Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your <u>name</u> and <u>study number</u> at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Monday 23 June 2003, 14:00 – 17:00 hours

Problem 1 (4 points)

A three storey office building is founded on a round cellar of reinforced concrete (Figure 1). Both the structure and the dominant loading are axisymmetric. We model the floor of the cellar as a linear elastic plate of thickness t and radius a (Figure 2). The Young's modulus is E and the Poison's ration is ν. The soil is modelled with homogeneously distributed springs with stiffness k [kN/m³]. The cellar wall is considered to be a rigid cylinder, therefore, the edges of the plate can translate vertically but cannot rotate. The plate is loaded by a distributed load p [kN/m²] and a line load f [kN/m] at the edge.

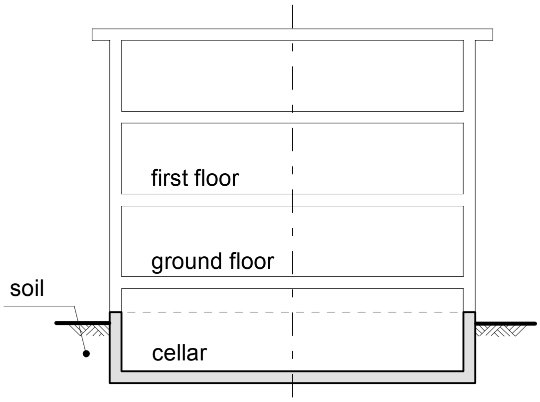


Figure 1. Cross-section of a round office building

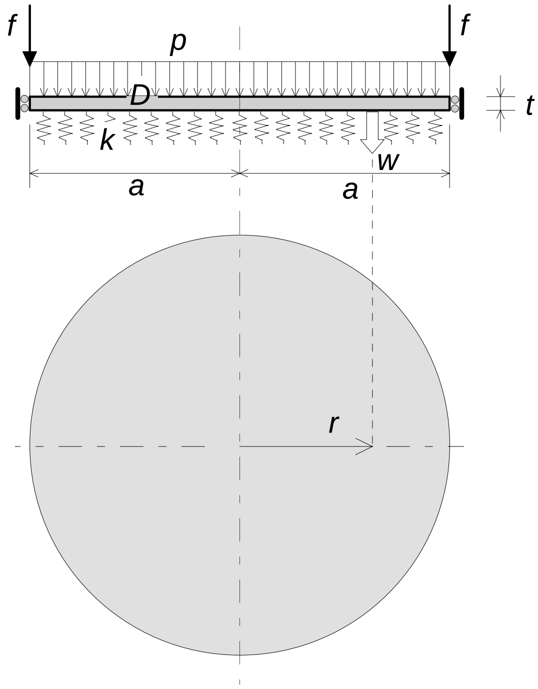


Figure 2. Axisymmetric plate

a As you know the differential equation of a axisymmetric plate is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right)=\frac{p}{D}$$

where w is the displacement

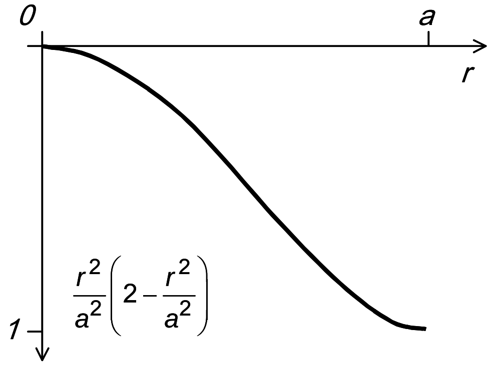
and $D=\dfrac{Et^3}{12(1-\nu^2)}$ the plate stiffness of the floor.

How can we introduce the distributed springs in this differential equation?

b An analytical solution exists for the differential equation that you formulated in question **a**. However, the solution is difficult to handle because it contains complex Bessel-functions. Therefore we want to approximate the solution using potential energy.

Give the formula for the potential energy of the spring supported plate. Evaluate the formula to a function of the displacement w and the curvatures κ_{rr} and $\kappa_{\theta\theta}$ or the curvatures κ_{xx} κ_{yy} and ρ_{xy} .

c We choose the following function for the deflection w of the plate.



$$w = C_1 + C_2 \frac{r^2}{a^2}\left(2 - \frac{r^2}{a^2}\right)$$

Formulate the boundary conditions of this function. Does the function meet these conditions? Explain your answer.

d Substitution of w in the formula of the potential energy gives the following result.

$$E_{pot} = \pi a^2 k \left(C_2^2 \tfrac{4}{15} \frac{40+\beta}{\beta} + \tfrac{1}{2} C_1^2 + \tfrac{2}{3} C_1 C_2 - C_1\left(\frac{p}{k} + \frac{2f}{ak}\right) - 2C_2\left(\frac{p}{3k} + \frac{f}{ak}\right) \right)$$

where $\beta = \dfrac{ka^4}{D}$.

Derive two equations from which the constants C_1 and C_2 can be solved. (You do not need to solve C_1 and C_2.)

e The constants have been solved for you with the following result.

$$C_1 = \frac{p}{k} + 3\frac{f}{ak}\frac{160-\beta}{240+\beta}$$

$$C_2 = \tfrac{15}{2}\frac{f}{ak}\frac{\beta}{240+\beta}$$

Calculate the moment m_{rr} in the middle (r = 0) and at the edge (r = a) of the plate.

Problem 2 (3 points)

A symmetric box-girder beam is loaded in torsion. The dimensions are drawn in Figure 3. The cross-section areas of the cells are A_1 and A_2. These are calculated between the dashed lines in the centres of the walls. The wall thickness t is small compared to the width b.

a Calculate the torsion stiffness GI_w.

b Calculate the largest shear stress τ as function of the moment M_w.

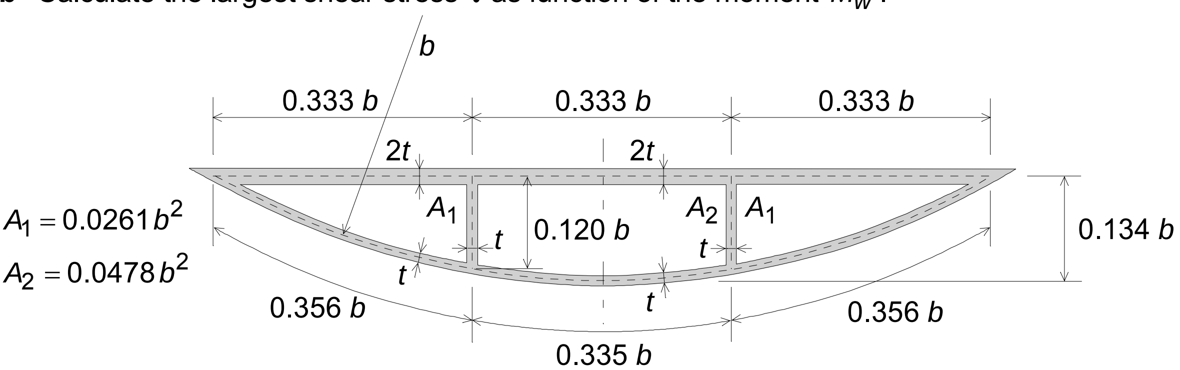


Figure 3. Cross-section of a box-girder beam

Problem 3 (3 points)

A thin axisymmetric plate is clamped at the edge $r = a$ and has a free edge at $r = b$ (Figure 4.). The plate stiffness is D and the Poisson's ratio is ν. The loading of the plate consists of a temperature gradient over the thickness of the plate. The temperature at the upper face of the plate is higher than at the bottom face. If the plate would have two free edges, stresses would not occur and the curvature of the plate would be $-\kappa_T$ in all directions and in each point of the plate ($\kappa_T > 0$).

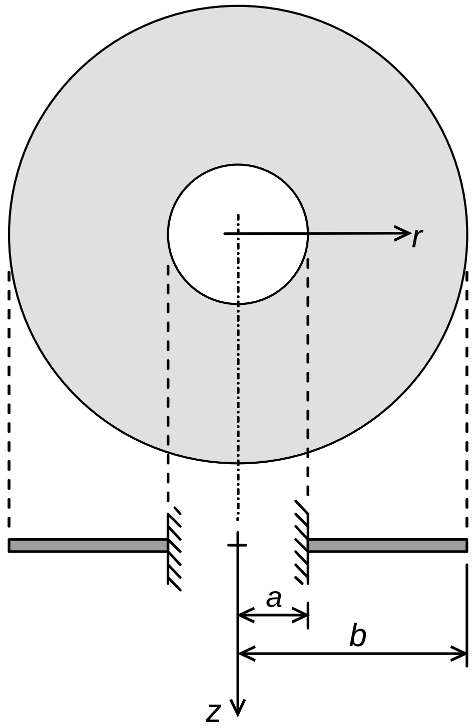


Figure 4. Axisymmetric plate

a The constitutive equations of a axisymmetric plate without temperature loading are

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta}\right)$$
$$m_{\theta\theta} = D\left(\nu\kappa_{rr} + \kappa_{\theta\theta}\right)$$

Show that these equations change into the following equations when the temperature loading is present.

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta} + (1+\nu)\kappa_T\right)$$
$$m_{\theta\theta} = D\left(\nu\kappa_{rr} + \kappa_{\theta\theta} + (1+\nu)\kappa_T\right)$$

b The shear force V_r can be determined by

$$V_r = D\,\mathrm{L}\varphi \qquad \text{(a)}$$

where

$$V_r = r\,v_r$$
$$\mathrm{L} = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r$$
$$\varphi = -\frac{dw}{dr}$$

The same shear force can be derived from

$$v_r = -D\frac{d}{dr}\nabla^2 w \qquad \text{(b)}$$

Show that equations (a) and (b) are equivalent.

c The homogeneous solution of the differential equation is

$$w(r) = C_1 + C_2 r^2 + C_3 \ln r + C_4 r^2 \ln r$$

If $C_1 = C_3 = C_4 = 0$ then only the term $C_2 r^2$ would remain. Show that in this case always $m_{rr} = m_{\theta\theta}$ and $v_r = 0$.
If $C_1 = C_2 = C_4 = 0$ then only the term $C_3 \ln r$ would remain. Assume that $\kappa_T = 0$. Show that in this case always $m_{rr} = -m_{\theta\theta}$ and $v_r = 0$.

d Discuss why the shear force in the problem at hand will be zero for all values of *r* in the interval $a \le r \le b$. Do not use the homogeneous solution of question **c**.

e Formulate the boundary conditions from which C_1, C_2, C_3 and C_4 can be solved. (You do not need to solve the constants.)

f The solution of the constants is

$$C_1 = (\ln a - \tfrac{1}{2})\frac{1+\nu}{n}\kappa_T$$
$$C_2 = \frac{1}{2a^2}\frac{1+\nu}{n}\kappa_T$$
$$C_3 = -\frac{1+\nu}{n}\kappa_T$$
$$C_4 = 0$$

where

$$n = \frac{1+\nu}{a^2} + \frac{1-\nu}{b^2}$$

Consequently, the displacement field is

$$w = \left(-\tfrac{1}{2} + \tfrac{1}{2}\frac{r^2}{a^2} - \ln\frac{r}{a}\right)\frac{1+\nu}{n}\kappa_T$$

Choose ν = 0 and *b* = 2*a* and check that boundary conditions are satisfied.

g Draw a graph of m_{rr} and $m_{\theta\theta}$ in the interval $a \le r \le 2a$. Write the value of the moments at both ends of the interval in the graph.

h Predict without analysis the moments if the clamped edge is replaced by a simply supported edge. Explain your answer.

Exam CT5141, 23 June 2003
Answers to Problem 1

a Differential Equation

The distributed springs can be introduced as a reduction of the distributed load. The new differential equation becomes therefore

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right)=\frac{p-k\,w}{D}$$

This can be written somewhat more elegantly.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right)+\frac{k\,w}{D}=\frac{p}{D}$$

b Potential Energy

De potential energy consists of 1) the strain energy in the plate, 2) the strain energy in the springs, 3) the position energy of the distributed load *q* and 4) the position energy due to the edge load *f*.

$$E_{pot}=\tfrac{1}{2}\int_A\left(m_{xx}\kappa_{xx}+m_{yy}\kappa_{yy}+m_{xy}\rho_{xy}\right)dA+\int_A\tfrac{1}{2}k\,w^2\,dA-\int_A p\,w\,dA-\int_s f\,w\,ds$$

where *A* is the area of the plate and *s* the edge of the plate.
We can also write this in polar coordinates.

$$E_{pot}=\int_{r=0}^{a}\left(\tfrac{1}{2}m_{rr}\kappa_{rr}+\tfrac{1}{2}m_{\theta\theta}\kappa_{\theta\theta}+\tfrac{1}{2}kw^2-pw\right)2\pi r\,dr-2\pi a\,f\,w(a)$$

The constitutive equations are

$$m_{rr}=D(\kappa_{rr}+\nu\kappa_{\theta\theta})$$
$$m_{\theta\theta}=D(\kappa_{\theta\theta}+\nu\kappa_{rr})\,.$$

Substitution of these in the potential energy function gives

$$E_{pot}=\int_{r=0}^{a}\left(\tfrac{1}{2}D(\kappa_{rr}+\nu\kappa_{tt})\kappa_{rr}+\tfrac{1}{2}D(\kappa_{\theta\theta}+\nu\kappa_{rr})\kappa_{\theta\theta}+\tfrac{1}{2}kw^2-pw\right)2\pi r\,dr-2\pi a\,f\,w(a)$$

which can be simplified to

$$E_{pot}=\int_{r=0}^{a}\left(D(\tfrac{1}{2}\kappa_{rr}^2+\nu\kappa_{\theta\theta}\kappa_{rr}+\tfrac{1}{2}\kappa_{\theta\theta}^2)+\tfrac{1}{2}kw^2-pw\right)2\pi r\,dr-2\pi a\,f\,w(a)\,.$$

c Boundary Conditions

The function *w* needs to be kinematically admissible. So, it should fulfil the kinematic boundary conditions. These follow from symmetry and the fixed edge rotation.

$$\left.\frac{dw}{dr}\right|_{r=0} = 0 \quad \text{and} \quad \left.\frac{dw}{dr}\right|_{r=a} = 0$$

The slope of the plate is $\dfrac{dw}{dr} = C_2 \dfrac{4r}{a^2}\left(1 - \dfrac{r^2}{a^2}\right)$

$$r = 0 \quad \rightarrow \quad \frac{dw}{dr} = 0$$

$$r = a \quad \rightarrow \quad \frac{dw}{dr} = 0$$

Therefore, the function correctly fulfils the kinematic boundary conditions.

Encore (not an exam question)
De dynamic boundary conditions are

$$r = 0 \quad \rightarrow \quad v_r = 0$$
$$r = a \quad \rightarrow \quad v_r = f$$

These need not be fulfilled for application of the principle of minimum potential energy.

d Coefficients

Potential energy must be minimal with respect to the coefficients that describe the displacements. In the minimum the derivatives equal zero.

$$E_{pot} = \pi a^2 k \left(C_2^2 \tfrac{4}{15} \frac{40+\beta}{\beta} + \tfrac{1}{2} C_1^2 + \tfrac{2}{3} C_1 C_2 - C_1 \left(\frac{p}{k} + \frac{2f}{ak}\right) - 2C_2 \left(\frac{p}{3k} + \frac{f}{ak}\right) \right)$$

$$\begin{cases} \dfrac{\partial E_{pot}}{\partial C_1} = \pi a^2 k \left(C_1 + \tfrac{2}{3} C_2 - \dfrac{p}{k} - \dfrac{2f}{ak} \right) = 0 \\ \dfrac{\partial E_{pot}}{\partial C_2} = \pi a^2 k \left(2C_2 \tfrac{4}{15} \dfrac{40+\beta}{\beta} + \tfrac{2}{3} C_1 - \dfrac{2p}{3k} - \dfrac{2f}{ak} \right) = 0 \end{cases}$$

From this C_1 and C_2 can be solved.

e Moments

$$w = C_1 + C_2 \left(2\frac{r^2}{a^2} - \frac{r^4}{a^4} \right)$$

$$\frac{dw}{dr} = C_2 \left(4\frac{r}{a^2} - 4\frac{r^3}{a^4} \right)$$

$$\kappa_{rr} = -\frac{d^2 w}{dr^2} = -C_2 \left(4\frac{1}{a^2} - 12\frac{r^2}{a^4} \right) \qquad \kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = -C_2 \left(4\frac{1}{a^2} - 4\frac{r^2}{a^4} \right)$$

$$m_{rr} = D(\kappa_{rr} + \nu\kappa_{\theta\theta}) = D\left(-C_2 \left(4\frac{1}{a^2} - 12\frac{r^2}{a^4} \right) - \nu C_2 \left(4\frac{1}{a^2} - 4\frac{r^2}{a^4} \right) \right) = -\frac{4D}{a^2} C_2 \left(1 + \nu - (3+\nu)\frac{r^2}{a^2} \right) =$$

$$= -\frac{4D}{a^2} \frac{15}{2} \frac{f}{ak} \frac{\beta}{240+\beta} \left(1 + \nu - (3+\nu)\frac{r^2}{a^2} \right) = -30 f a \frac{1}{240+\beta} \left(1 + \nu - (3+\nu)\frac{r^2}{a^2} \right)$$

$$\boxed{m_{rr}(0) = -30 f a \frac{(1+\nu)}{240+\beta}} \qquad \boxed{m_{rr}(a) = 60 f a \frac{1}{240+\beta}}$$

Encore 1 (not an exam question)
Note that the distributed load p does not contribute to the moments. Apparently p is directly carried by the distributed springs.

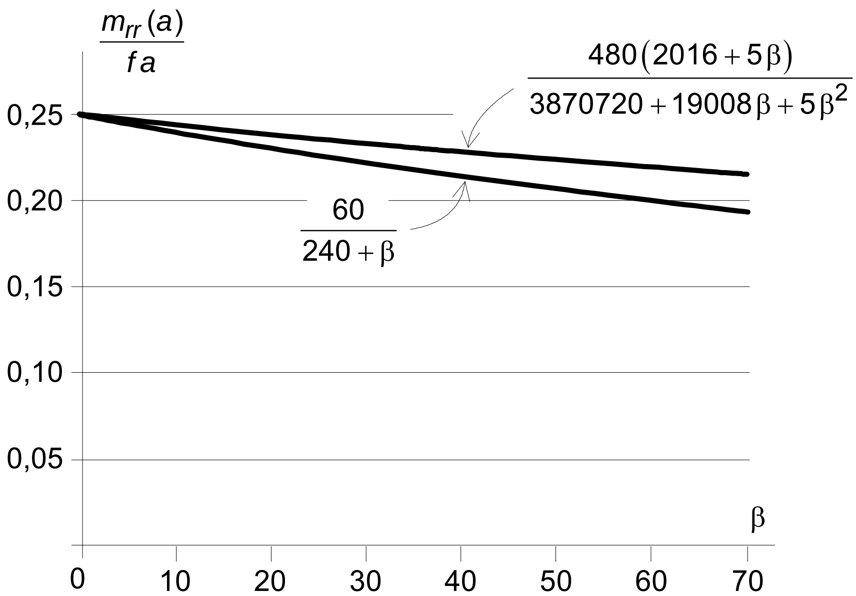
Note too that the moment in the middle of the plate is negative. At that location the reinforcement shall be placed in the top of the floor. At the edge the reinforcement shall be placed in the bottom of the floor.

Encore 2
The deflection can also be approximated by a polynomial of a higher order.

$$w = C_1 + C_2 \frac{r^2}{a^2} + C_3 \frac{r^4}{a^4} + C_4 \frac{r^6}{a^6}$$

Cleary, this provides a better approximation. Processing of the kinematic boundary conditions gives $C_4 = -\frac{1}{3} C_2 - \frac{2}{3} C_3$. The largest moment $m_{rr}(a)$ becomes

$m_{rr}(a) = \dfrac{f\,a\,480(2016+5\beta)}{3870720+19008\beta+5\beta^2}$. Below both approximations are plotted for practical values of β.



The largest difference is approximately 10 %.

Encore 3
We can use a simple trick to estimate the moments in the floor. Assume that the support reaction from the soil onto cellar floor is evenly distributed. Now we can ignore the distributed springs and use the normal plate theory.
The total load onto the floor is $F = f\,2\pi a + p\pi a^2$. The area of the floor is $A = \pi a^2$. Therefore, the evenly distributed support reaction is $\dfrac{F}{A} = \dfrac{f\,2\pi a + p\pi a^2}{\pi a^2} = \dfrac{f\,2}{a} + p$.
The moments in the floor are (Lecture book page 130)

$$m_{rr}(0) = \frac{(1+\nu)}{16}\left(p - \frac{f\,2}{a} - p\right)a^2 = -\tfrac{1}{8}(1+\nu) f a$$

$$m_{rr}(a) = -\frac{1}{8}\left(p - \frac{f2}{a} - p\right)a^2 = \tfrac{1}{4} f a$$

which corresponds to the result of question **e** for β = 0. As the graph above shows the trick is conservative and gives an error of at most 15 %.

Answers to Problem 2

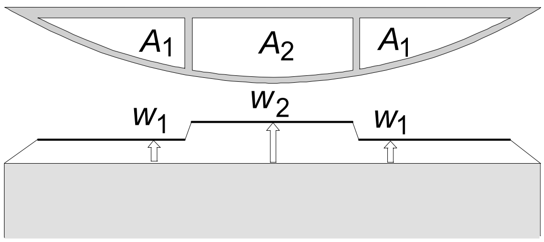
a Torsion Stiffness

We use the membrane analogy with weight-less plates. Due to symmetry just two equilibrium equations exist.

Equilibrium plate 1 $\quad 0.0261 b^2 p = 0.333\, b \dfrac{s w_1}{2t} + 0.356\, b \dfrac{s w_1}{t} - 0.120\, b \dfrac{s(w_2 - w_1)}{t}$

Equilibrium plate 2 $\quad 0.0478\, b^2 p = 0.333 b \dfrac{s w_2}{2t} + 0.335 b \dfrac{s w_2}{t} + 2 \times 0.120 b \dfrac{s(w_2 - w_1)}{t}$

These equations can be evaluated



$$0.0261 \frac{bpt}{s} = 0.643 w_1 - 0.120 w_2$$
$$0.0478 \frac{bpt}{s} = -0.240 w_1 + 0.742 w_2$$

from which w_1 and w_2 can be solved.

$$w_1 = 0.0560 \frac{bpt}{s}$$
$$w_2 = 0.0825 \frac{bpt}{s}$$

From the membrane we go to the ϕ bubble using the following substitutions.

$$w = \phi$$
$$p = 2\theta$$
$$s = \frac{1}{G}$$

Therefore,

$$\phi_1 = 0.112 G b t \theta$$
$$\phi_2 = 0.165 G b t \theta \,.$$

The torsion moment M_t is two times the volume of the ϕ-bubble.

$$M_t = 2(2A_1\phi_1 + A_2\phi_2) = 2(2 \times 0.0261 b^2 0.112 G b t \theta + 0.0478\, b^2 0.165 G b t \theta)$$
$$M_t = 0.0275\, G b^3 t \theta \qquad (\blacklozenge)$$

For a wire frame model of the box-girder beam we have $M_t = G I_t \theta$. So, the torsion stiffness is

$$\boxed{GI_t = 0.0275\, Gb^3 t}\,.$$

b Largest Shear Stress

The shear stress τ is the slope of the ϕ bubble. The largest slope occurs in the middle of the lower flange of the box-girder.

$$\tau_{max} = \frac{\phi_2}{t} = \frac{0.165 Gbt\theta}{t} = 0.165 Gb\theta$$

Equation (♦) can be rewritten as $\dfrac{M_t}{0.0275\, b^2 t} = Gb\theta$. Using this we obtain the shear stress

$$\boxed{\tau_{max} = \frac{M_t}{0.167\, b^2 t}}\,.$$

Answers to Problem 3

a Constitutive Equations

The deformation due to stresses is the total deformation minus the deformation due to temperature.

$$m_{rr} = D\big((\kappa_{rr} - (-\kappa_T)) + \nu(\kappa_{\theta\theta} - (-\kappa_T))\big) = D\big(\kappa_{rr} + \nu\kappa_{\theta\theta} + (1+\nu)\kappa_T\big)$$
$$m_{\theta\theta} = D\big(\nu(\kappa_{rr} - (-\kappa_T)) + (\kappa_{\theta\theta} - (-\kappa_T))\big) = D\big(\nu\kappa_{rr} + \kappa_{\theta\theta} + (1+\nu)\kappa_T\big)$$

b Shear Force

Substitution of

$$V_r = r\, v_r$$
$$\mathrm{L} = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r$$
$$\varphi = -\frac{dw}{dr}$$

into

$$V_r = D\mathrm{L}\varphi$$

gives

$$r\, v_r = Dr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\left(-\frac{dw}{dr}\right)$$
$$v_r = -D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w$$

Since the operator ∇^2 is defined as

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}$$

we can write this as

$$v_r = -D\frac{d}{dr}\nabla^2 w$$

which was to be proved.

c Terms

Term $w = C_2 r^2$

$$\frac{dw}{dr} = 2C_2 r$$

$$\frac{d^2w}{dr^2} = 2C_2$$

$$\kappa_{rr} = -\frac{d^2w}{dr^2} = -2C_2$$

$$\kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = -2C_2$$

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta} + (1+\nu)\kappa_T\right) = D\left(-2C_2 - \nu 2C_2 + (1+\nu)\kappa_T\right)$$

$$m_{\theta\theta} = D\left(\nu\kappa_{rr} + \kappa_{\theta\theta} + (1+\nu)\kappa_T\right) = D\left(-\nu 2C_2 - 2C_2 + (1+\nu)\kappa_T\right)$$

Therefore, $m_{rr} = m_{\theta\theta}$

$$v_r = \frac{d}{dr}m_{rr} + \frac{m_{rr}}{r} - \frac{m_{\theta\theta}}{r} = 0 + \frac{m_{rr}}{r} - \frac{m_{\theta\theta}}{r} = 0$$

Term $w = C_3 \ln r$

$$\frac{dw}{dr} = C_3\frac{1}{r}$$

$$\frac{d^2w}{dr^2} = -C_3\frac{1}{r^2}$$

$$\kappa_{rr} = -\frac{d^2w}{dr^2} = C_3\frac{1}{r^2}$$

$$\kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = -C_3\frac{1}{r^2}$$

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta} + (1+\nu)\kappa_T\right) = D\left(C_3\frac{1}{r^2} - \nu C_3\frac{1}{r^2} + 0\right) = D(1-\nu)\frac{C_3}{r^2}$$

$$m_{\theta\theta} = D\left(\nu\kappa_{rr} + \kappa_{\theta\theta} + (1+\nu)\kappa_T\right) = D\left(\nu C_3\frac{1}{r^2} - C_3\frac{1}{r^2} + 0\right) = -D(1-\nu)\frac{C_3}{r^2}$$

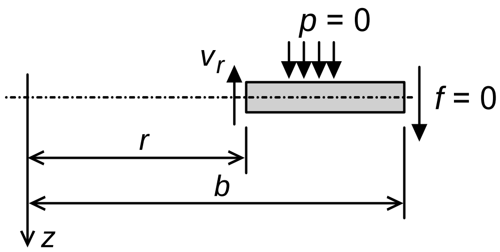
Therefore, $m_{rr} = -m_{\theta\theta}$

$$v_r = \frac{d}{dr}m_{rr} + \frac{m_{rr}}{r} - \frac{m_{\theta\theta}}{r} = -2D(1-\nu)\frac{C_3}{r^3} + \frac{D(1-\nu)\frac{C_3}{r^2}}{r} - \frac{-D(1-\nu)\frac{C_3}{r^2}}{r}$$

$$= -2D(1-\nu)\frac{C_3}{r^3} + \frac{2D(1-\nu)\frac{C_3}{r^2}}{r} = 0$$

d Equilibrium

Consider equilibrium in the z direction of a plate part between r and b ($r \geq a$). The plate does not have an edge load f at $r = b$ and does not have a distributed load p over the area of the plate. Hence $v_r = 0$.



e Boundary Conditions

$$r = a \rightarrow \begin{array}{l} w = 0 \\ \dfrac{dw}{dx} = 0 \end{array} \qquad r = b \rightarrow \begin{array}{l} m_{rr} = 0 \\ v_r = 0 \end{array}$$

f Check

Substitution of $\nu = 0$ and $b = 2a$ into $n = \dfrac{1+\nu}{a^2} + \dfrac{1-\nu}{b^2}$ gives $n = \dfrac{1}{a^2} + \dfrac{1}{4a^2} = \dfrac{5}{4a^2}$

$$w = \left(-\tfrac{1}{2} + \tfrac{1}{2}\frac{r^2}{a^2} - \ln\frac{r}{a} \right) \tfrac{4}{5} a^2 \kappa_T$$

$$\frac{dw}{dr} = \left(\frac{r}{a^2} - \frac{1}{\frac{r}{a}}\frac{1}{a} \right) \tfrac{4}{5} a^2 \kappa_T = \left(\frac{r}{a^2} - \frac{1}{r} \right) \tfrac{4}{5} a^2 \kappa_T$$

$$r = a \rightarrow \begin{array}{l} w = \left(-\tfrac{1}{2} + \tfrac{1}{2}\dfrac{a^2}{a^2} - \ln\dfrac{a}{a} \right) \tfrac{4}{5} a^2 \kappa_T = \left(-\tfrac{1}{2} + \tfrac{1}{2} - 0 \right) \tfrac{4}{5} a^2 \kappa_T = 0 \\ \dfrac{dw}{dr} = \left(\dfrac{a}{a^2} - \dfrac{1}{a} \right) \tfrac{4}{5} a^2 \kappa_T = 0 \end{array} \qquad \text{correct}$$

$$\frac{d^2w}{dr^2} = \left(\frac{1}{a^2} + \frac{1}{r^2} \right) \tfrac{4}{5} a^2 \kappa_T$$

$$\kappa_{rr} = -\frac{d^2w}{dr^2} = -\left(\frac{1}{a^2} + \frac{1}{r^2} \right) \tfrac{4}{5} a^2 \kappa_T$$

$$\kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = \left(-\frac{1}{a^2} + \frac{1}{r^2} \right) \tfrac{4}{5} a^2 \kappa_T$$

$$m_{rr} = D(\kappa_{rr} + \kappa_T) = D\left(-\left(\frac{1}{a^2} + \frac{1}{r^2} \right) \tfrac{4}{5} a^2 \kappa_T + \kappa_T \right) = \kappa_T D \left(\tfrac{1}{5} - \tfrac{4}{5}\frac{a^2}{r^2} \right)$$

$$m_{\theta\theta} = D(\kappa_{\theta\theta} + \kappa_T) = D\left(\left(-\frac{1}{a^2} + \frac{1}{r^2} \right) \tfrac{4}{5} a^2 \kappa_T + \kappa_T \right) = \kappa_T D \left(\tfrac{1}{5} + \tfrac{4}{5}\frac{a^2}{r^2} \right)$$

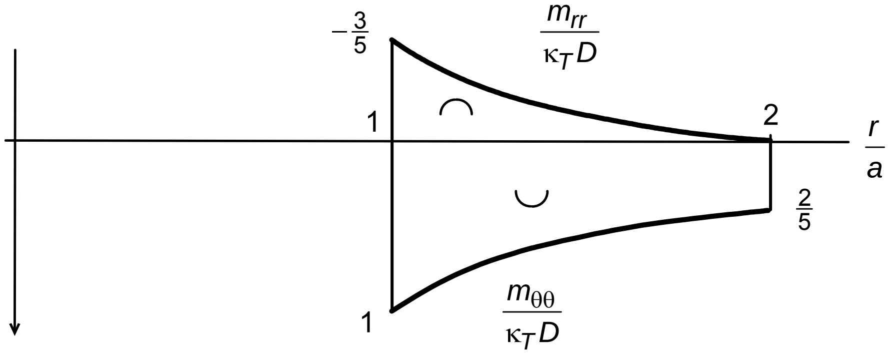
$$v_r = \frac{d}{dr} m_{rr} + \frac{m_{rr}}{r} - \frac{m_{\theta\theta}}{r} = \kappa_T D \left(\tfrac{8}{5}\frac{a^2}{r^3} \right) + \frac{\kappa_T D \left(\tfrac{1}{5} - \tfrac{4}{5}\frac{a^2}{r^2} \right)}{r} - \frac{\kappa_T D \left(\tfrac{1}{5} + \tfrac{4}{5}\frac{a^2}{r^2} \right)}{r}$$

$$= \kappa_T D \left(\tfrac{8}{5}\frac{a^2}{r^3} \right) + \frac{\kappa_T D \left(\tfrac{1}{5} - \tfrac{4}{5}\frac{a^2}{r^2} - \tfrac{1}{5} - \tfrac{4}{5}\frac{a^2}{r^2} \right)}{r} = \kappa_T D \left(\tfrac{8}{5}\frac{a^2}{r^3} \right) + \frac{\kappa_T D \left(-\tfrac{8}{5}\frac{a^2}{r^2} \right)}{r} = 0$$

$$r = b \rightarrow \qquad m_{rr} = \kappa_T D\left(\tfrac{1}{5} - \tfrac{4}{5}\frac{a^2}{(2a)^2}\right) = 0 \quad \text{correct}$$

$$v_r = 0$$

g Plot



h Simply Supported

In case of a simply supported edge at $r = a$ there is nothing that restricts the temperature induced curvature. Therefore the moments will be zero everywhere in the plate.