**Delft University of Technology**
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

**Exam CT5141 Theory of Elasticity**
Wednesday 21 January 2004, 9:00 – 12:00 hours

**Problem 1** (2.5 points)

Elasticity theory provides a solution for the radial stress $\sigma_{rr}$ that occurs in a bend of rectangular cross-section due to a moment $M$ (Figure 1)

$$\sigma_{rr} = -\tfrac{1}{2} E\, \varepsilon_i\, f_{rr}\ ,$$

where

$$f_{rr} = \frac{ab}{b^2 - a^2}\left( \frac{ab}{r^2} \ln\frac{b}{a} - \frac{a}{b}\ln\frac{r}{a} + \frac{b}{a}\ln\frac{r}{b} \right).$$

The moment in the cross-section can be derived as

$$M = -\tfrac{1}{2} E\, t \varepsilon_i C\ ,$$

where

$$C = -\tfrac{1}{4}(b^2 - a^2) + \frac{a^2 b^2}{b^2 - a^2}\left( \ln\frac{b}{a} \right)^2 .$$

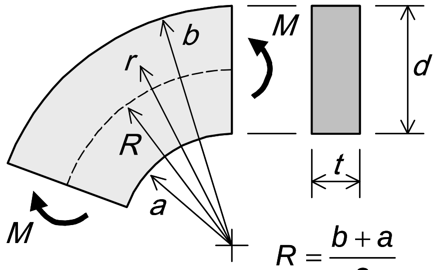


Figure 1. Curved beam

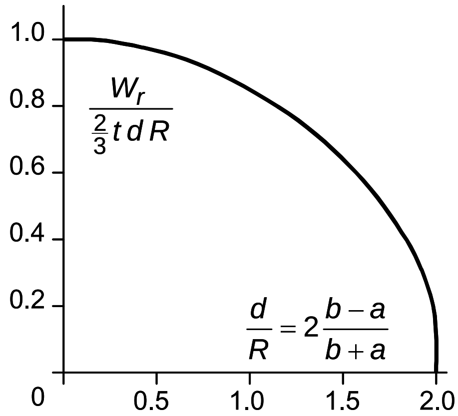


Figure 2. Function $W_r$

**a** Show that $\sigma_{rr}$ has a maximum at

$$r = ab\sqrt{\frac{2\ln\frac{b}{a}}{b^2 - a^2}}\ .$$

**b** Derive the formula for the elastic section modulus $W_r$ (Dutch: weerstandsmoment).

$$\sigma_{rr\,\max} = \frac{M}{W_r}$$

**c** The derived formula for $W_r$ has been plotted in Figure 2. It shows that $W_r$ goes to zero when $d$ goes to $2R$. Apparently, in this situation $\sigma_{rr\,\max}$ becomes infinite. Is this physically correct? Explain your answer.

**d** The first two non-zero terms of the Taylor expansion of $W_r$ to $d$ in zero are

$$\boxed{W_r \approx \tfrac{2}{3} t\, d\, R - \tfrac{13}{135} \frac{t\, d^3}{R}}\ .$$

It can be shown that this approximation has a deviation between -0.7% and 0% for $0<d<R$. Does this formula give correct results for straight beams? Explain your answer.

**e** Laminated wood is an orthotropic material. Therefore, Young's moduli of a wooden bend in the radial and circumferential directions differ. New formulae for the stresses can be derived including this effect. Write down the kinematic equations, constitutive equations and equilibrium equations that need be used in this derivation. (You do not need to derive a formula.)

**Problem 2** (2.5 points)

A triangular plate of thickness $t$ carries an evenly distributed load $q$ (Figure 3). The plate is simply supported at two edges. We assume that the kinematic, constitutive and equilibrium equations are linear. $D$ is the plate stiffness and $\nu$ is Poisson's rato. In this problem a formula is derived for the deflection of the plate using potential energy.

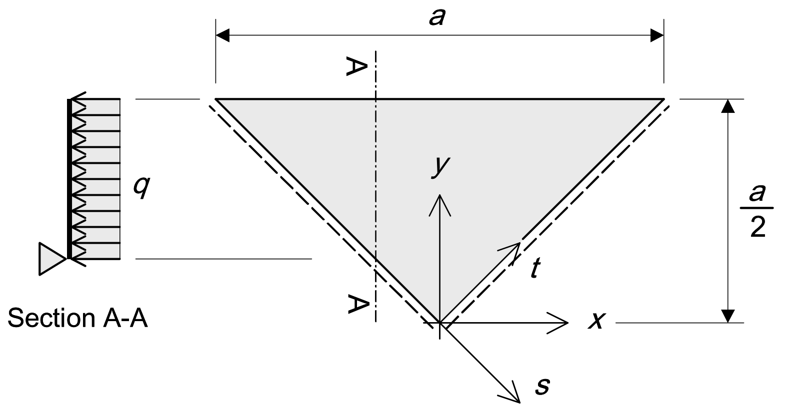


Figure 3. Triangular plate

The following general deflection function is assumed.

$$
\begin{aligned}
w = {} & a_1 + \\
& + a_2 x + a_3 y + \\
& + a_4 x^2 + a_5 xy + a_6 y^2 + \\
& + a_7 x^3 + a_8 x^2 y + a_9 x y^2 + a_{10} y^3 + \\
& + a_{11} x^4 + a_{12} x^3 y + a_{13} x^2 y^2 + a_{14} x y^3 + a_{15} y^4
\end{aligned}
$$

where $a_i$ are coefficients that need to be determined.

**a** Formulate the boundary conditions of the plate. Which boundary conditions need to be fulfilled for application of the principle of minimal potential energy?

**b** When the boundary conditions are processed the deflection function reduces to

$$w = (y^2 - x^2)\left(a_6 + a_9 x + a_{10} y + a_{13} x^2 + a_{14} xy + a_{15}(x^2 + y^2)\right).$$

Formulate the potential energy of the floor. Write it such that it can be evaluated by a mathematical computer program.

**c** The result of the computer evaluation is a large function of the following variables.

$$E_{\text{pot}} = E_{\text{pot}}(a, D, \nu, q, a_6, a_9, a_{10}, a_{13}, a_{14}, a_{15})$$

With respect to which variables needs the potential energy be minimal? How are the equations derived from which the coefficients can be solved?

**d** The largest deflection and moments are expected in the middle of the free edge $x$ = 0, $y$ = $a$/2. The computer solves this deflection as

$$w_{\max} = \frac{1}{3072} \frac{q\,a^4}{D(1-\nu)} \frac{232 - 167\nu + 7\nu^2}{21 - 10\nu - 2\nu^2} \,.$$

This can be approximated to

$$\boxed{w_{\max} = \frac{1}{278} \frac{q\,a^4}{D} \frac{1 - \frac{5}{22}\nu}{1-\nu}} \,.$$

The moments in this point are

$$m_{xx} = \tfrac{1}{64} qa^2 \frac{50 + 13\nu - 26\nu^2 + \nu^3}{21 - 10\nu - 2\nu^2}$$
$$m_{yy} = \tfrac{1}{32} qa^2 \frac{1 + 6\nu}{21 - 10\nu - 2\nu^2}$$
$$m_{xy} = 0 \,.$$

However, the moment $m_{yy}$ on the free edge should be zero. Explain this inconsistency.

**Problem 3** (2.5 points)

A prismatic prestressed concrete girder carries an eccentric force of 85 kN (Figure 4). Young's modulus $E$ = 30000 MPa and Poisson's ratio $\nu$ = 0.15. Due to the prestressing cracks do not occur. A computer program for section analysis has determined the cross-section properties of this girder.

| | |
|---|---|
| Centroid | $x$ = 157 mm, $y$ = 268 mm |
| Moments of Inertia | $I_{xx} = 3379\ 10^6$ mm$^4$ |
| | $I_{yy} = 382\ 10^6$ mm$^4$ |
| | $I_{xy} = 362\ 10^6$ mm$^4$ |
| Polar Moment of Inertia | $I_p = 3761\ 10^6$ mm$^4$ |
| Torsion Constant (Torsion Moment of Inertia) | $I_t = 717\ 10^6$ mm$^4$ |
| Warping Constant | $C_w = 765\ 10^{10}$ mm$^6$ |
| Shear Centre | $x$ = 180 mm, $y$ = 196 mm |

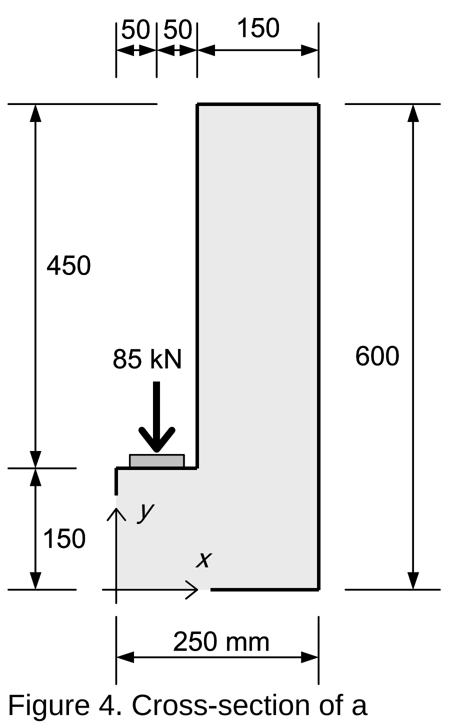

Figure 4. Cross-section of a prestressed concrete girder

**a** What percentage is the torsion stiffness of this girder larger than the torsion stiffness of a rectangular cross-section 150 x 600 mm?

**b** Explain shortly how the program computes the torsion properties of the cross-section.

**c** Calculate the torsion moment load onto the girder due to the eccentric force.

**d** What is an important restriction of the torsion theory of De Saint Venant?

**e** What is the warping constant $C_w$ used for?

**Problem 4** (2.5 points)

Derive the kinematic equation $\kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr}$ of an axisymmetric plate loaded in bending.

**Exam CT5141, 21 January 2004**
**Answers to Problem 1**

**a** Maximum

In the lecture book $f_{rr}$ has been plotted. The maximum value of $f_{rr}$ can be derived from from

$$\frac{df_{rr}}{dr} = 0 .$$

$$\frac{df_{rr}}{dr} = \frac{ab}{b^2 - a^2}\left( -2\frac{ab}{r^3} ln\frac{b}{a} - \frac{a}{b}\frac{1}{\frac{r}{a}}\frac{1}{a} + \frac{b}{a}\frac{1}{\frac{r}{b}}\frac{1}{b} \right)$$

$$= \frac{ab}{b^2 - a^2}\left( -2\frac{ab}{r^3} ln\frac{b}{a} + \frac{1}{r}(\frac{b}{a} - \frac{a}{b}) \right)$$

$$= \frac{ab}{b^2 - a^2}\left( -2\frac{ab}{r^3} ln\frac{b}{a} + \frac{1}{r}\frac{b^2 - a^2}{ab} \right)$$

Substitution of the solution

$$r = ab\sqrt{\frac{2\,ln\frac{b}{a}}{b^2 - a^2}}$$

gives

$$\frac{df_{rr}}{dr} = \frac{ab}{b^2 - a^2}\left( -2\frac{ab}{\left[ ab\sqrt{\frac{2\,ln\frac{b}{a}}{b^2 - a^2}} \right]^3} ln\frac{b}{a} + \frac{1}{ab\sqrt{\frac{2\,ln\frac{b}{a}}{b^2 - a^2}}}\frac{b^2 - a^2}{ab} \right)$$

$$= \frac{ab}{b^2 - a^2}\left( -2\frac{1}{a^2 b^2}\sqrt{\frac{b^2 - a^2}{2\,ln\frac{b}{a}}}^3 ln\frac{b}{a} + \frac{1}{ab}\sqrt{\frac{b^2 - a^2}{2\,ln\frac{b}{a}}}\frac{b^2 - a^2}{ab} \right)$$

$$= \frac{ab}{b^2 - a^2}\left( -2\frac{1}{a^2 b^2}\sqrt{\frac{b^2 - a^2}{2\,ln\frac{b}{a}}}\frac{b^2 - a^2}{2\,ln\frac{b}{a}} ln\frac{b}{a} + \frac{1}{ab}\sqrt{\frac{b^2 - a^2}{2\,ln\frac{b}{a}}}\frac{b^2 - a^2}{ab} \right) = 0 \quad \text{Q.E.D.}$$

**b** Section Modulus

$$f_{rr\ \max} = \frac{ab}{b^2-a^2}\left(\frac{ab}{a^2b^2\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}^{\,2}}\ln\frac{b}{a}-\frac{a}{b}\ln\frac{ab\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}}{a}+\frac{b}{a}\ln\frac{ab\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}}{b}\right)$$

$$=\frac{ab}{b^2-a^2}\left(\frac{1}{ab}\frac{b^2-a^2}{2\ln\frac{b}{a}}\ln\frac{b}{a}-\frac{a}{b}\ln\left(b\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}\right)+\frac{b}{a}\ln\left(a\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}\right)\right)$$

$$=\tfrac{1}{2}+\frac{ab}{b^2-a^2}\left(-\frac{a}{b}\ln\left(b\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}\right)+\frac{b}{a}\ln\left(a\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}\right)\right)$$

$$=\tfrac{1}{2}+\frac{-a^2\ln\left(b\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}\right)+b^2\ln\left(a\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}\right)}{b^2-a^2}$$

$$=\tfrac{1}{2}+\frac{-a^2\ln b-a^2\ln\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}+b^2\ln a+b^2\ln\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}}{b^2-a^2}$$

$$=\tfrac{1}{2}+\ln\sqrt{\frac{2\ln\frac{b}{a}}{b^2-a^2}}+\frac{b^2\ln a-a^2\ln b}{b^2-a^2}$$

$$=\tfrac{1}{2}+\tfrac{1}{2}\ln\frac{2\ln\frac{b}{a}}{b^2-a^2}+\frac{b^2\ln a-a^2\ln b}{b^2-a^2}$$

$$W_r=\frac{M}{\sigma_{rr\ \max}}=\frac{-\frac{1}{2}Et\varepsilon_i C}{-\frac{1}{2}E\varepsilon_i f_{rr\ \max}}=\frac{tC}{f_{rr\ \max}}=t\frac{-\frac{1}{4}(b^2-a^2)+\frac{a^2b^2}{b^2-a^2}\left(\ln\frac{b}{a}\right)^2}{\frac{1}{2}+\frac{1}{2}\ln\frac{2\ln\frac{b}{a}}{b^2-a^2}+\frac{b^2\ln a-a^2\ln b}{b^2-a^2}}$$

**c** Infinite Stress

A re-entrant corner (Dutch: inwendige hoek) cannot be sharper than the molecules from which the material is made of. So, there always is an internal radius $a$. The graph shows that when $d$ is a fraction smaller than $2R$ then $W_r$ is not zero. Therefore, infinite stresses will not occur. Nonetheless, the stress can become large and damage can be expected in sharp re-entrant corners.

**d** Approximation

For straight beams $R$ is very large. When $R$ goes to infinity the first term of the approximation formula goes to infinity and the second term to zero. Therefore, $W_r$ goes to infinity and $\sigma_{rr}$ goes to zero, which is physically correct. Therefore, the approximation formula gives the correct result for straight beams.

**e** Orthotropic
kinematic equations

$$\varepsilon_{rr} = \frac{du}{dr}$$
$$\varepsilon_{\theta\theta} = \frac{u}{r}$$

constitutive equations

$$\varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta} - \nu\sigma_{rr}}{E_{||}} + \varepsilon_i$$
$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E_{\perp}} - \frac{\nu\sigma_{\theta\theta}}{E_{||}}$$

equilibrium equation

$$\sigma_{\theta\theta} - \frac{d}{dr}(r\sigma_{rr}) = 0$$

**Answers to Problem 2**

**a** Boundary Conditions
kinematic boundary conditions

$$y = x \quad \to w = 0$$
$$y = -x \to w = 0$$

dynamic boundary conditions

$$y = x \quad \to m_{ss} = 0$$
$$y = -x \to m_{tt} = 0$$
$$y = \tfrac{1}{2}a \to m_{yy} = 0, \quad v_y = 0$$

For application of the principle of minimum potential energy the displacement function needs to fulfil the kinematic boundary conditions.

**b** Potential Energy

$$E_{\text{pot}} = \int_{y=0}^{\frac{1}{2}a} \int_{x=-y}^{y} \tfrac{1}{2}m_{xx}\kappa_{xx} + \tfrac{1}{2}m_{yy}\kappa_{yy} + \tfrac{1}{2}m_{xy}\rho_{xy} - qw \;\; dx\,dy$$

**c** Minimal

The potential energy needs to be minimal with respect to the parameters $a_6, a_9, a_{10}$, $a_{13}, a_{14}, a_{15}$ that describe the deformation. The equations are derived from

$$\frac{\partial E_{\text{pot}}}{\partial a_6} = 0, \quad \frac{\partial E_{\text{pot}}}{\partial a_9} = 0, \quad \frac{\partial E_{\text{pot}}}{\partial a_{10}} = 0, \quad \frac{\partial E_{\text{pot}}}{\partial a_{13}} = 0, \quad \frac{\partial E_{\text{pot}}}{\partial a_{14}} = 0, \quad \frac{\partial E_{\text{pot}}}{\partial a_{15}} = 0 .$$

**d** Inconsistency

The calculated moment $m_{yy}$ is not zero on the free edge because apparently it is an approximation.

Encore (not an exam problem)

The deflection function can be extended.

$$\begin{aligned} w = a_1 &+ \\ &+ a_2 x + a_3 y + \\ &+ a_4 x^2 + a_5 xy + a_6 y^2 + \\ &+ a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + \\ &+ a_{11} x^4 + a_{12} x^3 y + a_{13} x^2 y^2 + a_{14} xy^3 + a_{15} y^4 + \\ &+ a_{16} x^5 + a_{17} x^4 y + a_{18} x^3 y^2 + a_{19} x^2 y^3 + a_{20} xy^4 + a_{21} y^5 + \\ &+ a_{22} x^6 + a_{23} x^5 y + a_{24} x^4 y^2 + a_{25} x^3 y^3 + a_{26} x^2 y^4 + a_{27} xy^5 + a_{28} y^6 \end{aligned}$$

The calculated maximum deflection is now

$$w_{\max} = \tfrac{1}{12288} \frac{q a^4}{D(1-\nu)} \frac{3339791 - 1212410\nu - 205718\nu^2 + 17612\nu^3}{75381 - 8156\nu - 10590\nu^2 - 1060\nu^3} .$$

It can be shown that the approximation

$$w_{\max} = \tfrac{1}{278} \frac{q a^4}{D} \frac{1 - \frac{5}{22}\nu}{1-\nu} ,$$

has a deviation between -0.2% and 0.2% for $0 < \nu < 0.5$.

The formula for the maximum moment along the free edge is

$$m_{xx\,\max} = \tfrac{1}{512} a^2 q \frac{1490021 + 775764\nu - 634546\nu^2 - 46594\nu^3 + 8260\nu^4}{75381 - 8156\nu - 10590\nu^2 - 1060\nu^3} ,$$

which can be approximated as

$$\boxed{m_{xx\,\max} = \tfrac{15}{386} q a^2 (1 - \tfrac{10}{19}\nu)} .$$

It can be shown that the deviation of this formula is between –0.6% and 0.5% for $0 < \nu < 0.5$. It is noted that for $\nu < 0.1$ the absolute value of moment $m_{xx}$ and $m_{yy}$ in the origin ($x$ = 0, $y$ = 0) are somewhat larger than this value.

**Answers to Problem 3**

**a** Torsion Stiffness

girder $G\boldsymbol{I}_t = 717 \text{x} 10^6 G$

rectangle $G\boldsymbol{I}_t = 0.281 G b h^3 = 0.281 G \; 600 \text{x} 150^3 = 569 \text{x} 10^6 G$

larger $\frac{717 - 569}{569} 100 = 26\%$

**b** Program

The program solves a partial differential equation using the finite element method. This differential equation can either describe the $\phi$ bubble (force method) or the warping function $\psi$ (displacement method). From the result the torsion constant $\boldsymbol{I}_t$, warping constant $C_w$ and the Saint Venant shear stresses $\sigma_{zx}, \sigma_{zy}$ are computed.

**c** Torsion Loading

The eccentricity of the force is the horizontal distance from the force to the shear centre.
$M_t$ = 85 x (0.180 – 0.050) = 11.05 kNm

**d** Restriction

The torsion theory of De Saint Venant assumes that cross-sections can warp freely. This is often not the case in practice.

**e** Warping Constant

The warping constant $C_w$ is used in the torsion theory of Vlasov. This theory describes the effects of warping restraints and distributed torsion loading.

**Answer to Problem 4**

In Figure 5 can be seen that

$$\frac{R}{\psi r} = \frac{R + \frac{1}{2} h}{\psi (r + \varphi \frac{1}{2} h)}.$$

This can be evaluated to $R = \frac{r}{\varphi}$.

Since $\kappa_{\theta\theta} = -\frac{1}{R}$ and $\varphi = \frac{dw}{dr}$ we find $\kappa_{\theta\theta} = -\frac{1}{r} \frac{dw}{dr}$. Q.E.D.

Encore (Not an exam question)

In the previous derivation small rotations $\varphi$ are assumed. We can also do the derivation for any rotation.

$$\frac{R}{\psi r} = \frac{R + \frac{1}{2} h}{\psi (r + \sin\varphi \frac{1}{2} h)}.$$

This can be evaluated to

$$R = \frac{r}{\sin\varphi}.$$

Since $\kappa_{\theta\theta} = -\frac{1}{R}$ and $\tan\varphi = \frac{dw}{dr}$ we find

$$\kappa_{\theta\theta} = -\frac{1}{r}\sin\left(\arctan\frac{dw}{dr}\right).$$

This can be evaluated to

$$\kappa_{\theta\theta} = \frac{-\frac{1}{r}\frac{dw}{dr}}{\sqrt{1+\left(\frac{dw}{dr}\right)^2}}.$$

The first two terms of the Taylor expansion in $\frac{dw}{dr} = 0$ gives

$$\kappa_{\theta\theta} \approx -\frac{1}{r}\frac{dw}{dr} + \frac{1}{2}\frac{1}{r}\left(\frac{dw}{dr}\right)^3.$$

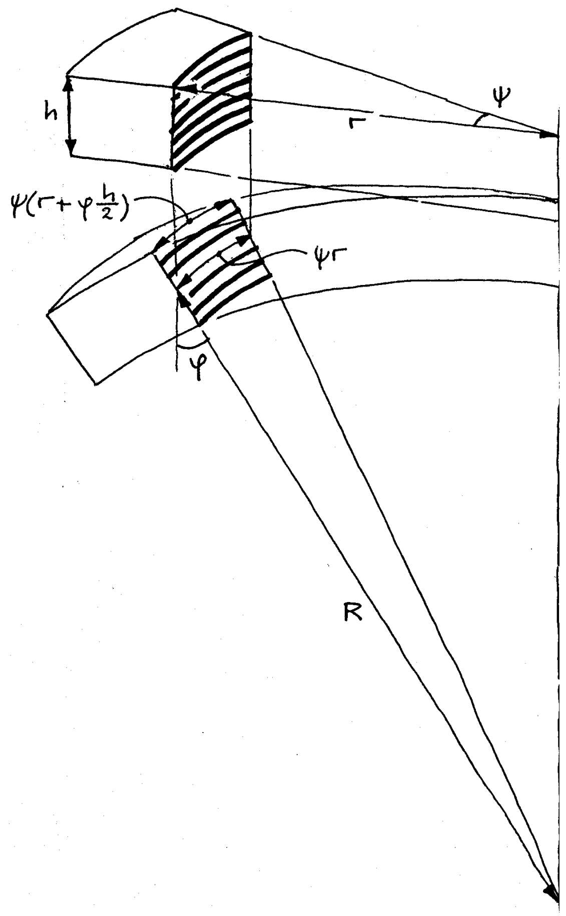

Figure 5. Deformation of an axisymmetric plate