2} = \delta\varphi_2$.
Partial integration for the second time provides:

$$\int_1^2 EI\,u_{,xx}\delta u_{,xx}\,dx = \int_1^2 EI\,u_{,xxxx}\delta u\,dx + EI\,u_{,xxx_1}\delta u_1 - EI\,u_{,xxx_2}\delta u_2 + EI\,u_{,xx_2}\delta\varphi_2$$

In the right-hand side δu_1 is equal to zero because u_1 is prescribed. Substitution of this result into (5.18) yields:

$$\delta E_{pot} = -\int_1^2 \left(-EI\,u_{,xxxx} + f\right)\delta u\,dx - \left(EI\,u_{,xxx_2} + F_2\right)\delta u_2 - \left(-EI\,u_{,xx_2} + T_2\right)\delta\varphi_2 = 0 \tag{5.19}$$

This relation only can be satisfied for <u>all</u> variations if along the beam it holds:

$$\boxed{-EI\,u_{,xxxx} + f = 0} \tag{5.20}$$

and also at the end 2:

$$\boxed{EI\,u_{,xxx_2} + F_2 = 0 \quad ; \quad -EI\,u_{,xx_2} + T_2 = 0} \tag{5.21}$$

These are exactly the same results as derived by the direct method in this section.

Solution

Also in this case a solution is provided for a uniformly distributed load f. At end 2 the force F_2 and the moment T_2 are set to zero. A particular solution then is:

$$u(x)_{\text{part}} = \frac{f}{24EI}x^4$$

The homogeneous solution equals:

$$u(x)_{\text{hom}} = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

From the total solution given by:

$$u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \frac{f}{24EI}x^4$$

The constants a_1, a_2, a_3 and a_4 are solved from the four boundary conditions:

$$x = 0 \;\rightarrow\; \begin{cases} u_1 = 0 \\ u_{,x} = 0 \end{cases} \quad ; \quad x = l \;\rightarrow\; \begin{cases} EI\,u_{,xx} = 0 \\ EI\,u_{,xxx} = 0 \end{cases}$$

This delivers:

$$a_1 = 0 \quad ; \quad a_2 = 0 \quad ; \quad a_3 = \frac{fl}{4EI} \quad ; \quad a_4 = -\frac{fl^2}{6EI}$$

So that as displayed in Fig. 5.5, the required displacement function becomes:

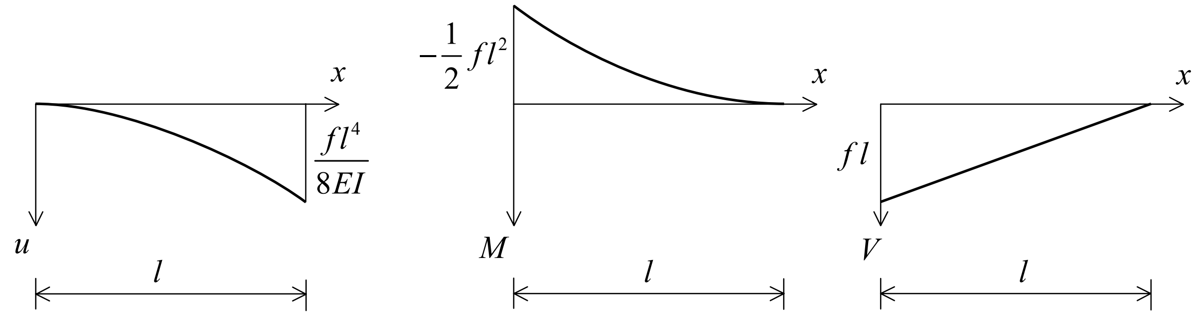


Fig. 5.5: Deflection, moment and shear force along the beam.

$$u(x) = \frac{fl^4}{EI}\left(\frac{1}{4}\frac{x^2}{l^2} - \frac{1}{6}\frac{x^3}{l^3} + \frac{1}{24}\frac{x^4}{l^4}\right) \tag{5.22}$$

From which it follows:

$$M(x) = -EI\,u_{,xx} = fl^2\left(-\frac{1}{2} + \frac{x}{l} - \frac{1}{2}\frac{x^2}{l^2}\right)$$

The derivative of which equals:

$$V(x) = M_{,x} = fl\left(1 - \frac{x}{l}\right)$$

6 Approximated solutions

In chapter 2, the concept of weighted residuals was introduced where the equilibrium equation was replaced by the following requirement:

$$\iiint_V R\,\delta u\,dV + \iint_{S_p} r\,\delta u\,dS = 0 \tag{6.1}$$

This condition has to be satisfied for <u>every</u> kinematically admissible variation δu, which is only the case if R is zero in all point of the volume V and r is zero in all points of the surface S_p. Therefore, the fulfilment of this condition delivers the exact solution.
The same idea holds for the replacement of the compatibility condition in chapter 3 by the requirement that:

$$\iiint_V R\,\delta\sigma\,dV + \iint_{S_u} r\,\delta p\,dS = 0 \tag{6.2}$$

This condition has to be satisfied for <u>all</u> statically admissible variations $\delta\sigma$. This again means that R is zero in all points of V and r is zero in all points of S_u, and also in this case the exact solution is obtained.

The replacing functional conditions (6.1) and (6.2) do not necessarily have to be utilised to obtain the exact solution. They also offer the possibility to generate approximated solutions. This can be achieved as follows. In chapter 2 and 3, the replacing functional conditions are converted and reduced to the minimum of potential energy and the minimum of complementary energy. In both cases, the condition is that a functional $I(x,y,z)$ has to be minimised, i.e.:

$$I(x,y,z) = \text{ minimal} \tag{6.3}$$

For the principle of minimum potential energy the functional I is equal to E_{pot} and for the principle of minimum complementary energy the functional I is equal to E_{compl}. The minimisation of E_{pot} and E_{compl} provides the solutions of $u(x,y,z)$ and $\sigma(x,y,z)$, respectively. In general, the solution is indicated by $s(x,y,z)$. When a method of approximation is applied, the solution is written as the sum of a finite number (n) of separate functions:

$$s(x,y,z) = \sum_{i=1}^{n} a_i b_i(x,y,z) \tag{6.4}$$

where a_i is the participation factor for the function $b_i(x,y,z)$.
Every function has to satisfy the imposed conditions. In the displacement method $s(x,y,z)$ represents the displacement field. For this field, the requirement holds that it has to be compatible in V, and that it has to satisfy the kinematic boundary conditions on S_u. In the

force method $s(x,y,z)$ represents the stress field. For this field, the condition of equilibrium in V holds, and that the dynamic boundary conditions are satisfied on S_p.
The still unknown parameters a_i can be solved by substitution of $s(x,y,z)$ into $I(x,y,z)$, followed by the minimisation of $I(x,y,z)$ under the condition that for each of the parameters the requirement holds that:

$$\frac{\partial I(x,y,z)}{\partial a_i} = 0 \tag{6.5}$$

This exercise provides a number of equations equal to the amount of unknown parameters. Generally, not the exact minimum value of the functional $I(x,y,z)$ shall be found, but a neighbouring (algebraically) larger value. In the case that the exact solution indeed is found, the collection of functions in (6.4) is called a *complete set*. In coming sections this will be elucidated by a number of examples from the displacement method. A beam submitted to extension and bending will be discussed. For these linear structures it is also quite simple to find the exact solution. However, it should be clear that in practice the method is applied only if it is very difficult or impossible to determine the exact solution.

In order to interpret what exactly is happening, the following can be reflected on. When the solution is obtained from a complete set of functions, the exact solution will be found and for this exact solution the residuals R and r in (6.1) and (6.2) will be zero in every point of the volume V and in every point of the surface S_p or S_u, respectively. When the solution is not a complete set of functions, R and r do not become zero in all points of V and S_p or S_u, respectively. The zeroing of the residuals is achieved only in average sense, and weighted over the volume V and the surface S_p or S_u, the respective residuals will be zero. This weighting procedure takes place through variations of the used finite set of functions.

The method is most frequently applied in connection with the displacement method. Quite old and well known are the methods of Rayleigh (1870) and/or Ritz (1908). A more modern version is the finite element method (FEM). This method became popular (again) with the development of the computer capacity.

6.1 Beam subjected to extension (displacement method)

A prismatic cantilever beam is considered with axial stiffness EA as shown in Fig.6.1. Along the beam a uniformly distributed load f per unit of length is acting. The displacement $u(x)$ and normal force $N(x)$ along the beam will be determined twice, firstly with one parameter a_1 and secondly with two parameters a_1 and a_2.

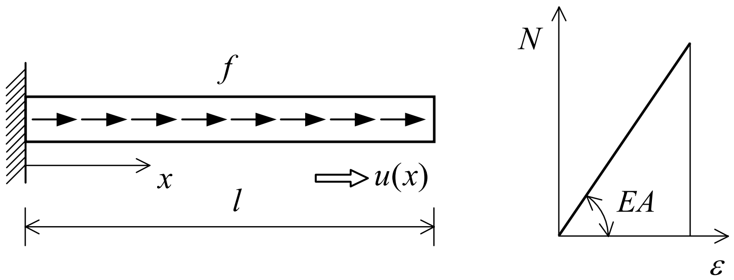


Fig. 6.1: Bar subjected to a uniformly distributed load in axial direction.

Solution with one parameter
In this case the function $u(x)$ becomes:

$$u(x) = a_1 b_1(x)$$

For $b_1(x)$ the function x/l is chosen. This function satisfies the kinematic boundary condition of zero displacement at the restrained end $x = 0$, i.e. (see Fig.6.2):

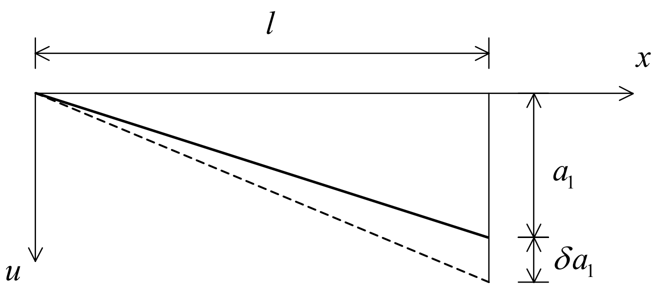


Fig. 6.2: Kinematically admissible displacement field.

$$u(x) = a_1 \frac{x}{l}$$

The strain equals:

$$\varepsilon = u_{,x} = \frac{a_1}{l}$$

and the normal force:

$$N = EA\varepsilon = \frac{EA}{l} a_1$$

The potential energy is:

$$E_{pot} = \frac{1}{2}\int_0^l N\varepsilon\, dx - \int_0^l f u\, dx = \frac{1}{2}\int_0^l \frac{EA}{l^2} a_1^2 dx - \int_0^l f a_1 \frac{x}{l} dx = \frac{1}{2}\frac{EA}{l} a_1^2 - \frac{1}{2} f l\, a_1$$

Variation with respect to a_1 delivers:

$$\delta E_{pot} = \left(\frac{EA}{l} a_1 - \frac{1}{2} f l \right) \delta a_1 = 0 \quad \rightarrow \quad \frac{EA}{l} a_1 - \frac{1}{2} f l = 0$$

From which it follows:

$$a_1 = \frac{1}{2}\frac{f l^2}{EA}$$

The displacement and normal force become:

$$u(x) = \frac{1}{2}\frac{fl}{EA}x \quad ; \quad N(x) = EAu_{,x} = \frac{1}{2}fl$$

In the drawings of Fig. 6.3 this solution is compared with the exact one of section 5.1. It can be seen that the approximation of the displacement is reasonable. At the end, even the exact displacement is predicted. However, the normal force is only calculated properly at the middle of the beam.

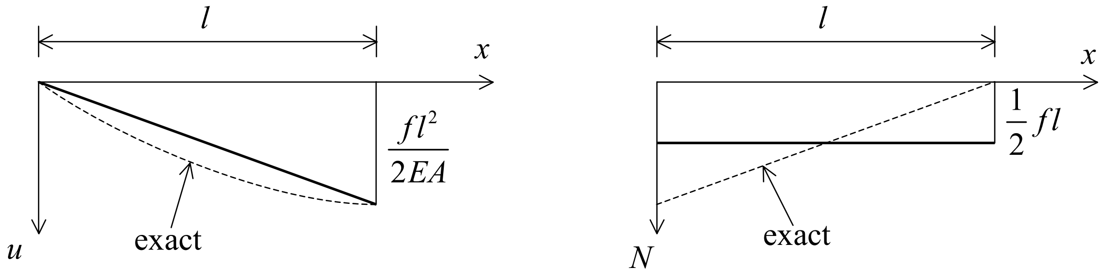


Fig. 6.3: Deflection and normal force along the bar for one parameter.

Solution with two parameters

An extra function $(x/l)^2$ is added to the displacement field:

$$u(x) = a_1\frac{x}{l} + a_2\frac{x^2}{l^2}$$

The strain equals:

$$\varepsilon = u_{,x} = \frac{1}{l}\left(a_1 + 2a_2\frac{x}{l}\right)$$

and the normal force:

$$N = EA\varepsilon = \frac{EA}{l}\left(a_1 + 2a_2\frac{x}{l}\right)$$

The potential energy is:

$$E_{pot} = \frac{1}{2}\int_0^l EA\varepsilon^2\,dx - \int_0^l fu\,dx = \frac{EA}{2l^2}\int_0^l \left(a_1 + 2a_2\frac{x}{l}\right)^2 dx - \int_0^l f\left(a_1\frac{x}{l} + a_2\frac{x^2}{l^2}\right)dx \quad \rightarrow$$

$$E_{pot} = \frac{EA}{2l^2}\int_0^l \left(a_1^2 + 4a_1a_2\frac{x}{l} + 4a_2^2\frac{x^2}{l^2}\right)dx - f\int_0^l \left(a_1\frac{x}{l} + a_2\frac{x^2}{l^2}\right)dx \quad \rightarrow$$

$$E_{pot} = \frac{EA}{l}\left(\frac{1}{2}a_1^2 + a_1a_2 + \frac{2}{3}a_2^2\right) - fl\left(\frac{1}{2}a_1 + \frac{1}{3}a_2\right)$$

Variation with respect to a_1 and a_2 delivers:

$$\delta E_{pot} = \frac{\partial E_{pot}}{\partial a_1}\delta a_1 + \frac{\partial E_{pot}}{\partial a_2}\delta a_2 = 0 \quad \rightarrow$$

$$\delta E_{pot} = \left(\frac{EA}{l}\left(a_1 + a_2\right) - \frac{1}{2} fl\right)\delta a_1 + \left(\frac{EA}{l}\left(a_1 + \frac{4}{3}a_2\right) - \frac{1}{3} fl\right)\delta a_2 = 0 \quad \rightarrow$$

$$\frac{EA}{l}\left(a_1 + a_2\right) = \frac{1}{2} fl \quad ; \quad \frac{EA}{l}\left(a_1 + \frac{4}{3}a_2\right) = \frac{1}{3} fl$$

From which it follows:

$$a_1 = \frac{fl^2}{EA} \quad ; \quad a_2 = -\frac{1}{2}\frac{fl^2}{EA}$$

The solution for the displacement and normal force become:

$$u(x) = \frac{fl^2}{EA}\left(\frac{x}{l} - \frac{1}{2}\frac{x^2}{l^2}\right) \quad ; \quad N(x) = EAu_{,x} = fl\left(1 - \frac{x}{l}\right)$$

This is the exact solution, which means that for this simple example obviously the complete set of functions was used (see Fig. 6.4).

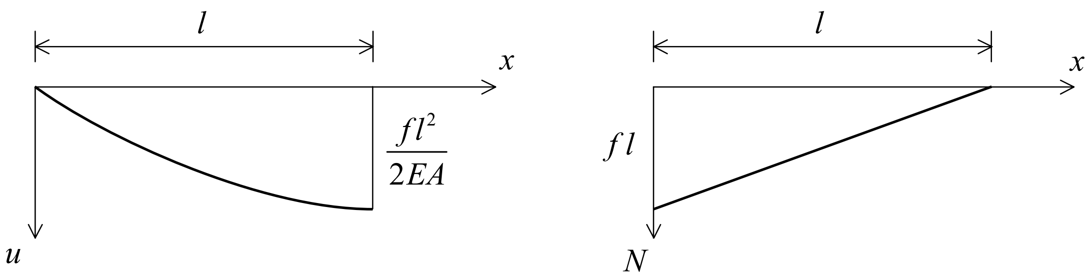


Fig. 6.4: Deflection and normal force along the bar for two parameters.

6.2 Beam subjected to bending (displacement method)

Again the beam is prismatic with flexural stiffness EI and length l. As shown in Fig. 6.5, a distributed load is applied perpendicular to the beam axis. In this example the displacement

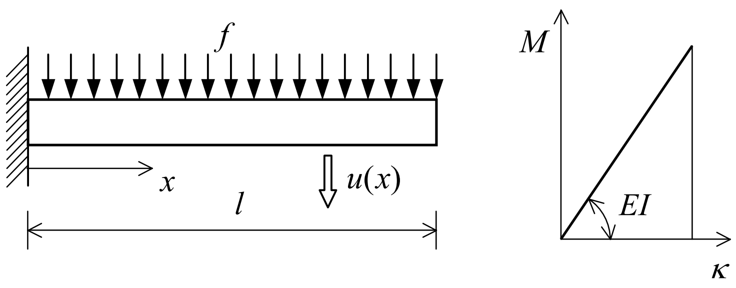


Fig. 6.5: Bar subjected to a uniformly distributed load in transverse direction.

$u(x)$ perpendicular to the beam axis will be calculated, as well as the internal moment $M(x)$. This exercise is carried out for respectively one parameter a_1, two parameters a_1, a_2 and three parameters a_1, a_2, a_3.

Solution with one parameter
The displacement relation $u(x)$ is given by:

$$u(x) = a_1\, b_1(x)$$

For $b_1(x)$ the function $(x/l)^2$ is chosen. The linear function is not considered because it describes a rigid body movement without causing any deformations. The function $(x/l)^2$ satisfies the kinematic boundary conditions at the fixed end, where both u and $u_{,x}$ have to be zero.
Thus, the displacement field equals (Fig. 6.6):

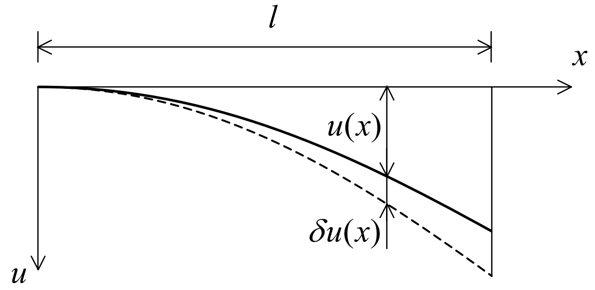


Fig. 6.6: Kinematically admissible displacement field.

$$u(x) = a_1 \frac{x^2}{l^2}$$

The curvature becomes:

$$\kappa = -u_{,xx} = -\frac{2a_1}{l^2}$$

and the moment:

$$M = EI\,\kappa = -\frac{2EI}{l^2} a_1$$

The potential energy becomes:

$$E_{pot} = \frac{1}{2}\int_0^l EI\kappa^2\,dx - \int_0^l fu\,dx = \frac{1}{2}\int_0^l \frac{EI}{l^4}\left(-2a_1\right)^2 dx - \int_0^l f a_1 \frac{x^2}{l^2}\,dx = \frac{2EI}{l^3} a_1^2 - \frac{1}{3} f l\, a_1$$

Variation with respect to a_1 delivers:

$$\delta E_{pot} = \left(\frac{4EI}{l^3} a_1 - \frac{1}{3} fl \right) \delta a_1 = 0 \quad \rightarrow \quad a_1 = \frac{1}{12} \frac{fl^4}{EI}$$

Therefore, the solution reads:

$$u(x) = \frac{fl^2}{12EI} x^2$$

The moment and transverse force are:

$$M(x) = -EI\,u_{,xx} = -\frac{1}{6} fl^2 \quad ; \quad V(x) = M_{,x} = 0$$

Compared with the exact solution from section 5.2, the approximation of the displacement field is not too good and the approximation of the moment is even worse. For the transverse force no value at all has been found (see Fig. 6.7).

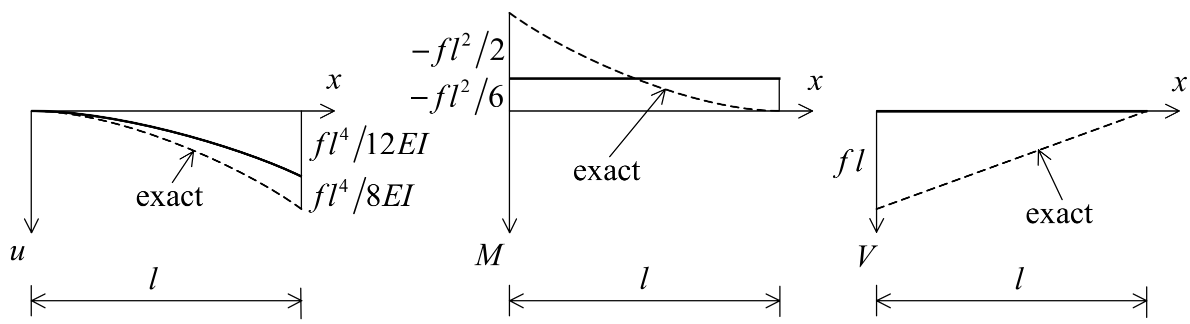


Fig. 6.7: Deflection, moment and shear force for one parameter.

Solution with two parameters
A third order term is added to the displacement field:

$$u(x) = a_1 \frac{x^2}{l^2} + a_2 \frac{x^3}{l^3}$$

The curvature now equals:

$$\kappa = -u_{,xx} = -\frac{1}{l^2} \left(2a_1 + 6a_2 \frac{x}{l} \right)$$

and the moment:

$$M = EI\,\kappa = -\frac{EI}{l^2} \left(2a_1 + 6a_2 \frac{x}{l} \right)$$

The potential energy becomes:

$$E_{pot} = \frac{1}{2}\int_0^l EI\kappa^2\,dx - \int_0^l fu\,dx = \frac{1}{2}\frac{EI}{l^4}\int_0^l \left(2a_1 + 6a_2\frac{x}{l}\right)^2 dx - f\int_0^l \left(a_1\frac{x^2}{l^2} + a_2\frac{x^3}{l^3}\right)dx \quad \rightarrow$$

$$E_{pot} = \frac{1}{2}\frac{EI}{l^3}\left(4a_1^2 + 12a_1a_2 + 12a_2^2\right) - fl\left(\frac{1}{3}a_1 + \frac{1}{4}a_2\right)$$

Variation with respect to a_1 and a_2 delivers:

$$\delta E_{pot} = \left(\frac{1}{2}\frac{EI}{l^3}(8a_1 + 12a_2) - \frac{1}{3}fl\right)\delta a_1 + \left(\frac{1}{2}\frac{EI}{l^3}(12a_1 + 24a_2) - \frac{1}{4}fl\right)\delta a_2 = 0$$

The parameters become:

$$\left.\begin{aligned} 4a_1 + 6a_2 &= \frac{1}{3}\frac{fl^4}{EI} \\ 6a_1 + 12a_2 &= \frac{1}{4}\frac{fl^4}{EI} \end{aligned}\right\} \quad \rightarrow \quad a_1 = \frac{5}{24}\frac{fl^4}{EI} \quad ; \quad a_2 = -\frac{1}{12}\frac{fl^4}{EI}$$

Therefore, the solution reads:

$$u(x) = \frac{fl^4}{12EI}\left(\frac{5}{2}\frac{x^2}{l^2} - \frac{x^3}{l^3}\right)$$

The moment and transverse force are:

$$M(x) = -EI\,u_{,xx} = \frac{1}{2}fl^2\left(-\frac{5}{6} + \frac{x}{l}\right) \quad ; \quad V(x) = M_{,x} = \frac{fl}{2}$$

The deflection is already very accurate (at the end it is even exact). The moment is improved and for the transverse force a value different from zero is found (see Fig. 6.8).

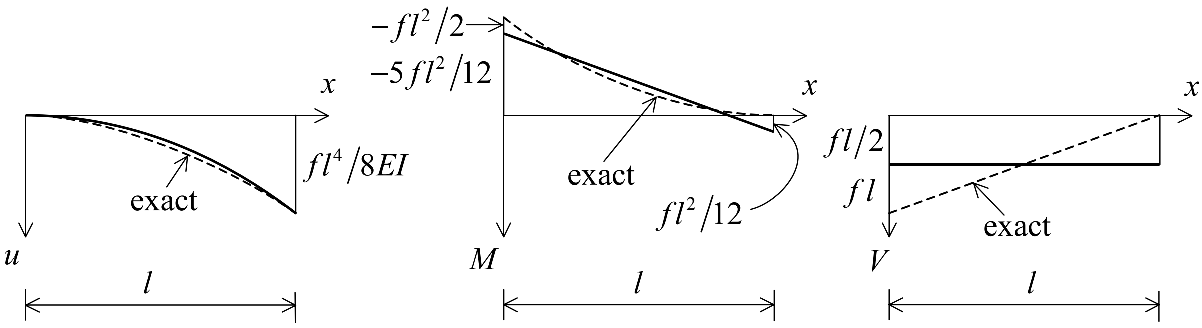


Fig. 6.8: Deflection, moment and shear force for two parameters.

Solution with three parameters
By adding a third parameter a fourth order polynomial is obtained:

$$u(x) = a_1 \frac{x^2}{l^2} + a_2 \frac{x^3}{l^3} + a_3 \frac{x^4}{l^4}$$

The curvature now equals:

$$\kappa = -u_{,xx} = -\frac{1}{l^2}\left(2a_1 + 6a_2 \frac{x}{l} + 12a_3 \frac{x^2}{l^2} \right)$$

and the moment:

$$M = EI\,\kappa = -\frac{EI}{l^2}\left(2a_1 + 6a_2 \frac{x}{l} + 12a_3 \frac{x^2}{l^2} \right)$$

The potential energy becomes:

$$E_{pot} = \frac{1}{2}\frac{EI}{l^4} \int_0^l \left(2a_1 + 6a_2 \frac{x}{l} + 12a_3 \frac{x^2}{l^2} \right)^2 dx - f \int_0^l \left(a_1 \frac{x^2}{l^2} + a_2 \frac{x^3}{l^3} + a_3 \frac{x^4}{l^4} \right) dx \quad \rightarrow$$

$$E_{pot} = \frac{1}{2}\frac{EI}{l^3}\left(4a_1^2 + 12a_1a_2 + 16a_1a_3 + 12a_2^2 + 36a_2a_3 + \frac{144}{5}a_3^2 \right) - fl\left(\frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 \right)$$

Variation with respect to a_1, a_2 and a_3 delivers:

$$\left.\begin{array}{l} 4a_1 + 6a_2 + 8a_3 = \frac{1}{3}\frac{fl^4}{EI} \\ 6a_1 + 12a_2 + 18a_3 = \frac{1}{4}\frac{fl^4}{EI} \\ 8a_1 + 18a_2 + \frac{144}{5}a_3 = \frac{1}{5}\frac{fl^4}{EI} \end{array}\right\} \quad \rightarrow \quad a_1 = \frac{1}{4}\frac{fl^4}{EI} \quad ; \quad a_2 = -\frac{1}{6}\frac{fl^4}{EI} \quad ; \quad a_3 = \frac{1}{24}\frac{fl^4}{EI}$$

The solution reads:

$$u(x) = \frac{fl^4}{EI}\left(\frac{1}{4}\frac{x^2}{l^2} - \frac{1}{6}\frac{x^3}{l^3} + \frac{1}{24}\frac{x^4}{l^4} \right)$$

The moment and transverse force are:

$$M(x) = -EI\,u_{,xx} = fl^2\left(-\frac{1}{2} + \frac{x}{l} - \frac{1}{2}\frac{x^2}{l^2} \right) \quad ; \quad V(x) = M_{,x} = fl\left(1 - \frac{x}{l} \right)$$

These are the exact solutions for $u(x)$, $M(x)$ and $V(x)$, which have been calculated earlier in section 5.2. In this problem, three functions are sufficient to make the set of trial functions complete. The addition of more functions (of a higher order) makes no sense. However, if this still is done, than the calculated coefficients a_i of these terms will be zero.

The examples also clearly demonstrate that the approximation is worse as higher derivatives are considered. The addition of higher order polynomials does not have a large effect on the displacement field (no derivative), but does help in the improvement of the moment and transverse force (second and third derivative).

6.3 Approximations for the stiffness and flexibility matrices

Now, the subject discussed in section 4.6 will be reconsidered. In that section the flexibility and stiffness matrices were derived for a prismatic beam by making use of:

$$c_{ij} = \frac{\partial^2 E_c}{\partial M_i \partial M_j} \quad ; \quad k_{ij} = \frac{\partial^2 E_s}{\partial \varphi_i \partial \varphi_j} \tag{6.6}$$

The procedure discussed above can also be used for finding approximations for the flexibility and stiffness matrices. This will be shown for a tapered beam having a flexural stiffness, $EI(x)$, which is varying along the beam axis (see Fig. 6.9). At $x = 0$ and at $x = l$ the flexural stiffnesses are $2EI$ and EI, respectively. A linearly varying function is chosen:

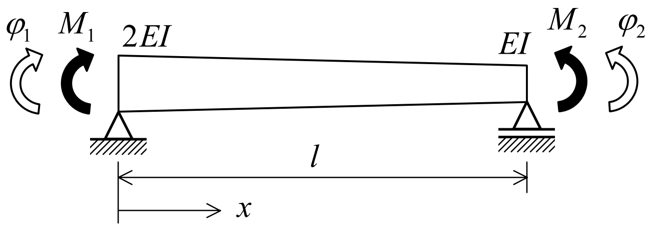


Fig. 6.9: Tapered beam subjected to bending.

$$EI(x) = \left(2 - \frac{x}{l}\right) EI$$

Flexibility matrix

For the calculation of the flexibility matrix again the same procedure is applied:

$$M(x) = \left(1 - \frac{x}{l}\right) M_1 + \frac{x}{l} M_2 = b_1(x) M_1 + b_2(x) M_2$$

The complementary energy E_c becomes:

$$E_c = \frac{1}{2}\int_0^l \frac{M^2(x)}{EI(x)} dx = \frac{1}{2EI}\left(\int_0^l \frac{\left(1-\frac{x}{l}\right)^2}{\left(2-\frac{x}{l}\right)} dx\, M_1^2 + 2\int_0^l \frac{\frac{x}{l}\left(1-\frac{x}{l}\right)}{\left(2-\frac{x}{l}\right)} dx\, M_1 M_1 + \int_0^l \frac{\left(\frac{x}{l}\right)^2}{\left(2-\frac{x}{l}\right)} dx\, M_2^2 \right)$$

The three integrals can be solved numerically by application of for example the trapezoidal rule or Simpson's rule. From the more accurate Simpson's rule applied to one interval it is found:

$$E_c = \frac{l}{EI}\left(0.097M_1^2 + 0.113M_1M_2 + 0.137M_2^2\right)$$

The components of the flexibility matrix are:

$$c_{11} = \frac{\partial^2 E_c}{\partial M_1^2} = 0.194\frac{l}{EI} \quad ; \quad c_{12} = \frac{\partial^2 E_c}{\partial M_1 \partial M_2} = 0.113\frac{l}{EI} \quad ; \quad c_{22} = \frac{\partial^2 E_c}{\partial M_2^2} = 0.274\frac{l}{EI}$$

In matrix notation:

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \frac{l}{EI}\begin{bmatrix} 0.194 & 0.113 \\ 0.113 & 0.274 \end{bmatrix}\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} \tag{6.7}$$

Stiffness matrix
For the computation of the stiffness matrix again the following deflection function is chosen:

$$w(x) = x\left(1-\frac{x}{l}\right)^2 \varphi_1 + \frac{x^2}{l}\left(1-\frac{x}{l}\right)\varphi_2 = b_1(x)\varphi_1 + b_2(x)\varphi_2$$

$$\kappa(x) = -w_{,xx} = \left(\frac{4}{l}-\frac{6x}{l^2}\right)\varphi_1 - \left(\frac{2}{l}-\frac{6x}{l^2}\right)\varphi_2$$

The potential energy becomes:

$$E_s = \frac{1}{2}\int_0^l EI(x)\kappa^2(x)\,dx \quad \rightarrow$$

$$E_s = \frac{EI}{2}\left(\int_0^l \left(2-\frac{x}{l}\right)\left(\frac{4}{l}-\frac{6x}{l^2}\right)^2 dx\,\varphi_1^2 - 2\int_0^l \left(2-\frac{x}{l}\right)\left(\frac{4}{l}-\frac{6x}{l^2}\right)\left(\frac{2}{l}-\frac{6x}{l^2}\right)dx\,\varphi_1\varphi_2 + \int_0^l \left(2-\frac{x}{l}\right)\left(\frac{2}{l}-\frac{6x}{l^2}\right)^2 dx\,\varphi_2^2\right)$$

Again, the integrals can be determined analytically, but can also be obtained by a numerical procedure. It follows:

$$E_s = \frac{EI}{l}\left(3.50\varphi_1^2 - 3.00\varphi_1\varphi_2 + 2.50\varphi_2^2\right)$$

The components of the stiffness matrix become:

$$k_{11} = \frac{\partial^2 E_s}{\partial \varphi_1^2} = 7.00\frac{EI}{l} \quad ; \quad k_{12} = \frac{\partial^2 E_s}{\partial \varphi_1 \partial \varphi_2} = -3.00\frac{EI}{l} \quad ; \quad k_{22} = \frac{\partial^2 E_s}{\partial \varphi_2^2} = 5.00\frac{EI}{l}$$

In matrix notation:

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \frac{EI}{l}\begin{bmatrix} 7.00 & -3.00 \\ -3.00 & 5.00 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} \tag{6.8}$$

If this relation is inverted, it is found:

$$\frac{l}{EI}\begin{bmatrix} 0.192 & 0.115 \\ 0.115 & 0.269 \end{bmatrix}\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} \tag{6.9}$$

This result should be identical to (6.7), but some discrepancies can be observed. This means that at least one of the matrices is an approximation.
In the exact formulation, for the beam subjected to bending it holds:

$$\frac{d^2M}{dx^2} = 0 \quad ; \quad \frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2} \right) = 0$$

Which confirms that the linear momentum distribution is exact and the third order function for the deflection is not correct. The flexibility matrix given by (6.7) is the exact one, but the stiffness matrix in (6.8) is an approximation. The exact stiffness matrix, which is the inverse of the flexibility matrix in (6.7) equals:

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \frac{EI}{l}\begin{bmatrix} 6.78 & -2.80 \\ -2.80 & 4.80 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} \tag{6.10}$$

It can be seen that the approximated stiffness matrix is quite good. The error in the terms on the main diagonal is only 3 to 4 percent.

Remark 1
The approximated stiffness matrix is applied in modern computer programmes for sheet pilings. The sheet pile wall is considered to be a flexural member on an elastic foundation. The piling itself is prismatic but the subgrade modulus may vary along the piling.

Remark 2
The integrals in E_c can be given a physical interpretation, by using the "*moment-area*" concept. When only the moment M_1 is considered it can be written:

$$M(x) = \left(1 - \frac{x}{l} \right) M_1$$

and the "*reduced moment*":

$$M_{red} = \frac{M(x)}{EI(x)} = \frac{\left(1-\frac{x}{l}\right)}{\left(2-\frac{x}{l}\right)} \frac{M_1}{EI}$$

When M_{red} is considered to be an external load, then the support reaction following from this load equals the angular deflection. At end 1 the support reaction is:

$$\int_0^l \left\{ M_{red}\, dx \left(1-\frac{x}{l}\right) \right\}$$

and at node 2:

$$\int_0^l \left\{ M_{red}\, dx\, \frac{x}{l} \right\}$$

So:

$$\varphi_1 = \int_0^l \frac{\left(1-\frac{x}{l}\right)^2}{\left(2-\frac{x}{l}\right)} dx\, \frac{M_1}{EI} = c_{11} M_1$$

$$\varphi_2 = \int_0^l \frac{\left(1-\frac{x}{l}\right)\left(\frac{x}{l}\right)}{\left(2-\frac{x}{l}\right)} dx\, \frac{M_1}{EI} = c_{21} M_1$$

These results are in agreement with the previously found values of c_{11} and c_{21}.

6.4 The method Ritz presented as finite element method

In the method Ritz the displacement $u(x)$ is considered as a superposition of functions:

$$u(x) = \sum a_i\, b_i(x) \tag{6.11}$$

Each function $b_i(x)$ is defined over the entire volume of the structure and has to satisfy the kinematic boundary conditions. Normally, considerably high-order and complex functions $b_i(x)$ are used, so that a limited amount of degrees of freedom a_i will do. This means that only a small number of equations have to be solved.

With the rise of the computer, the solution of large systems of equations became possible. New computational methods were developed such as the *Finite Element Method* (*FEM*). Now deliberately a high number of degrees of freedom a_i are defined, but very simple functions

$b_i(x)$ are considered, called *basis functions* or *shape functions*. This concept will be demonstrated for a bar subjected to extension. The actually smoothly distributed displacement $u(x)$ can be approximated by a polygon. In four points of the bar unknown displacement are introduced as degrees of freedom, given by u_1, u_2, u_3 and u_4 (see Fig. 6.10). Between the

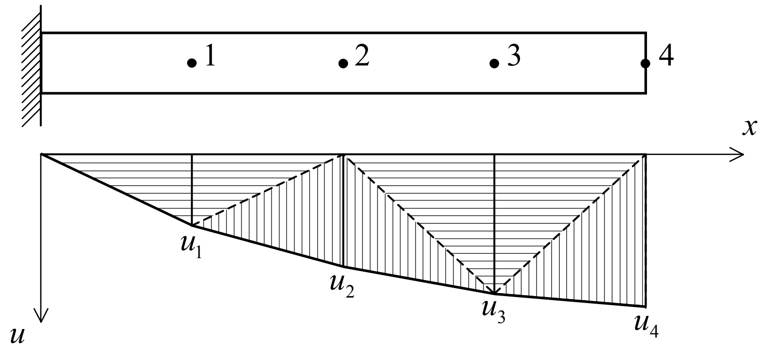


Fig. 6.10: Displacement field approximated by a polygon.

points 1, 2, 3 and 4 the displacement is obtained by linear interpolation. The magnitudes of u_1, u_2, u_3 and u_4 follow the requirement that the potential energy has to be stationary for variations of these four degrees of freedom.

The in this way defined displacement field can be considered to be built up out of four "base fields" called *trial functions* or *test functions*:

$$
\begin{aligned}
u_1 b_1(x) &= u_1 b_1^{\boxed{1}}(x) + u_1 b_1^{\boxed{2}}(x) \\
u_2 b_2(x) &= u_2 b_2^{\boxed{2}}(x) + u_2 b_2^{\boxed{3}}(x) \\
u_3 b_3(x) &= u_3 b_3^{\boxed{3}}(x) + u_3 b_3^{\boxed{4}}(x) \\
u_4 b_4(x) &= u_4 b_4^{\boxed{4}}(x)
\end{aligned}
\tag{6.12}
$$

where the numbers in the rectangular frames refer to the fields between the respective points as can be seen in Fig. 6.11. For the total displacement field it now can be written:

$$
u(x) = \sum_{\text{nodes}} u_i b_i(x) \tag{6.13}
$$

Again it can be seen that $u(x)$ is the sum of a set of functions. However, in this case the shape functions $b_i(x)$ are very simple. Each shape function is piecewise linear per element, has a maximum value of unity and satisfies the kinematic boundary condition.
The total area of the structure is now divided in sub-areas, the so-called *elements* indicated by framed numbers. In this example there are four elements. In the finite element method the common boundary of two elements is called the *element edge*. On the element edges the *nodes* or *nodal points* 1 up to 4 are situated.
As described above the displacement field is a summation of the contribution of the nodes and a function $b_i(x)$, which may extend over more than one element. These contributions can be reorganised such that the displacement field becomes a summation over the several elements (also see Fig. 6.12):

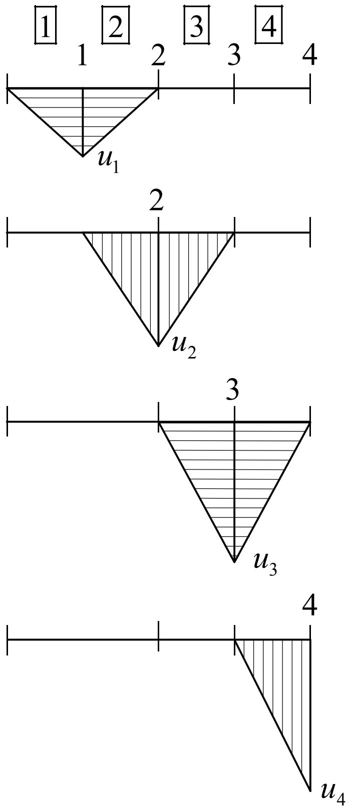


Fig. 6.11: The four trial functions.

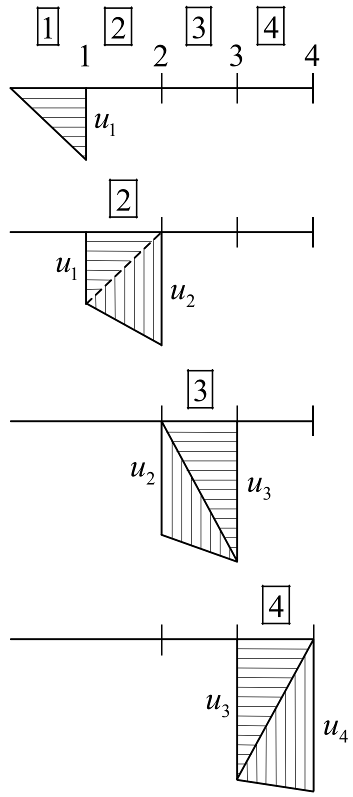


Fig. 6.12: Displacement fields in the elements.

$$
\begin{aligned}
&u_1 b_1(x) = B^{\boxed{1}}(x)\, u^{\boxed{1}} \\
&u_1 b_1^{\boxed{2}}(x) + u_2 b_2^{\boxed{2}}(x) = \left\{ b_1^{\boxed{2}}(x) \quad b_2^{\boxed{2}}(x) \right\} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \mathbf{B}^{\boxed{2}}(x)\, \mathbf{u}^{\boxed{2}} \\
&u_2 b_2^{\boxed{3}}(x) + u_3 b_3^{\boxed{3}}(x) = \left\{ b_2^{\boxed{3}}(x) \quad b_3^{\boxed{3}}(x) \right\} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \mathbf{B}^{\boxed{3}}(x)\, \mathbf{u}^{\boxed{3}} \\
&u_3 b_3^{\boxed{4}}(x) + u_4 b_4^{\boxed{4}}(x) = \left\{ b_3^{\boxed{4}}(x) \quad b_4^{\boxed{4}}(x) \right\} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \mathbf{B}^{\boxed{4}}(x)\, \mathbf{u}^{\boxed{4}}
\end{aligned}
\tag{6.14}
$$

The total displacement field becomes:

$$
u(x) = \sum_{\text{elements}} \boldsymbol{B}^e(x)\, \boldsymbol{u}^e \tag{6.15}
$$

The big advantage of this approach is that the potential energy can be calculated per element. The total potential energy then becomes a simple addition of the elemental contributions:

$$
E_{pot} = \sum_{\text{elements}} E^e_{pot}(u^e) \tag{6.16}
$$

Now, above-mentioned example with the four elements will be worked out further. For an element e between the nodes l and r it holds (see Fig. 6.13):

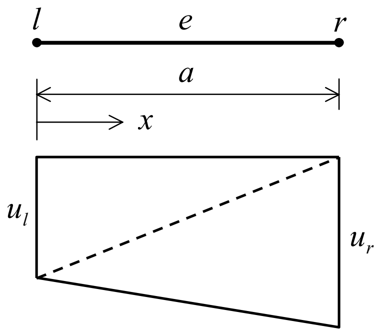


Fig. 6.13: Displacement field in an arbitrary element.

$$b_l^e(x) = 1 - \frac{x}{a} \quad ; \quad b_r^e(x) = \frac{x}{a}$$

The displacement becomes:

$$u(x) = \boldsymbol{B}(x)\,\boldsymbol{u} = \left\{ b_l^e(x) \;\; b_r^e(x) \right\} \begin{Bmatrix} u_l \\ u_r \end{Bmatrix} = b_l^e(x) u_l + b_r^e(x) u_r = \left(1 - \frac{x}{a}\right) u_l + \frac{x}{a} u_r$$

The strain can be obtained by differentiation:

$$\varepsilon(x) = \frac{du(x)}{\partial x} = \frac{1}{a}\left(-u_l + u_r\right)$$

The potential energy of the element equals:

$$E_s^e = \int_0^a \frac{1}{2} EA\,\varepsilon^2\,dx = \frac{1}{2}\frac{EA}{a}\left(-u_l + u_r\right)^2$$

Introduction of the stiffness factor:

$$K = \frac{EA}{a}$$

provides:

$$E_s^e = \frac{1}{2} K \left(u_l^2 - 2u_l u_r + u_r^2\right)$$

The potential energy of position of an axial distributed load $f(x)$ equals:

$$E_p^e = -\int_0^a f(x)\,u(x)\,dx = -\int_0^a f(x)\,b_l(x)\,dx * u_l - \int_0^a f(x)\,b_r(x)\,dx * u_r$$

For constant $f(x) = f$ it is found:

$$E_p^e = f u_l \int_0^a \left(1 - \frac{x}{a}\right) dx - f u_r \int_0^a \frac{x}{a} dx = -\frac{1}{2} f a u_l - \frac{1}{2} f a u_r = -\frac{1}{2} F u_l - \frac{1}{2} F u_r$$

where $F = f a$ has been introduced.
The total potential energy of the element equals:

$$E_{pot}^e = E_s^e + E_p^e = \frac{1}{2} K \left(u_l^2 - 2u_l u_r + u_r^2\right) - \frac{1}{2} F u_l - \frac{1}{2} F u_r$$

The elemental potential energies of the example become:

$$\begin{aligned}
E_{pot}^{(1)} &= \frac{1}{2} K \left(u_1^2\right) && - \frac{1}{2} F u_1 \\
E_{pot}^{(2)} &= \frac{1}{2} K \left(u_1^2 - 2u_1 u_2 + u_2^2\right) - \frac{1}{2} F u_1 && - \frac{1}{2} F u_2 \\
E_{pot}^{(3)} &= \frac{1}{2} K \left(u_2^2 - 2u_2 u_3 + u_3^2\right) - \frac{1}{2} F u_2 && - \frac{1}{2} F u_3 \\
E_{pot}^{(4)} &= \frac{1}{2} K \left(u_3^2 - 2u_3 u_4 + u_4^2\right) - \frac{1}{2} F u_3 && - \frac{1}{2} F u_4
\end{aligned}$$

Summation provides the total potential energy:

$$E_{pot} = K\left(u_1^2 - u_1 u_2 + u_2^2 - u_2 u_3 + u_3^2 - u_3 u_4 + \frac{1}{2} u_4^2\right) - F\left(u_1 + u_2 + u_3 + \frac{1}{2} u_4\right)$$

Minimisation offers:

$$\begin{aligned}
&\frac{\partial E_{pot}}{\partial u_1} = K\left(2u_1 - u_2\right) - F = 0 \quad ; \quad \frac{\partial E_{pot}}{\partial u_3} = K\left(-u_2 + 2u_3 - u_4\right) - F = 0 \\
&\frac{\partial E_{pot}}{\partial u_2} = K\left(-u_1 + 2u_2 - u_3\right) - F = 0 \quad ; \quad \frac{\partial E_{pot}}{\partial u_4} = K\left(-u_3 + u_4\right) - F/2 = 0
\end{aligned} \tag{6.17}$$

In matrix notation this set of equations can be rewritten as:

$$K \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F \\ F \\ F \\ F/2 \end{Bmatrix} \tag{6.18}$$

providing the following solution:

$$
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 3.5F/K \\ 6.0F/K \\ 7.5F/K \\ 8.0F/K \end{Bmatrix} \tag{6.19}
$$

The found displacement and the normal force distributions are shown in Fig. 6.14.

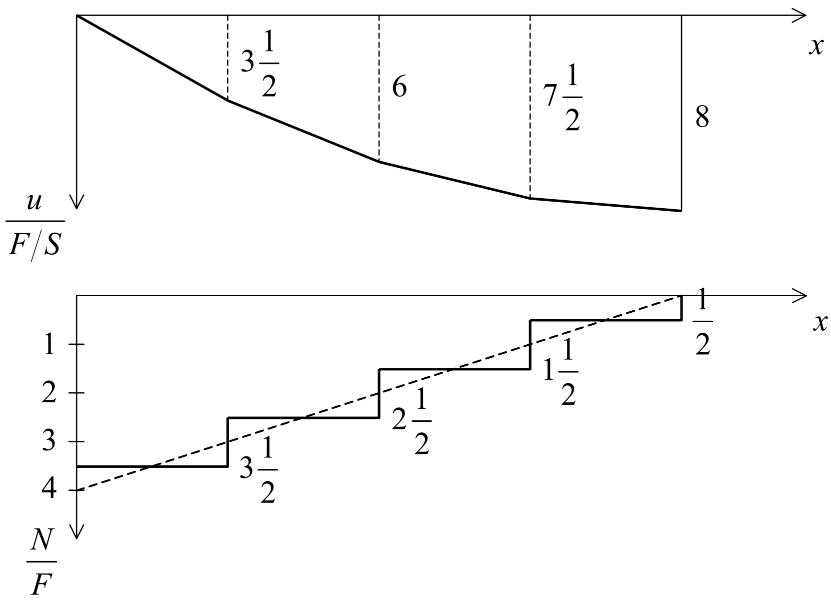


Fig. 6.14: Calculated displacement and normal force distributions.

6.5 Finite Element Method on basis of the virtual work equation

The example with four elements as discussed in section 6.4 will be considered again. In this case however, the *principle of weighted residuals* of Galerkin for the equilibrium will be used. Again a volume load P and a surface load p at the end of the bar are introduced (also see sections 2.2 and 2.3). The cross-sectional area of the bar is set to unity, i.e. $A=1$. In the nodes 1 up to 4 the internal stresses on the cut faces are σ_1 up to σ_4. Fig. 6.15 shows the bar divided into elements. The equilibrium equations valid inside the elements, on the cut faces and on the end face of the bar are indicated as well.
The Galerkin condition incorporating all these equilibrium equations reads:

$$
\begin{aligned}
&\int_0^{a^{\boxed{1}}}\left(\sigma^{\boxed{1}}_{,x}+P\right)\delta u\,dx+\int_0^{a^{\boxed{2}}}\left(\sigma^{\boxed{2}}_{,x}+P\right)\delta u\,dx+\int_0^{a^{\boxed{3}}}\left(\sigma^{\boxed{3}}_{,x}+P\right)\delta u\,dx+\int_0^{a^{\boxed{4}}}\left(\sigma^{\boxed{4}}_{,x}+P\right)\delta u\,dx+\\
&+\left\{\begin{array}{l}\left(-\sigma_1^{\boxed{1}}+\sigma_1\right)\delta u_1\\ \left(-\sigma_1+\sigma_1^{\boxed{2}}\right)\delta u_1\end{array}\right.+\left\{\begin{array}{l}\left(-\sigma_2^{\boxed{2}}+\sigma_2\right)\delta u_2\\ \left(-\sigma_2+\sigma_2^{\boxed{3}}\right)\delta u_2\end{array}\right.+\left\{\begin{array}{l}\left(-\sigma_3^{\boxed{3}}+\sigma_3\right)\delta u_3\\ \left(-\sigma_3+\sigma_3^{\boxed{4}}\right)\delta u_3\end{array}\right.+\left(-\sigma_4^{\boxed{4}}+p\right)\delta u_4=0
\end{aligned} \tag{6.20}
$$

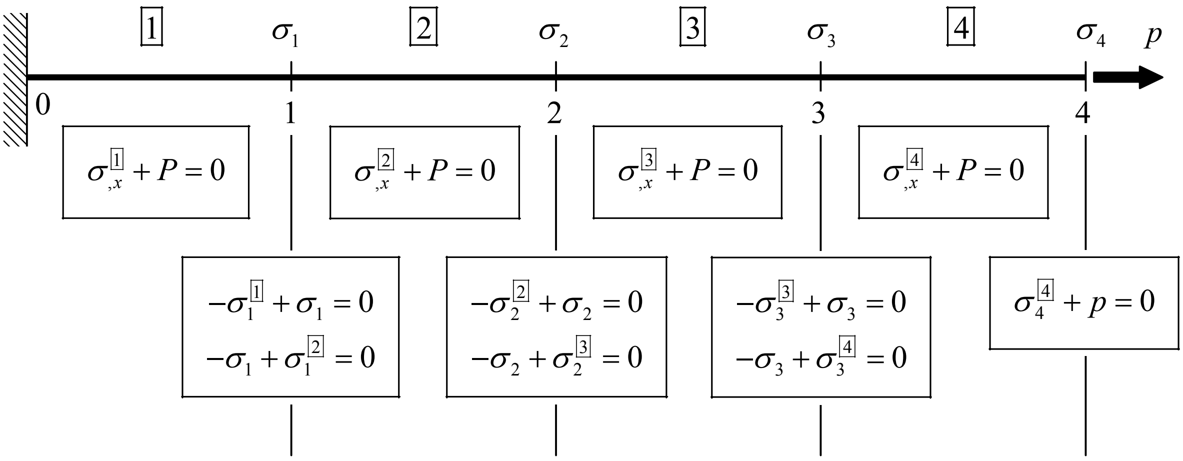


Fig. 6.15: Governing equations of a bar divided into four elements.

The variation of the displacements is zero at the left end where the displacement is prescribed. By partial integration for each of the four elements it holds:

$$\int_0^{a^{\boxed{i}}} \sigma_{,x}^{\boxed{i}} \delta u\, dx = -\int_0^{a^{\boxed{i}}} \sigma^{\boxed{i}} \delta\varepsilon\, dx - \sigma_{i-1}^{\boxed{i}} \delta u_{i-1} + \sigma_i^{\boxed{i}} \delta u_i \tag{6.21}$$

This relation transforms the Galerkin condition into (all signs are changed):

$$\underbrace{\sum_{i=1}^{4} \left(\int_0^{a^{\boxed{i}}} \sigma\, \delta\varepsilon\, dx - \int_0^{a^{\boxed{i}}} P\, \delta u\, dx \right)}_{\substack{\text{contibution of volume} \\ \text{summation over the 4 elements}}} \quad \underbrace{-p\, \delta u_4}_{\substack{\text{contribution of} \\ \text{the edge } S_p}} \quad = 0 \tag{6.22}$$

In the example of section 6.4 a realistic value of A is assumed and a uniformly distributed line load f. No load at the free end of the bar is present. The integral condition then becomes:

$$\sum_{i=1}^{4} \left(\int_0^{a^{\boxed{i}}} N \delta\varepsilon\, dx - \int_0^{a^{\boxed{i}}} f\, \delta u\, dx \right) = 0 \quad \rightarrow \quad \sum_{i=1}^{4} \left(\delta E_s^{\boxed{i}} + \delta E_p^{\boxed{i}} \right) = 0 \tag{6.23}$$

For element e situated between the nodes l and r it holds:

$$u(x) = \left(1 - \frac{x}{a}\right) u_l + \frac{x}{a} u_r \quad ; \quad \delta u(x) = \left(1 - \frac{x}{a}\right) \delta u_l + \frac{x}{a} \delta u_r$$

$$\varepsilon(x) = \frac{1}{a}\left(-u_l + u_r\right) \quad ; \quad \delta\varepsilon(x) = \frac{1}{a}\left(-\delta u_l + \delta u_r\right)$$

$$N(x) = \frac{EA}{a}\left(-u_l + u_r\right)$$

The variation of the potential energies becomes:

$$\delta E_s^e = \int_0^a N\delta\varepsilon\, dx = \frac{EA}{a}\left(-u_l + u_r\right)\frac{1}{a}\left(-\delta u_l + \delta u_r\right)a = \frac{EA}{a}\left(u_l - u_r\right)\delta u_l + \frac{EA}{a}\left(-u_l + u_r\right)\delta u_r$$

$$\delta E_p^e = -\int_0^a f\,\delta u\, dx = -f\int_0^a \left(1 - \frac{x}{a}\right)\delta u_l\, dx - f\int_0^a \frac{x}{a}\delta u_r\, dx = -\frac{1}{2}fa\,\delta u_l - \frac{1}{2}fa\,\delta u_r$$

Substitution of $K = EA/a$ and $F = fa$ for one element it totally provides:

$$\delta E^e = \delta E_s^e + \delta E_p^e = K\left(u_l - u_r\right)\delta u_l + K\left(-u_l + u_r\right)\delta u_r - \frac{1}{2}F\delta u_l - \frac{1}{2}F\delta u_r$$

For the four elements of the example this delivers (note that $u_0 = 0$ and $\delta u_0 = 0$:

$$\begin{aligned}
\delta E^{\boxed{1}} &= \qquad\qquad\qquad\; +Ku_1\delta u_1 \qquad\qquad\qquad\quad -\frac{1}{2}F\delta u_1 \\
\delta E^{\boxed{2}} &= K\left(u_1 - u_2\right)\delta u_1 + K\left(-u_1 + u_2\right)\delta u_2 - \frac{1}{2}F\delta u_1 - \frac{1}{2}F\delta u_2 \\
\delta E^{\boxed{3}} &= K\left(u_2 - u_3\right)\delta u_2 + K\left(-u_2 + u_3\right)\delta u_3 - \frac{1}{2}F\delta u_2 - \frac{1}{2}F\delta u_3 \\
\delta E^{\boxed{4}} &= K\left(u_3 - u_4\right)\delta u_3 + K\left(-u_3 + u_4\right)\delta u_4 - \frac{1}{2}F\delta u_3 - \frac{1}{2}F\delta u_4 \quad + \\
\hline
\delta E &= \left\{K\left(2u_1 - u_2\right) - F\right\}\delta u_1 + \left\{K\left(-u_1 + 2u_2 - u_3\right) - F\right\}\delta u_2 + \\
&\quad + \left\{K\left(-u_2 + 2u_3 - u_4\right) - F\right\}\delta u_3 + \left\{K\left(-u_3 + u_4\right) - \frac{1}{2}F\right\}\delta u_4 = 0
\end{aligned}$$

Since δE has to be zero for all allowable variations δu_1 up to δu_4, all terms between braces have to be zero. This provides the same relations as derived in the previous section by the principle of minimum of potential energy (see (6.17) and the matrix equation (6.18)).

6.6 Epilogue

In the approach of the principle of minimum potential energy, the contribution of an element can be written as:

$$E_{pot}^e = \frac{1}{2}\left\{u_l \quad u_r\right\}\begin{bmatrix} EA/a & -EA/a \\ -EA/a & EA/a \end{bmatrix}\begin{Bmatrix} u_l \\ u_r \end{Bmatrix} - \left\{u_l \quad u_r\right\}\begin{Bmatrix} fa/2 \\ fa/2 \end{Bmatrix} \tag{6.24}$$

The Galerkin approach provided:

$$\delta E^e = \left\{\delta u_l \quad \delta u_r\right\}\begin{bmatrix} EA/a & -EA/a \\ -EA/a & EA/a \end{bmatrix}\begin{Bmatrix} u_l \\ u_r \end{Bmatrix} - \left\{\delta u_l \quad \delta u_r\right\}\begin{bmatrix} fa/2 \\ fa/2 \end{bmatrix} \tag{6.25}$$

In matrix notation these relations read:

$$E_{pot}^{e} = \frac{1}{2}\boldsymbol{u}^{T}\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^{T}\boldsymbol{f} \tag{6.26}$$

$$\delta E^{e} = \delta\boldsymbol{u}^{T}\boldsymbol{K}\,\boldsymbol{u} - \delta\boldsymbol{u}^{T}\boldsymbol{f} \tag{6.27}$$

It can be seen that the stiffness matrix $\boldsymbol{K}$ as well as the nodal force vector $\boldsymbol{f}$ can be obtained by both methods. Because an elastic system is involved, both methods lead to the same answer.
However, The virtual work (Galerkin) approach has become more popular than the principle of minimum potential energy. The advantage of the virtual work equation is especially demonstrated in the application of non-linear and non-elastic material behaviour.

7 Application of the Finite Element Method on a one-dimensional system

In chapter 2 of the lecture notes "Direct Methods" a bar subjected to extension was analysed, which was spring supported in axial direction (see Fig. 7.1). The solution was determined by

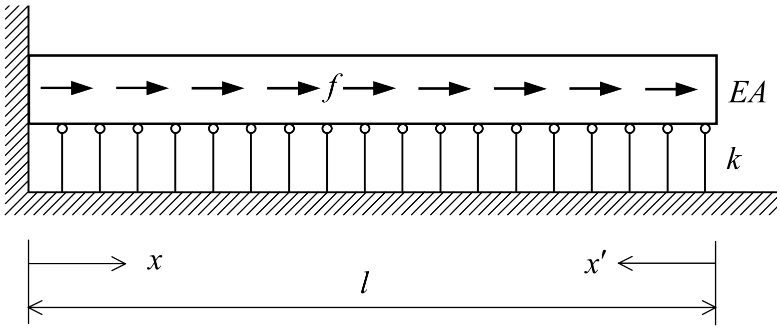


Fig. 7.1: Spring-supported bar, which is uniformly loaded in axial direction.

both the displacement and the force method, in both cases for a large length l. The general solution valid for any length l can be calculated by:

$$u(x) = \left(-\frac{1}{1+e^{-2\alpha}} e^{-x/\lambda} - \frac{e^{-\alpha}}{1+e^{-2\alpha}} e^{-x'/\lambda} + 1 \right) \frac{f}{k} \tag{7.1}$$

$$N(x) = \left(\frac{1}{1+e^{-2\alpha}} e^{-x/\lambda} - \frac{e^{-\alpha}}{1+e^{-2\alpha}} e^{-x'/\lambda} \right) \lambda f \tag{7.2}$$

$$s(x) = \left(-\frac{1}{1+e^{-2\alpha}} e^{-x/\lambda} - \frac{e^{-\alpha}}{1+e^{-2\alpha}} e^{-x'/\lambda} + 1 \right) f \tag{7.3}$$

where:

u	=	displacement [m]
N	=	normal force [N]
s	=	spring load [N/m]
$\lambda = \sqrt{EA/k}$	=	characteristic length [m]
EA	=	extensional stiffness [N]
k	=	spring stiffness [N/m^2]
$\alpha = l/\lambda$	=	dimensionless length [-]

When $\alpha \gg 1$, the damping term with x' from the right side disappears. But when $\alpha \approx 1$, this term cannot be neglected. In Fig. 7.2 the solution has been drawn for $\alpha = 2\sqrt{6}$ and $\alpha = 2$. These exact solutions will be approximated by the finite element method. In section 7.1 this is done with the principle of minimum potential energy and in section 7.2 the principle of minimum complementary energy is used.

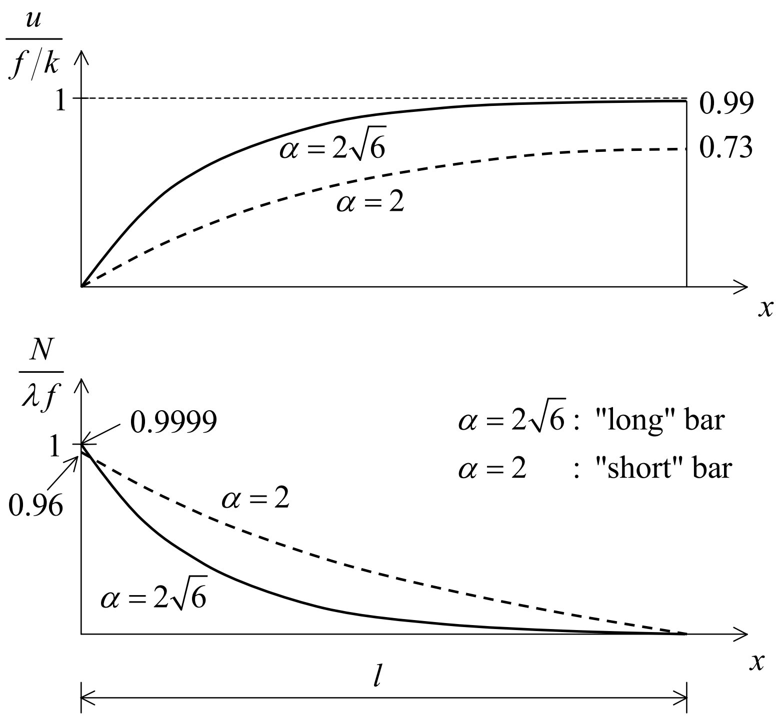


Fig. 7.2: Exact solutions for spring-supported bar.

7.1 Approximation by minimum of potential energy

The bar is divided into two elements. The degrees of freedom in the two nodes 1 and 2 are the displacements u_1 and u_2, respectively. The two elements are indicated by $\boxed{1}$ and $\boxed{2}$ (see Fig. 7.3). Inside the elements a linear field of displacements is assumed. A linear displacement field is the simplest distribution that satisfies the kinematic boundary condition(s).
For the whole system it holds:

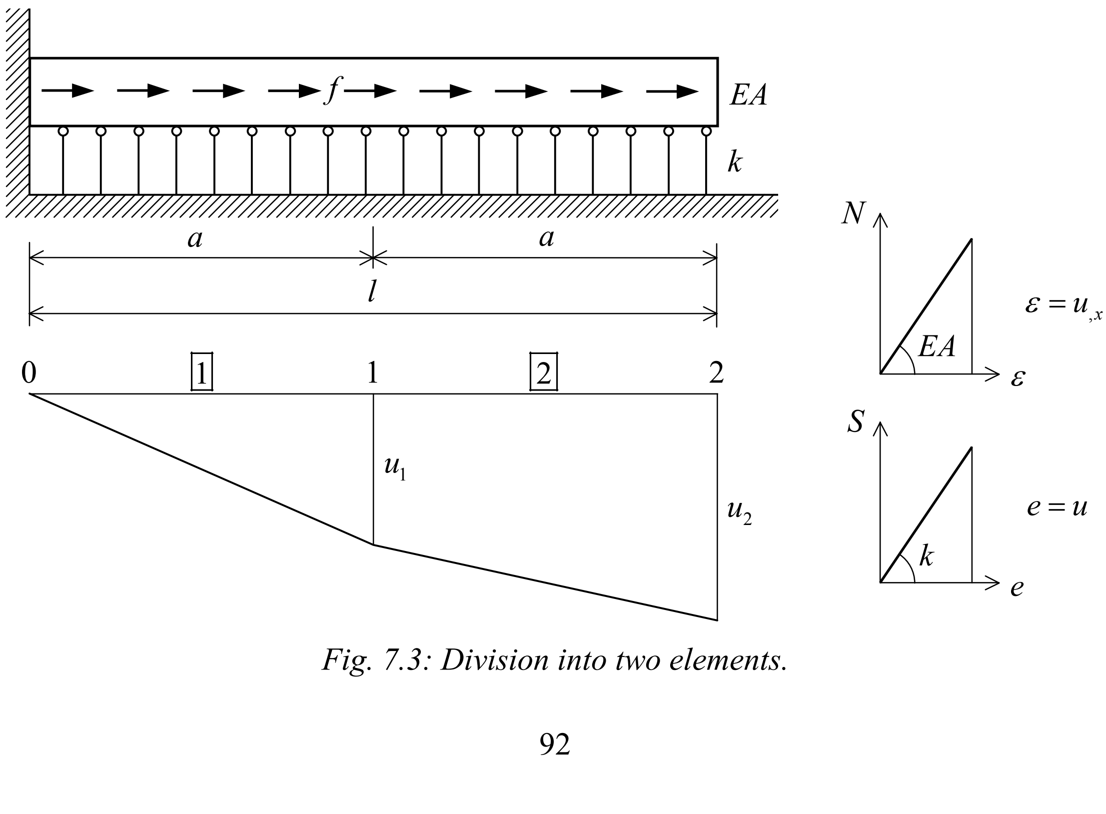


Fig. 7.3: Division into two elements.

$$E_{pot} = \underbrace{\int_0^l \frac{1}{2} EA\,\varepsilon^2\,dx}_{\text{energy in the bar}} + \underbrace{\int_0^l \frac{1}{2} k\,e^2\,dx}_{\text{energy in the springs}} - \underbrace{\int_0^l f u\,dx}_{\text{potential enrgy of position}} \tag{7.4}$$

The potential energy consists out of three parts, namely the contribution of the bar, the contribution of the springs and the contributions of the external load (position). These three contributions will be obtained for each of the elements separately and summed up afterwards.

Element $\boxed{\mathbf{1}}$

The displacement and strain fields are (see Fig. 7.4):

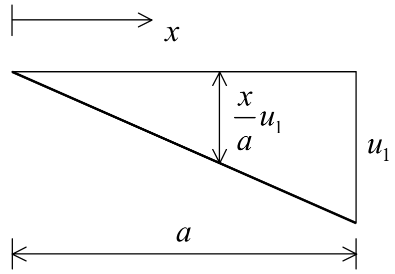


Fig. 7.4: Displacement field of element 1.

$$u(x) = \frac{x}{a} u_1 \quad ; \quad \varepsilon(x) = u_{,x} = \frac{u_1}{a} \tag{7.5}$$

The energy in the bar is:

$$E_s = \int_0^a \frac{1}{2} EA\,\varepsilon^2\,dx = \frac{1}{2} EA \left(\frac{u_1}{a} \right)^2 a \quad \rightarrow \quad \boxed{E_s = \frac{1}{2} \frac{EA}{a} u_1^2} \tag{7.6}$$

The energy in the springs is:

$$E_s = \int_0^a \frac{1}{2} k\,u^2\,dx = \frac{1}{2} k u_1^2 \int_0^a \left(\frac{x}{a} \right)^2 dx \quad \rightarrow \quad \boxed{E_s = \frac{1}{6} k\,a\,u_1^2} \tag{7.7}$$

The energy of the external load is:

$$E_p = -\int_0^a f u\,dx = -f u_1 \int_0^a \frac{x}{a} dx \quad \rightarrow \quad \boxed{E_p = -\frac{1}{2} f\,a\,u_1} \tag{7.8}$$

Element $\boxed{\mathbf{2}}$

The displacement and strain fields are (see Fig. 7.5):

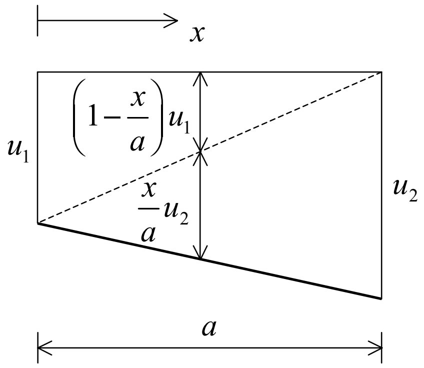


Fig. 7.5: Displacement field of element 2.

$$u(x) = \left(1 - \frac{x}{a}\right)u_1 + \frac{x}{a}u_2 \quad ; \quad \varepsilon(x) = u_{,x} = \frac{1}{a}\left(-u_1 + u_2\right) \tag{7.9}$$

The energy in the bar is:

$$E_s = \frac{1}{2}EA\int_0^a \left(\frac{-u_1 + u_2}{a}\right)^2 dx \quad \rightarrow \quad \boxed{E_s = \frac{1}{2}\frac{EA}{a}\left(u_1^2 - 2u_1u_2 + u_2^2\right)} \tag{7.10}$$

The energy in the springs is:

$$E_s = \frac{1}{2}k\int_0^a \left\{\left(1 - \frac{x}{a}\right)u_1 + \frac{x}{a}u_2\right\}^2 dx = \quad \rightarrow \quad \boxed{E_s = \frac{1}{6}k\,a\left(u_1^2 + u_1u_2 + u_2^2\right)} \tag{7.11}$$

The energy of the external load is:

$$E_p = -fu_1\int_0^a \left(1 - \frac{x}{a}\right)dx - -fu_2\int_0^a \frac{x}{a}dx \quad \rightarrow \quad \boxed{E_p = -\frac{1}{2}fa\left(u_1 + u_2\right)} \tag{7.12}$$

Total potential energy

The sum of all framed contributions equals:

$$E_{pot} = \left(\frac{EA}{a} + \frac{1}{3}ka\right)u_1^2 + \left(-\frac{EA}{a} + \frac{1}{6}ka\right)u_1u_2 + \left(\frac{1}{2}\frac{EA}{a} + \frac{1}{6}ka\right)u_2^2 - fau_1 - \frac{1}{2}fau_2 \tag{7.13}$$

This expression is stationary if the following is satisfied:

$$\frac{\partial E_{pot}}{\partial u_1} = 0 \quad \rightarrow \quad \left(2\frac{EA}{a} + \frac{2}{3}ka\right)u_1 + \left(-\frac{EA}{a} + \frac{1}{6}ka\right)u_2 = fa$$
$$\frac{\partial E_{pot}}{\partial u_2} = 0 \quad \rightarrow \quad \left(-\frac{EA}{a} + \frac{1}{6}ka\right)u_1 + \left(+\frac{EA}{a} + \frac{1}{3}ka\right)u_2 = \frac{1}{2}fa \tag{7.14}$$

Division of (7.14) by ka and introduction of:

$$\frac{EA}{ka^2} = \frac{4}{\alpha^2} \tag{7.15}$$

where α just as in the exact solution, is defined by:

$$\alpha = \frac{l}{\lambda} \tag{7.16}$$

provides the following two equations:

$$\left(\frac{8}{\alpha^2} + \frac{2}{3}\right)u_1 + \left(-\frac{4}{\alpha^2} + \frac{1}{6}\right)u_2 = \frac{f}{k} \quad ; \quad \left(-\frac{4}{\alpha^2} + \frac{1}{6}\right)u_1 + \left(\frac{4}{\alpha^2} + \frac{1}{3}\right)u_2 = \frac{1}{2}\frac{f}{k} \tag{7.17}$$

A solution will be obtained for the same values of α, for which the exact solutions have been determined as well:

$\alpha = 2\sqrt{6}$	behaviour as “long” bar
$\alpha = 2$	behaviour as “short” bar

“long” bar: $\alpha = 2\sqrt{6}$

In this case from the relations in (7.17) it follows:

$$u_1 = \frac{f}{k} \quad ; \quad u_2 = \frac{f}{k}$$

The normal forces then become:

$$N^{\boxed{1}} = EA\frac{u_1}{a} = \frac{EA}{a}\frac{f}{k} = \frac{\lambda}{a}(\lambda f) \quad ; \quad N^{\boxed{2}} = EA\left(\frac{-u_1 + u_2}{a}\right) = 0$$

For the ratio λ/a it can be written:

$$\frac{\lambda}{a} = \frac{2\lambda}{l} = \frac{2}{\alpha} = \frac{1}{\sqrt{6}} = 0.41$$

So, the normal forces can be simplified to:

$$N^{\boxed{1}} = 0.41\lambda f \quad ; \quad N^{\boxed{2}} = 0$$

Both the displacement and normal force distributions are shown in Fig. 7.6.

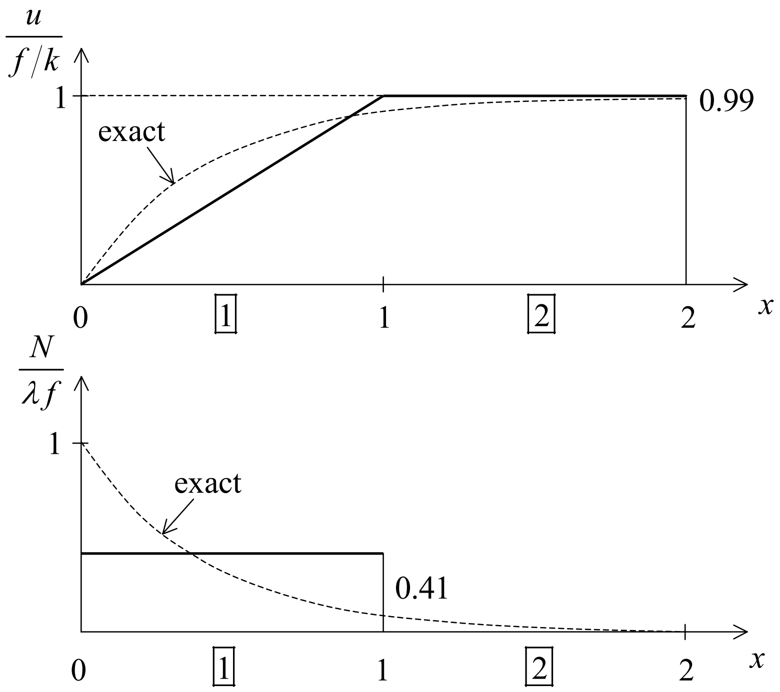


Fig. 7.6: Comparison with exact solution.

***"short" bar:* $\alpha = 2$**

Now the relations in (7.17) become:

$$\frac{8}{3}u_1 - \frac{5}{6}u_2 = \frac{f}{k} \quad ; \quad -\frac{5}{6}u_1 + \frac{4}{3}u_2 = \frac{1}{2}\frac{f}{k}$$

Solving the displacements provide:

$$u_1 = \frac{63}{103}\frac{f}{k} = 0.61\frac{f}{k} \quad ; \quad u_2 = \frac{78}{103}\frac{f}{k} = 0.76\frac{f}{k}$$

The normal forces are:

$$N^{\boxed{1}} = \frac{63}{103}\frac{\lambda}{a}(\lambda f) \quad ; \quad N^{\boxed{2}} = \frac{15}{103}\frac{\lambda}{a}(\lambda f)$$

For the ratio λ/a it holds:

$$\frac{\lambda}{a} = \frac{2}{\alpha} = 1$$

So, the normal forces can be simplified to:

$N^{\boxed{1}} = 0.61\lambda f \quad ; \quad N^{\boxed{2}} = 0.15\lambda f$

In Fig. 7.7 both the displacement and normal force distribution are shown again.

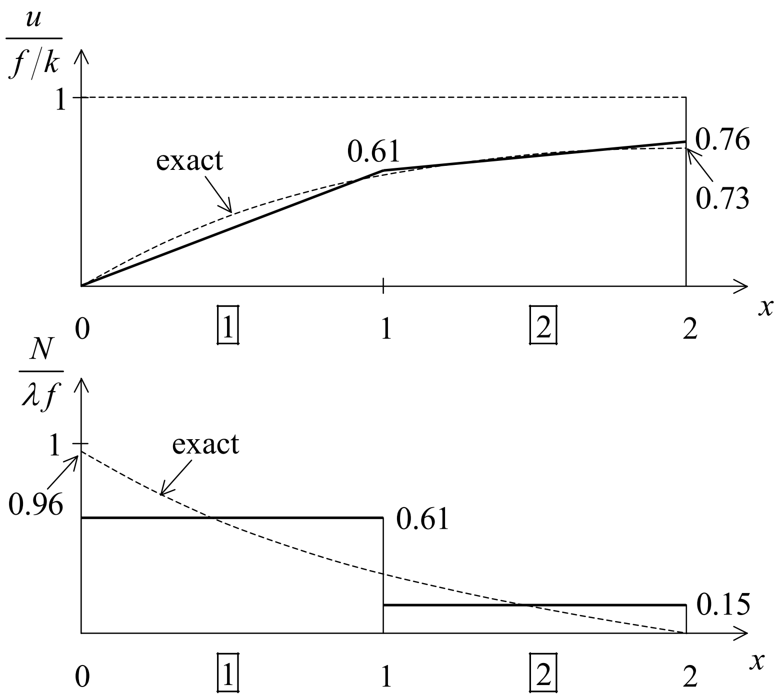


Fig. 7.7: Comparison with exact solution.

7.2 Approximation by minimum of complementary energy

Similarly to section 7.1 two elements $\boxed{1}$ and $\boxed{2}$ and two nodes 1 and 2 are used. Now a stress field is defined such that equilibrium is satisfied. In this example this is done by choosing two redundants φ_1 and φ_2 (see Fig. 7.8). For this structure this is the simplest distribution that satisfies the equilibrium equations. The redundant φ_1 is a constant distributed load between element $\boxed{1}$ and the springs, and φ_2 is the constant distributed load between element $\boxed{2}$ and the springs. In each point of the bar the value of the normal force N can be expressed in φ_1, φ_2 and f. The spring loads s are equal to the local value of φ.
The expression for the complementary energy reads:

$$E_{compl} = \underbrace{\int_0^l \frac{1}{2}\frac{N^2}{EA}dx}_{\text{energy in bar}} + \underbrace{\int_0^l \frac{1}{2}\frac{s}{k}dx}_{\text{energy in springs}} \tag{7.18}$$

No displacement term can be found in above energy expression, because no prescribed displacement different from zero is present. The two contributions of the bar and the springs are calculated for each of the elements separately.

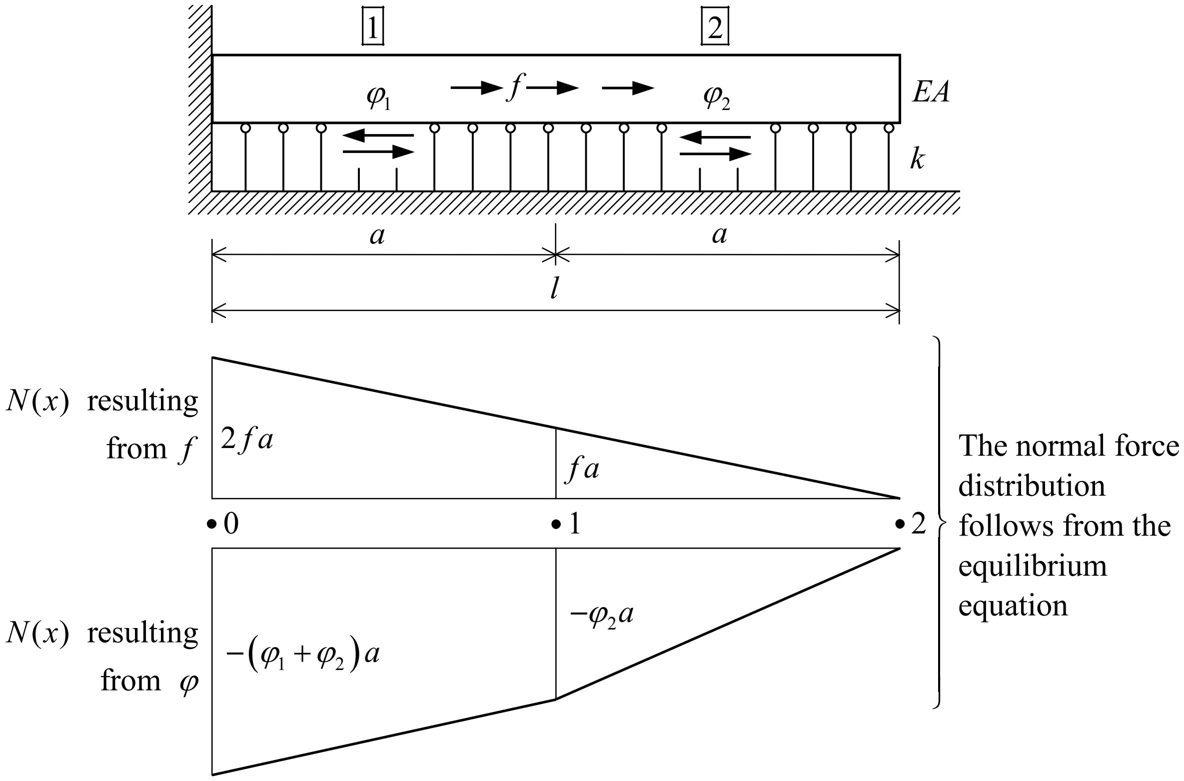


Fig. 7.8: Introduction of redundants.

Element $\boxed{\mathbf{1}}$

The normal force field equals (see Fig. 7.9):

$$N(x) = fa\left(2-\frac{x}{a}\right) - \varphi_1 a\left(1-\frac{x}{a}\right) - \varphi_2 a \tag{7.19}$$

The energy in the bar is:

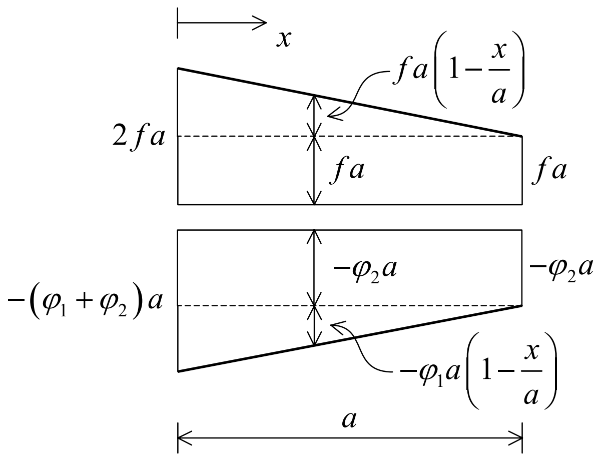


Fig. 7.9: Normal force fields of element 1.

$$E_c = \frac{1}{2EA}\int_0^a N^2(x)\,dx \quad \rightarrow$$

$$E_c = \frac{1}{2EA}\left\{ f^2a^2\int_0^a\left(2-\frac{x}{a}\right)^2 dx - 2f\varphi_1 a^2\int_0^a\left(2-\frac{x}{a}\right)\left(1-\frac{x}{a}\right)dx - 2f\varphi_2 a^2\int_0^a\left(2-\frac{x}{a}\right)dx + \right.$$

$$\left. \varphi_1^2a^2\int_0^a\left(1-\frac{x}{a}\right)^2 dx + 2\varphi_1\varphi_2 a^2\int_0^a\left(1-\frac{x}{a}\right)dx + \varphi_2^2a^2\int_0^a dx \right\} \quad \rightarrow$$

$$\boxed{E_c = \frac{1}{2}\frac{a^3}{EA}\left(\frac{7}{3}f^2 - \frac{5}{3}f\varphi_1 - 3f\varphi_2 + \frac{1}{3}\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2\right)} \tag{7.20}$$

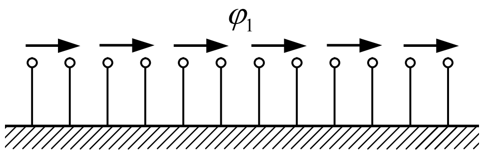


Fig. 7.10: Spring load.

The energy in the springs is (see Fig. 7.10):

$$\left.\begin{array}{l} E_c = \dfrac{1}{2}\displaystyle\int_0^a \frac{s^2}{k}\,dx \\ s = \varphi_1 \end{array}\right\} \rightarrow \boxed{E_c = \frac{1}{2}\frac{\varphi_1^2 a}{k}} \tag{7.21}$$

***Element* $\boxed{\mathbf{2}}$**

The normal force field equals (see Fig. 7.11):

$$N(x) = a\left(f - \varphi_2\right)\left(1-\frac{x}{a}\right) \tag{7.22}$$

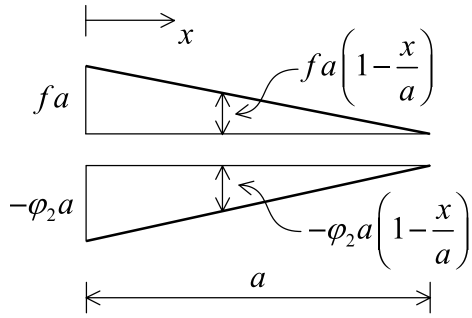


Fig. 7.11: Normal force fields of element 2.

The energy in the bar is:

$$E_c = \frac{1}{2EA}\int_0^a N^2(x)\,dx = \frac{a^2}{2EA}\int_0^a (f-\varphi_2)^2\left(1-\frac{x}{a}\right)^2 dx \quad \rightarrow$$

$$\boxed{E_c = \frac{1}{6}\frac{a^3}{EA}\left(f^2 - 2f\varphi_2 + \varphi_2^2\right)} \tag{7.23}$$

The energy in the springs is:

$$E_c = \frac{1}{2}\int_0^a \frac{\varphi_2^2}{k}\,dx \quad \rightarrow \quad \boxed{E_c = \frac{1}{2}\frac{\varphi_2^2 a}{k}} \tag{7.24}$$

Total complementary energy
The sum of all framed contributions equals:

$$E_{compl} = \frac{a^3}{EA}\left(\frac{4}{3}f^2 - \frac{5}{6}f\varphi_1 - \frac{11}{6}f\varphi_2 + \frac{1}{6}\varphi_1^2 + \frac{1}{2}\varphi_1\varphi_2 + \frac{2}{3}\varphi_2^2\right) + \frac{a}{k}\left(\frac{1}{2}\varphi_1^2 + \frac{1}{2}\varphi_2^2\right) \tag{7.25}$$

Variation with respect to φ_1 and φ_2 provides:

$$\begin{aligned}
\frac{\partial E_{compl}}{\partial \varphi_1} = 0 \quad &\rightarrow \quad \left(\frac{1}{3}\frac{a^3}{EA} + \frac{a}{k}\right)\varphi_1 + \frac{1}{2}\frac{a^3}{EA}\varphi_2 - \frac{5}{6}\frac{a^3}{EA}f = 0 \\
\frac{\partial E_{compl}}{\partial \varphi_2} = 0 \quad &\rightarrow \quad \frac{1}{2}\frac{a^3}{EA}\varphi_1 + \left(\frac{4}{3}\frac{a^3}{EA} + \frac{a}{k}\right)\varphi_2 - \frac{11}{6}\frac{a^3}{EA}f = 0
\end{aligned} \tag{7.26}$$

Multiplication of (7.26) and by EA/a^3 and introduction of:

$$\frac{EA}{ka^2} = \frac{4}{\alpha^2} \tag{7.27}$$

where α is defined by:

$$\alpha = \frac{l}{\lambda} \tag{7.28}$$

provides the following two equations:

$$\left(\frac{1}{3} + \frac{4}{\alpha^2}\right)\varphi_1 + \frac{1}{2}\varphi_2 = \frac{5}{6}f \quad ; \quad \frac{1}{2}\varphi_1 + \left(\frac{4}{3} + \frac{4}{\alpha^2}\right)\varphi_2 = \frac{11}{6}f \tag{7.29}$$

***"long" bar:* $\alpha = 2\sqrt{6}$**
In this case from the relations in (7.29) it follows:

$$\left.\begin{array}{l} \dfrac{1}{2}\varphi_1 + \dfrac{1}{2}\varphi_2 = \dfrac{5}{6}f \\ \dfrac{1}{2}\varphi_1 + \dfrac{3}{2}\varphi_2 = \dfrac{11}{6}f \end{array}\right\} \rightarrow \quad \varphi_1 = \frac{2}{3}f \quad ; \quad \varphi_2 = f$$

The normal forces then become:

$$N_0 = \frac{1}{3}fa \quad ; \quad N_1 = N_2 = 0$$

In order to compare this result with the exact solution the length a is expressed in λ:

$$a = \frac{1}{2}l = \frac{1}{2}\alpha\lambda = \sqrt{6}\lambda = 2.46\lambda$$

So, the normal forces can be simplified to:

$$N_0 = 0.82\,\lambda f \quad ; \quad N_1 = N_2 = 0$$

Above results are shown in Fig. 7.12.

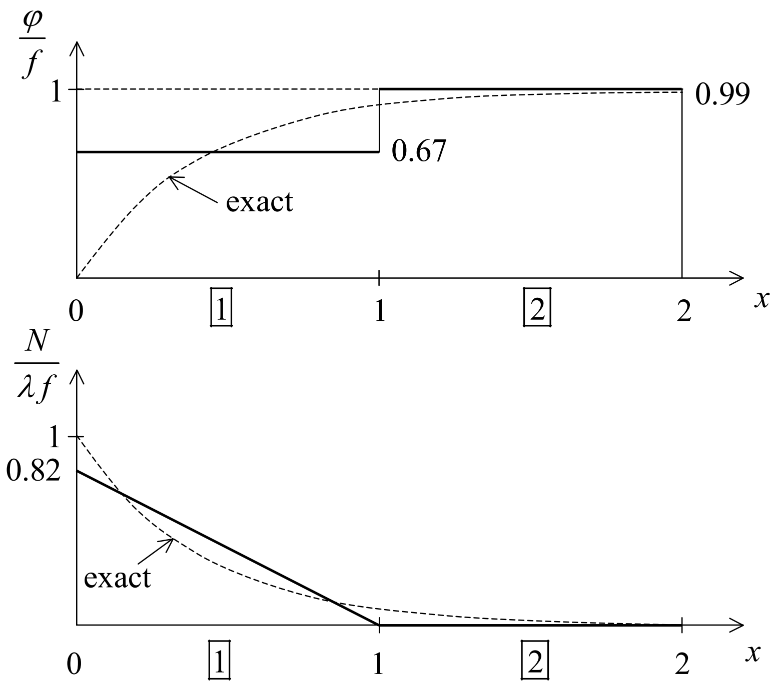


Fig. 7.12: Comparison with exact solution.

***"short" bar:* $\alpha = 2$**
Now the relations in (7.29) become:

$$\left.\begin{array}{l} \frac{4}{3}\varphi_1 + \frac{1}{2}\varphi_2 = \frac{5}{6}f \\ \frac{1}{2}\varphi_1 + \frac{7}{3}\varphi_2 = \frac{11}{6}f \end{array}\right\} \rightarrow \quad \varphi_1 = \frac{37}{103}f = 0.36f \quad ; \quad \varphi_2 = \frac{73}{103}f = 0.71f$$

Solving the normal forces provide:

$$N_0 = \frac{96}{103}fa \quad ; \quad N_1 = \frac{30}{103}fa \quad ; \quad N_2 = 0$$

with $a = \alpha\lambda/2 = \lambda$ this becomes:

$$N_0 = \frac{96}{103}f\lambda = 0.93f\lambda \quad ; \quad N_1 = \frac{30}{103}f\lambda = 0.29f\lambda \quad ; \quad N_2 = 0$$

In Fig. 7.13 the graphical representation of these results can be found.

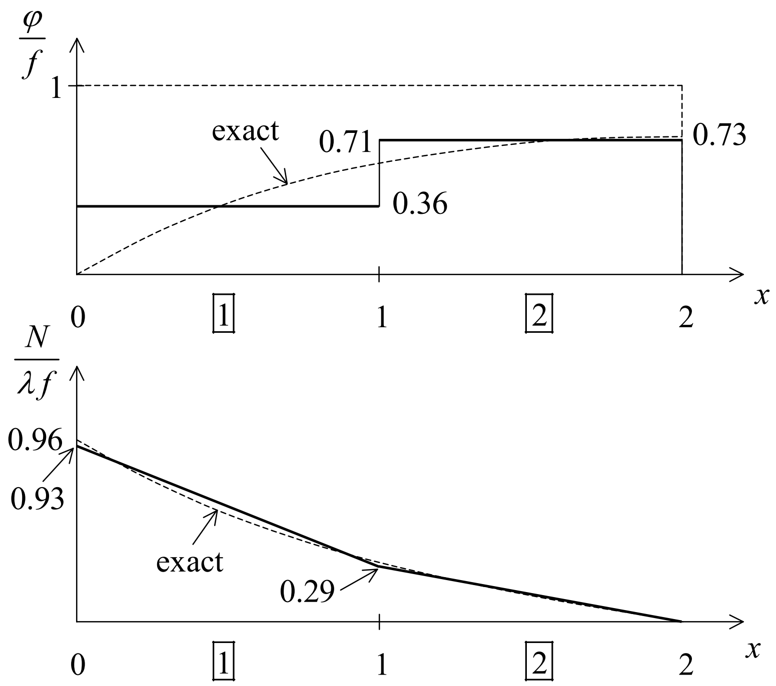


Fig. 7.13: Comparison with exact solution.

Calculation of the displacements

When the displacement u_2 in node 2 has to be determined, a (not known) force F_2 has to be applied at that position (see Fig. 7.14). This force is incorporated in the derivation as extra redundant together with φ_1 and φ_2 (when u_1 in node 1 has to be determined, a force F_1 has to be applied in node 1; when both u_1 and u_2 have to be determined, both F_1 and F_2 have to be applied).

The total complementary energy then becomes:

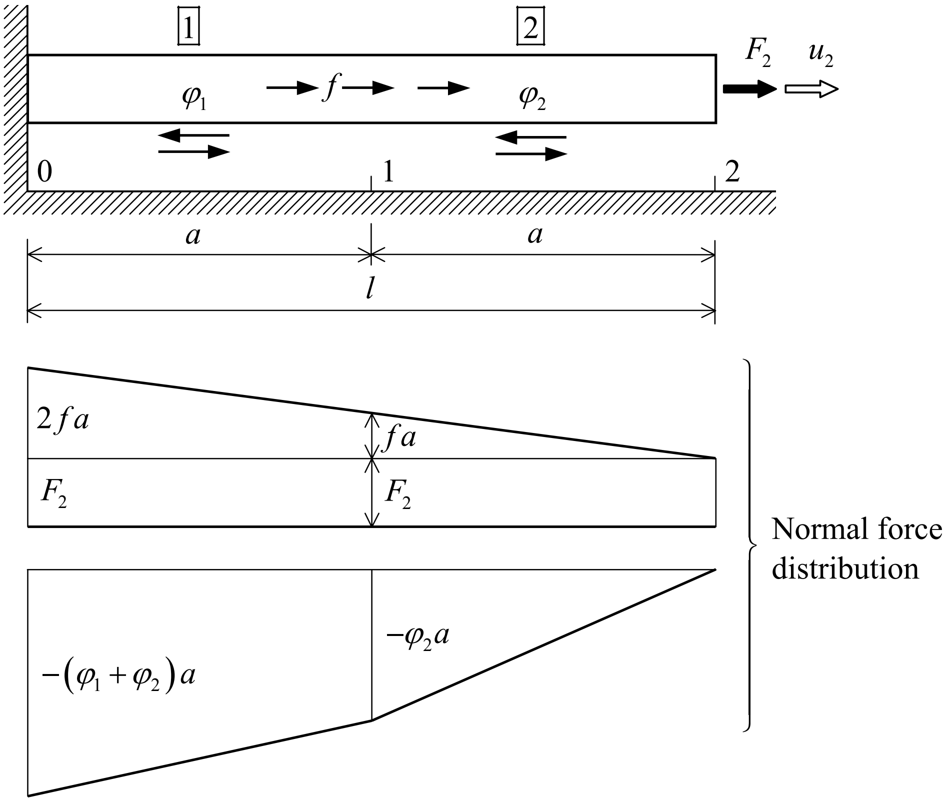


Fig. 7.14: Calculation of displacements.

$$E_{compl} = \frac{a^3}{EA}\left(\frac{4}{3}f^2 - \frac{5}{6}f\varphi_1 - \frac{11}{6}f\varphi_2 + \frac{1}{6}\varphi_1^2 + \frac{1}{2}\varphi_1\varphi_2 + \frac{2}{3}\varphi_2^2\right) + \frac{a}{k}\left(\frac{1}{2}\varphi_1^2 + \frac{1}{2}\varphi_2^2\right) + \\ + \frac{1}{EA}\left(2F_2 fa^2 - F_2\varphi_1 a^2 - F_2\varphi_2 a^2 + F_2^2 l\right) - F_2 u_2 \quad (7.30)$$

Subsequently variation with respect to φ_1, φ_2 and F_2 delivers three equations:

$$\begin{aligned} \left(\frac{2}{3}+\frac{4}{a^2}\right)\varphi_1 &+ \frac{1}{2}\varphi_2 - \frac{1}{a}F_2 = \frac{5}{6}f \\ \frac{1}{2}\varphi_1 &+ \left(\frac{4}{3}+\frac{4}{a^2}\right)\varphi_2 - \frac{1}{a}F_2 = \frac{11}{6}f \\ -\frac{1}{2}\varphi_1 &- \frac{3}{2}\varphi_2 + \frac{2}{a}F_2 = \frac{EA}{a^2}u_2 - 2f \end{aligned} \quad (7.31)$$

For $F_2 = 0$, the first two equations provide the same values for φ_1 and φ_2. For u_2 it then follows from the third equation:

$$u_2 = \left(2f - \frac{1}{2}\varphi_1 - \frac{3}{2}\varphi_2\right)\frac{a^2}{EA} = \left(\frac{2f - \frac{1}{2}\varphi_1 - \frac{3}{2}\varphi_2}{k}\right)\frac{a^2}{4}$$

For the “long” and “short” bar it respectively is found:

$$\alpha = 2\sqrt{6} \quad \rightarrow \quad \left. \begin{array}{l} \varphi_1 = \dfrac{2}{3} f \\ \varphi_2 = \ \ f \end{array} \right\} \quad \rightarrow \quad u_2 = \frac{\alpha^2}{24}\frac{f}{k} = \frac{f}{k} \qquad \text{(exact solution: } u_2 = 0.9999\frac{f}{k}\text{)}$$

$$\alpha = 2 \quad \rightarrow \quad \left. \begin{array}{l} \varphi_1 = \dfrac{37}{103} f \\ \varphi_2 = \dfrac{73}{103} f \end{array} \right\} \quad \rightarrow \quad u_2 = \frac{78}{103}\frac{f}{k}\frac{\alpha^2}{4} = 0.76\frac{f}{k} \qquad \text{(exact solution: } u_2 = 0.73\frac{f}{k}\text{)}$$

7.3 Epilogue

The results of both the displacement and the force methods are on the average calculated well. In the chosen computational example only two elements were defined. By increasing the amount of elements, the correspondence with the exact solution will become better.

In the displacement method the displacement u is calculated, followed by the calculation of the normal force N and the spring load s.
In the force method, firstly the redundant φ (equal to the spring load s) is obtained followed by the normal force N and displacement u.

In the displacement method the distribution of u (and therefore also s) is linear, and is the normal force N is constant per element.
In the force method this is quite the reverse. Then N is linear and φ (thus s) is constant across each element.

Both the displacement and the force methods lead in the chosen example to approximated solutions. In the displacement method the equilibrium x-direction will be violated, and in the force method compatibility between the bar and the springs will not be satisfied.
That the equilibrium is not correct in the displacement method can be shown by calculation of the normal force corresponding with the spring load s and comparison of this force with the previously obtained solution. For both examples this is shown in Fig. 7.15.

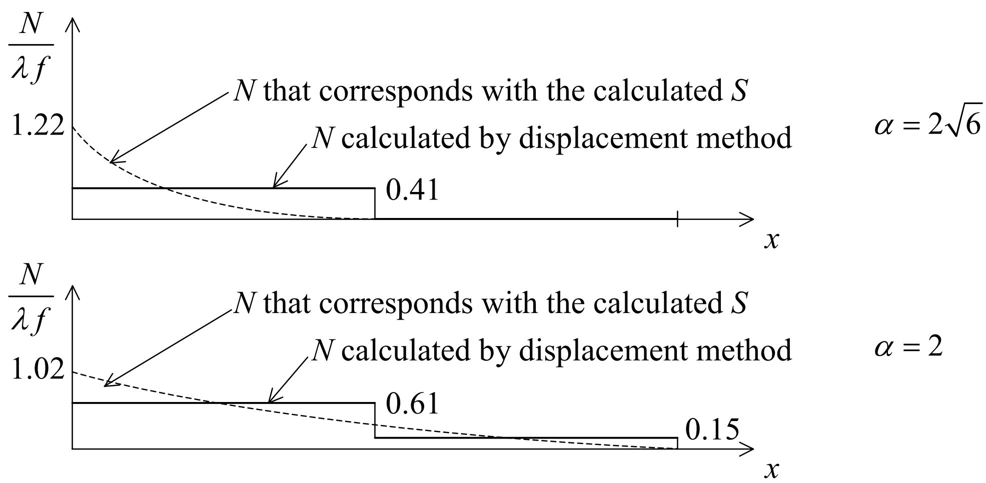


Fig. 7.15: Violation of equilibrium.

That compatibility is violated in the force method can clearly be seen, when the displacement u of the springs resulting from the spring load s is compared with the displacement u in the bar resulting from the normal force N, as shown in Fig. 7.16.

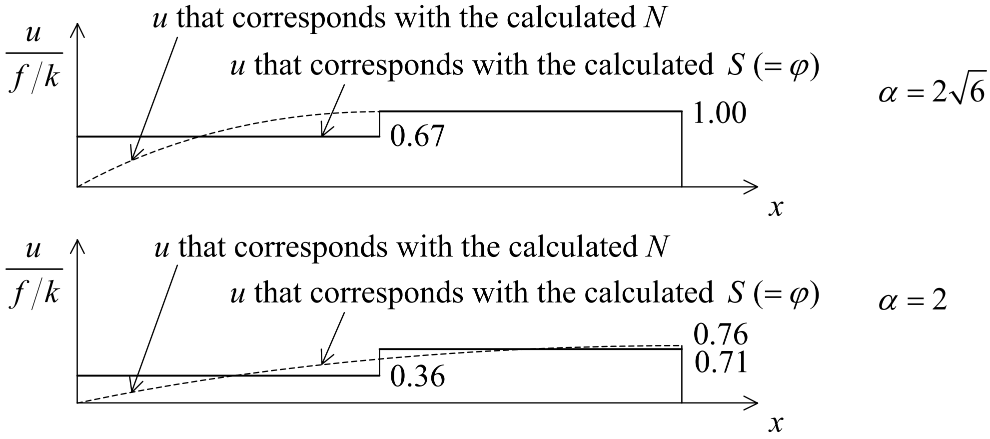


Fig. 7.16: Violation of compatibility.

A final remark should be made concerning the suitability of both methods for calculations on a computer. The displacement method has the advantage that per element it is dealt with degrees of freedom only belonging to that specific element. In the force method, the stress state in an element is normally also depending on redundants defined in other elements, which makes it difficult to programme a simple computer algorithm. For this reason the displacement method is given preference above the force method in computer applications.

8 Characteristics of the approximations

A linear-elastic system is considered loaded by n loads F_1 up to F_n, with corresponding displacements u_1 up to u_n. The loads may be point loads but also generalised loads are possible.

8.1 Displacement method

In the displacement method a displacement field is interpolated between the discrete degrees of freedom u_1 up to u_n. Doing so, a quadratic expression is found for the potential energy:

$$E_{pot} = \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{f} \tag{8.1}$$

where $\boldsymbol{u}$ is the vector containing all discrete displacements and $\boldsymbol{f}$ is the force vector. The matrix $\boldsymbol{K}$ is a stiffness matrix. The forces $\boldsymbol{f}$ are given (and therefore exact), and the (approximated) displacements $\boldsymbol{u}$ are to be found.
The potential energy is stationary if:

$$\frac{\partial E_{pot}}{\partial \boldsymbol{u}} = \boldsymbol{0} \quad \rightarrow \quad \boldsymbol{Ku} - \boldsymbol{f} = \boldsymbol{0} \quad \rightarrow \quad \boldsymbol{Ku} = \boldsymbol{f} \tag{8.2}$$

The stationary value of E_{pot} is:

$$E_{pot} = \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{f} = -\frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\,\boldsymbol{u} = -\frac{1}{2}\boldsymbol{u}^T \boldsymbol{f} \tag{8.3}$$

where the term $\frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\,\boldsymbol{u}$ is just the deformation energy. Generally, the exact minimum shall not be found for the assumed displacement field but only a neighbouring approximation. For

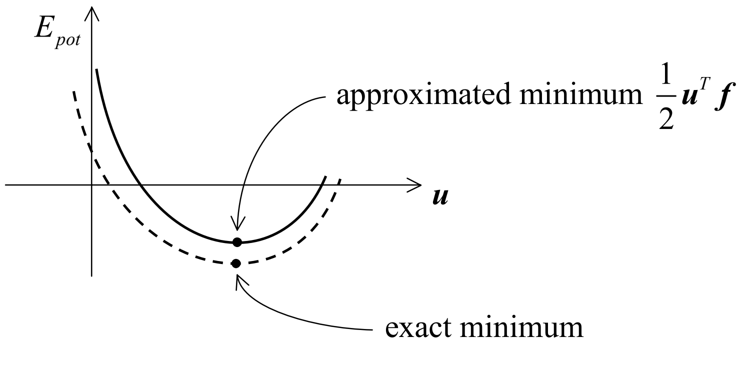


Fig. 8.1: Approximated and exact minimum of potential energy.

the term $\frac{1}{2}\boldsymbol{u}^T \boldsymbol{f}$ a value is found which is too small, as shown in Fig. 8.1. Since $\boldsymbol{f}$ is fixed, as a rule $\boldsymbol{u}$ will be too small. It also can be said that the deformation energy is underestimated. Because for the term $\frac{1}{2}\boldsymbol{u}^T \boldsymbol{K}\,\boldsymbol{u}$ a value is found that is too small.

8.2 Force method

In the force method, the displacements $\boldsymbol{u}$ are considered given and the associated forces $\boldsymbol{f}$ are calculated. On bases of equilibrium, a stress field is chosen that is corresponding with the forces $\boldsymbol{f}$ (the redundants). In this case the complementary energy becomes a quadratic expression:

$$E_{compl} = \frac{1}{2} \boldsymbol{f}^T \boldsymbol{C}\, \boldsymbol{f} - \boldsymbol{f}^T \boldsymbol{u} \tag{8.4}$$

where $\boldsymbol{C}$ is the flexibility or compliance matrix. The displacements $\boldsymbol{u}$ are given and the (approximated) forces $\boldsymbol{f}$ are to be found.
The complementary energy is stationary if:

$$\frac{\partial E_{compl}}{\partial \boldsymbol{f}} = \boldsymbol{0} \quad \rightarrow \quad \boldsymbol{C}\, \boldsymbol{f} - \boldsymbol{u} = \boldsymbol{0} \quad \rightarrow \quad \boldsymbol{C}\, \boldsymbol{f} = \boldsymbol{u} \tag{8.5}$$

The stationary value of E_{compl} equals:

$$E_{compl} = \frac{1}{2} \boldsymbol{f}^T \boldsymbol{C}\, \boldsymbol{f} - \boldsymbol{f}^T \boldsymbol{u} = -\frac{1}{2} \boldsymbol{f}^T \boldsymbol{C}\, \boldsymbol{f} = -\frac{1}{2} \boldsymbol{f}^T \boldsymbol{u} \tag{8.6}$$

where the term $\frac{1}{2} \boldsymbol{f}^{\mathrm{T}} \boldsymbol{C}\, \boldsymbol{f}$ is just the deformation energy. Because a stress field has been assumed the calculated value of $\frac{1}{2} \boldsymbol{f}^{\mathrm{T}} \boldsymbol{u}$ will be too small as shown in Fig. 8.2. Therefore, it

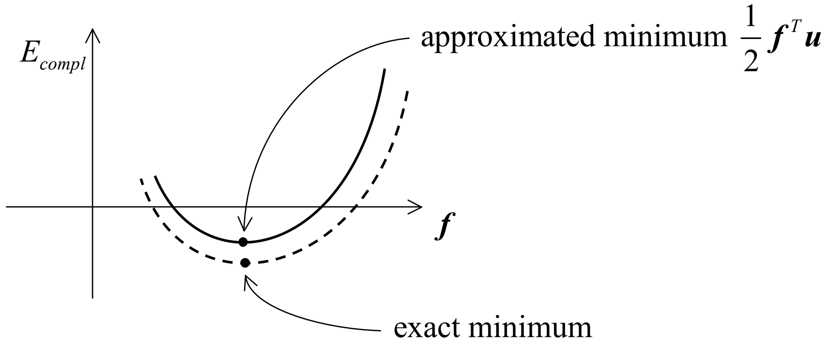


Fig. 8.2: Approximated and exact minimum of complementary energy.

has to be concluded that for a given $\boldsymbol{u}$ a force $\boldsymbol{f}$ is calculated, which is too small. In other words, when the correct load $\boldsymbol{f}$ is being used, the found displacements $\boldsymbol{u}$ will be too large. This also means that the deformation energy $\frac{1}{2} \boldsymbol{f}^T \boldsymbol{C}\, \boldsymbol{f}$ ($= \frac{1}{2} \boldsymbol{f}^{\mathrm{T}} \boldsymbol{u}$) then will be too large.

8.3 Conclusion

In approximated solutions obtained by the principle of minimum potential energy, the stiffness is overestimated. As a rule, the displacements are too small. Better expressed: the deformation energy E_s is underestimated.

In approximated solutions obtained by the principle of minimum complementary energy, the stiffness is underestimated. As a rule, the displacements are too large. Better expressed: the deformation energy E_c is overestimated.

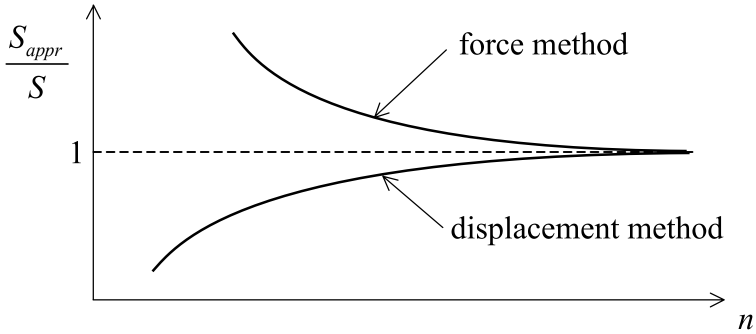


Fig. 8.3: Convergence of force method and displacement method towards exact solution.

When more and more elements n are used, meaning that the assumed displacement field or the assumed stress field are in better agreement with the exact solution, the two approximation methods will converge to the same exact solution. In this way an upper-bound and a lower-bound solution encloses the exact solution (see Fig. 8.3).