Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Wednesday 26 January 2005, 9:00 – 12:00 hours

Problem 1 (4 points)

Two metal strips are glued together (Figure 1). The strips have a thickness t_1 and t_2, respectively. Young's moduli are E_1 and E_2, respectively. The strips are loaded by an axial force N. The width of the strips is w.

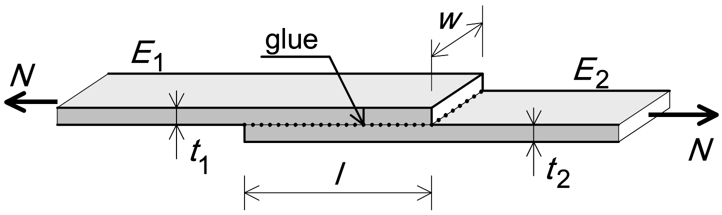


Figure 1. Strips glued together

We want to study the shear stress in the thin glue layer. Therefore, a model is made consisting of bars connected by distributed springs (Figure 2). The springs have a stiffness k.

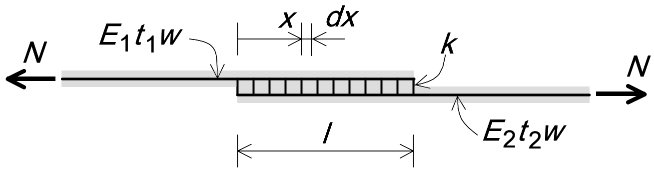


Figure 2. Model of the strip joint

a The model is not completely in equilibrium because the axial forces N are not aligned. When can this eccentricity be neglected?

b Show that the spring stiffness k can be calculated from

$$\frac{w}{k} = \frac{t_1}{2G_1} + \frac{d}{G_g} + \frac{t_2}{2G_2}$$

where G_1 and G_2 are the shear moduli of the metals. d is the thickness of the glue layer and G_g is the shear modulus of the glue (Figure 3).

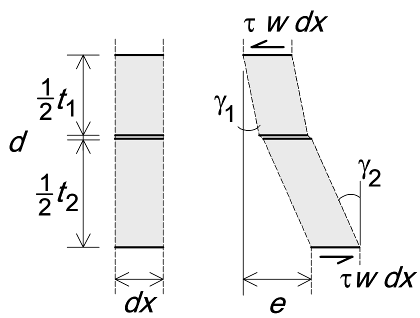
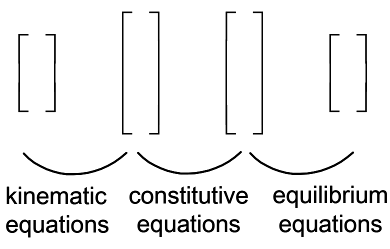


Figure 3. Slice of the strip joint

c Formulate the kinematic equations, constitutive equations and equilibrium equations of the model. Complete the following framework for the quantities involved.



d Derive a differential equation – or multiple differential equations – for solving this problem. Use either the displacement method or the force method (You do not need to do both).

e Formulate the boundary conditions that need be used in solving the differential equation.

Maple solves the differential equation and gives the following solution of the shear stress τ in the glue layer.

$$\tau = \frac{\beta N}{w} \frac{E_1 t_1 (e^{\beta(l+x)} + e^{\beta(l-x)}) + E_2 t_2 (e^{\beta x} + e^{\beta(2l-x)})}{(E_1 t_1 + E_2 t_2)(e^{\beta 2l} - 1)}$$

where

$$\beta^2 = \frac{k}{w} \frac{E_1 t_1 + E_2 t_2}{E_1 t_1 E_2 t_2}$$

In Figure 4 this stress is plotted for $E_1 = 2e5$ N/mm², $E_2 = 1e5$, $t_1 = 8$, $t_2 = 5$ and l = 80 mm (d = 0, ν = 0). It shows that the largest shear stresses occurs at x = 0 and $x = l$. In the middle the shear stress is almost zero. The length l has little influence on the results if it is much longer than the characteristic length $1/\beta$. In Figure 4 the value of $1/\beta$ = 7.34 mm. The maximum shear stresses are

$$\hat{\tau}_1 = \lim_{l \to \infty, x \downarrow 0} \tau = \frac{\beta N}{w} \frac{E_2 t_2}{E_1 t_1 + E_2 t_2}$$

$$\hat{\tau}_2 = \lim_{l \to \infty, x \uparrow l} \tau = \frac{\beta N}{w} \frac{E_1 t_1}{E_1 t_1 + E_2 t_2} .$$

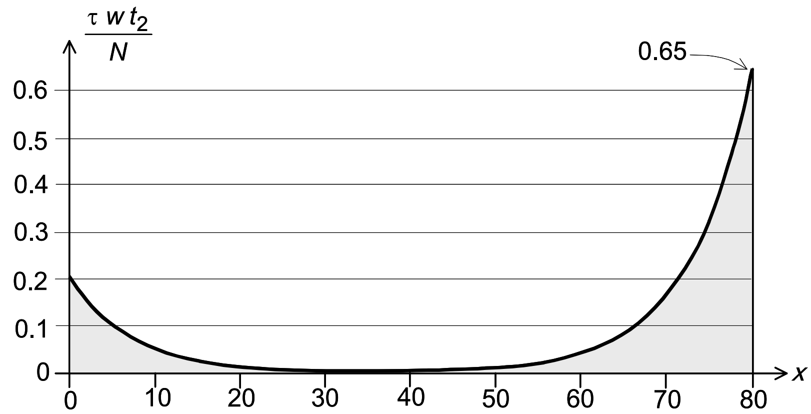


Figure 4. Shear stress in the glue layer

f Simplify the formulae for the maximum shear stress. Assume $E_1 = E_2 = E$, $t_1 = t_2 = t$, d = 0 and $G_1 = G_2 = \frac{1}{2} E$.

Problem 2 (3 points)

The following formula gives the deflection of a triangular plate at the middle of the free edge due to an evenly distributed perpendicular loading q (Figure 5).

$$w = \frac{qa^2b^2}{D(1\text{-}\nu)} \frac{1180 + 466\alpha - 216\alpha^2 - 320\nu\alpha - 27\nu}{100000}$$

Where $\alpha = \frac{a}{b}$. The error is less than 0.8 % for 0.2 < α < 1 and 0.1 < ν < 0.4. Note that this deflection is close to the maximum deflection.

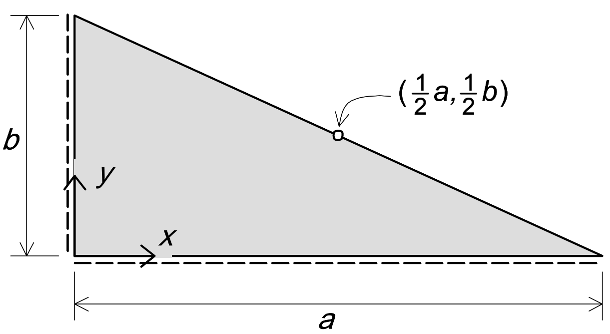


Figure 5. Triangular plate

a Explain how this formula can be derived using minimum potential energy.

b What is approximated in this derivation?

c Explain how the accuracy of this formula can be determined.

d Could the formula also be derived using minimum complementary energy? Explain your answer.

Problem 3 (3 points)

The differential equations for Saint Venant torsion can be solved for an equilateral triangle cross-section (Figure 6). The results are

$$\psi = \frac{z}{6a}(3y^2 - z^2)$$

$$\phi = \frac{G\theta}{6a}(a - y)(2a + y - z\sqrt{3})(2a + y + z\sqrt{3})\,.$$

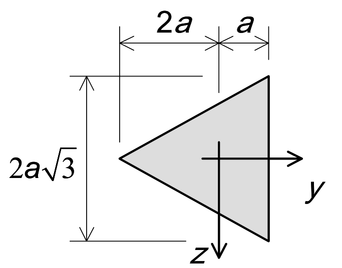


Figure 6. Equilateral triangle cross-section

a Calculate the maximum shear stress $\tau_{\max}$ in this cross-section as a function of the torsion moment M_t.

b Calculate the maximum normal stress $\sigma_{xx,\max}$ in this cross section as a function of the bi-moment M_ω.

The following results might be useful in the derivations.

$$\int_{-2a}^{a} \int_{\frac{y+2a}{-\sqrt{3}}}^{\frac{y+2a}{\sqrt{3}}} \psi^2 \, dz dy = \frac{3}{70} a^6 \sqrt{3}$$

$$\int_{-2a}^{a} \int_{\frac{y+2a}{-\sqrt{3}}}^{\frac{y+2a}{\sqrt{3}}} \phi \, dz dy = \frac{9}{10} G\theta a^4 \sqrt{3}$$

$$\frac{\partial \psi}{\partial y} = \frac{zy}{a}$$

$$\frac{\partial \psi}{\partial z} = \frac{y^2 - z^2}{2a}$$

$$\frac{\partial \phi}{\partial y} = G\theta \frac{z^2 - 2ay - y^2}{2a}$$

$$\frac{\partial \phi}{\partial z} = G\theta \frac{y - a}{a} z$$

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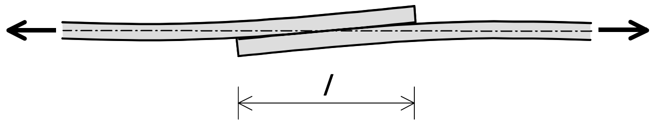
Answers to Problem 1

a Eccentricity

The joint will rotate to be in equilibrium. This will introduce extra stresses in the glue layer. The rotation will be small if $\frac{1}{2}t_1 + \frac{1}{2}t_2 \ll l$. Therefore, the eccentricity can be neglected if it is small compared to the length of the joint.



b Spring Stiffness

The following relations exist

$\tau w = ke$

$\tau = G_1 \gamma_1$

$\tau = G_2 \gamma_2$

$\tau = G_g \gamma_g$

$e = \gamma_1 \frac{1}{2} t_1 + \gamma_g d + \gamma_2 \frac{1}{2} t_2$

Substitution into the first relation gives

$$\frac{w}{k} = \frac{e}{\tau} = \frac{\gamma_1 \frac{1}{2} t_1 + \gamma_g d + \gamma_2 \frac{1}{2} t_2}{\tau} = \frac{\frac{\tau}{G_1} \frac{1}{2} t_1 + \frac{\tau}{G_g} d + \frac{\tau}{G_2} \frac{1}{2} t_2}{\tau} = \frac{t_1}{2G_1} + \frac{d}{G_g} + \frac{t_2}{2G_2}$$

Q.E.D.

c Kinematic Equations

$$\varepsilon_1 = \frac{du_1}{dx}$$

$$\varepsilon_2 = \frac{du_2}{dx} \qquad (\,*\,)$$

$$e = u_2 - u_1$$

Constitutive Equations

$$N_1 = E_1 t_1 w \varepsilon_1$$

$$N_2 = E_2 t_2 w \varepsilon_2 \qquad (\,**\,)$$

$$\tau w = ke$$

Equilibrium Equations

$$\frac{dN_1}{dx} + p_1 + \tau w = 0$$

$$\frac{dN_2}{dx} + p_2 - \tau w = 0$$

Framework

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ e \end{bmatrix}}_{\text{kinematic equations}} \quad \underbrace{\quad}_{\text{constitutive equations}} \quad \underbrace{\begin{bmatrix} N_1 \\ N_2 \\ \tau w \end{bmatrix} \quad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{\text{equilibrium equations}}$$

d Displacement Method

We substitute everything in the equilibrium equations.

$$\begin{cases} \dfrac{d}{dx} E_1 t_1 w \varepsilon_1 + p_1 + ke = 0 \\ \dfrac{d}{dx} E_2 t_2 w \varepsilon_2 + p_2 - ke = 0 \end{cases}$$

$$\begin{cases} E_1 t_1 w \dfrac{d^2 u_1}{dx^2} + p_1 + k(u_2 - u_1) = 0 \\ E_2 t_2 w \dfrac{d^2 u_2}{dx^2} + p_2 - k(u_2 - u_1) = 0 \end{cases}$$

Thus we obtain two coupled differential equation in u_1 and u_2.

Force Method

We derive a compatibility equation and substitute everything in there.

$$\frac{de}{dx} = \varepsilon_2 - \varepsilon_1$$

$$\frac{d}{dx}\frac{\tau w}{k} = \frac{N_2}{E_2 t_2 w} - \frac{N_1}{E_1 t_1 w} \qquad (***)$$

$$\frac{w}{k}\frac{d^2\tau}{dx^2} = \frac{1}{E_2 t_2 w}\frac{dN_2}{dx} - \frac{1}{E_1 t_1 w}\frac{dN_1}{dx}$$

$$\frac{w}{k}\frac{d^2\tau}{dx^2} = \frac{1}{E_2 t_2 w}\left(-p_2 + \tau w\right) - \frac{1}{E_1 t_1 w}\left(-p_1 - \tau w\right)$$

Since $p_1 = p_2 = 0$

$$\frac{w}{k}\frac{d^2\tau}{dx^2} - \left(\frac{1}{E_2 t_2} + \frac{1}{E_1 t_1}\right)\tau = 0$$

Thus we obtain one differential equation in the redundant τ.

e Boundary Conditions Displacement Method

$$x = 0 \quad \rightarrow \quad N_1 = N, \quad N_2 = 0$$

$$x = l \quad \rightarrow \quad N_1 = 0, \quad N_2 = N$$

Using equations * and ** gives

$$x = 0 \quad \rightarrow \quad \frac{du_1}{dx} = \frac{N}{E_1 t_1 w}, \quad \frac{du_2}{dx} = 0$$

$$x = l \quad \rightarrow \quad \frac{du_1}{dx} = 0, \qquad \frac{du_2}{dx} = \frac{N}{E_2 t_2 w}$$

Boundary Conditions Force Method

$x = 0 \quad \rightarrow \quad N_1 = N, \quad N_2 = 0$

$x = l \quad \rightarrow \quad N_1 = 0, \quad N_2 = N$

Using equation *** gives

$$x = 0 \quad \rightarrow \quad \frac{d\tau}{dx} = -\frac{kN}{E_1 t_1 w^2}$$

$$x = l \quad \rightarrow \quad \frac{d\tau}{dx} = \frac{kN}{E_2 t_2 w^2}$$

f Simple Formula

$$\beta^2 = \frac{k}{w}\frac{E_1 t_1 + E_2 t_2}{E_1 t_1 E_2 t_2} = \frac{k}{w}\frac{Et + Et}{EtEt} = \frac{2k}{wEt}$$

$$\frac{w}{k} = \frac{t_1}{2G_1} + \frac{d}{G_g} + \frac{t_2}{2G_2} = \frac{t}{2G} + \frac{t}{2G} = \frac{t}{G} = \frac{2t}{E}$$

$$\hat{\tau} = \frac{\beta N}{w}\frac{E_1 t_1}{E_1 t_1 + E_2 t_2} = \frac{\beta N}{w}\frac{Et}{Et + Et} = \frac{\beta N}{2w} = \sqrt{\frac{2k}{wEt}}\frac{N}{2w} = \sqrt{\frac{2E}{2tEt}}\frac{N}{2w}$$

$$\boxed{\hat{\tau} = \frac{N}{2wt}}$$

Apparently, the maximum shear stress in the joint is approximately half the tensile stress in the strips.

Answers to Problem 2

a Derivation

1) Assume a deflection function as a function of several parameters.
2) Make sure that this function fulfils the kinematic boundary conditions.
3) Formulate the potential energy.
4) Solve the parameters by making the potential energy minimal.
5) Substitute the parameters in the deflection function
6) Determine the maximum deflection.

b Approximation

The assumed deflection function probably is not able to perfectly describe the exact deflection. If it could describe the exact deflection the exact deflection would be found. If it cannot describe the exact deflection an approximation is found. So, the deflection has been approximated.

We can test whether the solved deflection is exact by substitution into the equilibrium equations and the dynamic boundary conditions. If it is not exact, equilibrium will not be fulfilled. So, we can also say that equilibrium has been approximated.

c Accuracy

We will find a more accurate approximation if we repeat the derivation with more terms in the deflection function. If this approximation differs little from the first one the solution is accurate.

d Complementary energy

In theory it would be possible to use complementary energy for a plate deflection problem but this would be much more difficult than potential energy. The reason is that in complementary energy a moment distribution m_{xx}, m_{yy}, m_{xy} in the plate would need to be formulated that fulfils equilibrium and the dynamic boundary conditions.

Answers to Problem 3

a Maximum Shear Stress

The maximum shear stress occurs where the slope of the ϕ-bubble is maximal. This occurs at the middles of the edges. For example $y = a$, $z = 0$. The stress will be directed along the edge. Therefore,

$$\tau_{max} = \sigma_{xz}(a,0) = -\left.\frac{\partial \phi}{\partial y}\right|_{y=a,z=0} = -G\theta \left.\frac{z^2 - 2ay - y^2}{2a}\right|_{y=a,z=0} = \tfrac{3}{2} G\theta a$$

The torsion moment is two times the volume of the ϕ-bubble.

$$M_t = 2\iint_A \phi \, dA = 2 \tfrac{9}{10} \sqrt{3} G\theta a^4 = \tfrac{9}{5} \sqrt{3} G\theta a^4$$

Combining the two latter equations we obtain

$$\tau_{max} = \tfrac{3}{2} G\theta a = \frac{3}{2} \frac{M_t}{\tfrac{9}{5}\sqrt{3}\, a^3}$$

$$\boxed{\tau_{max} = \tfrac{5}{18} \sqrt{3} \frac{M_t}{a^3}}.$$

b Maximum Normal Stress

The normal stress due to restrained warping is

$$\sigma_{xx} = \frac{\psi}{C_w} M_\omega$$

It is maximal when ψ is maximal. This occurs on the edges. For example on the edge $y = a$.

$$0 = \left.\frac{\partial \psi}{\partial z}\right|_{y=a} = \left.\frac{y^2 - z^2}{2a}\right|_{y=a} = \frac{a^2 - z^2}{2a} \qquad \Rightarrow \qquad z = -a, a$$

$$\psi_{\max} = \psi(a,a) = \left.\frac{z}{6a}(3y^2 - z^2)\right|_{y=z=a} = \tfrac{1}{3} a^2$$

The warping constant is

$$C_w = \iint_A \psi^2 dA = \tfrac{3}{70} a^6 \sqrt{3} \, .$$

Combining the latter three equations gives

$$\sigma_{xx,\max} = \frac{\Psi_{\max}}{C_w} M_\omega = \frac{\frac{1}{3} a^2}{\frac{3}{70} a^6 \sqrt{3}} M_\omega$$

$$\boxed{\sigma_{xx,\max} = \frac{70}{27} \sqrt{3} \frac{M_\omega}{a^4}}$$

Note that these analytical results are useful for checking finite element programs.