Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Friday 4 November 2005, 9:00 – 12:00 hours

Problem 1 (3 points)

A round plate is fixed at the edges (Fig. 1). The plate has a radius *b*. The plate is loaded by an evenly distributed load *p* over an area with a radius *a*. The plate material is isotropic, homogeneous and linear elastic.

a Give the differential equations for this plate.

b Formulate the boundary conditions ($r = 0$ and $r = b$) and transition conditions ($r = a$).

c The differential equations are solved by Maple. The solution is

$$w = \frac{Fb^2}{64\pi D}\begin{cases} 4 - 3\dfrac{a^2}{b^2} - 4\left(\dfrac{a^2}{b^2} + 2\dfrac{r^2}{b^2}\right)\ln\dfrac{b}{a} - \dfrac{r^2}{b^2}\left(2\dfrac{a^2}{b^2} - \dfrac{r^2}{a^2}\right) & \text{if } 0 \le r \le a \\ 4 + 2\dfrac{a^2}{b^2} + 4\left(\dfrac{a^2}{b^2} + 2\dfrac{r^2}{b^2}\right)\ln\dfrac{r}{b} - \dfrac{r^2}{b^2}\left(2\dfrac{a^2}{b^2} + 4\right) & \text{if } a < r \le b \end{cases}$$

where $F = \pi a^2 p$.

Derive a formula for the maximum deflection of the plate.

d Give the equations to derive the moments in the plate. (You do not need to evaluate the equations).

e Figure 2 shows the moment distribution in the plate for different values of *a*. Is the principle of De Saint Venant valid in this situation? Explain your answer.

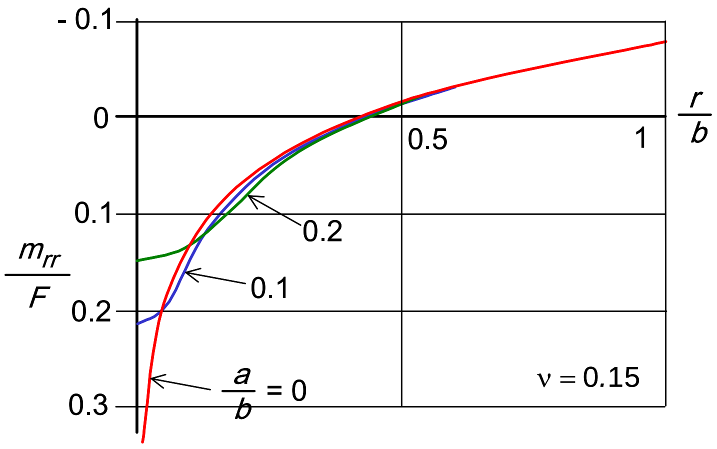


Figure 2. Moment Distributions

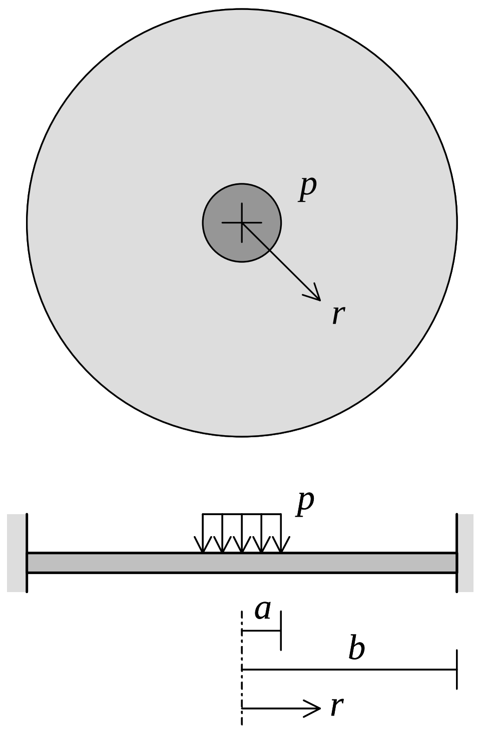


Figure 1. Round Plate

Problem 2 (4 points)

A rubber joint is placed between two stiff construction parts (Fig. 3). The joint width is *a* [mm] and the joint depth is *b* [mm]. Young's modulus of the rubber is *E* [N/mm²]. The joint carries an evenly distributed load *p* [N/mm²].

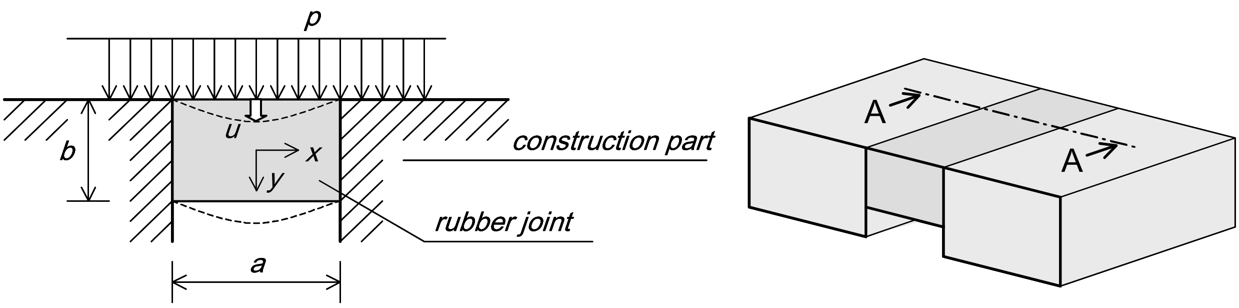


Figure 3. Section A – A

We want to derive a formula for the deflection *u* [mm] of the joint. In order to apply the principle of minimum potential energy, the following displacement field is chosen.

$$u_x = 0$$

$$u_y = \left(1 + 2\frac{x}{a}\right)\left(1 - 2\frac{x}{a}\right)u$$

a Is this displacement field kinematically admissible? Explain your answer.

b Give the expression for the potential energy for a joint part with a length *c* [mm]. (You do not need to evaluate the expression.)

c The potentental energy can be evaluated to

$$E_{pot} = \tfrac{8}{3}\frac{bcGu^2}{a} - \tfrac{2}{3}pacu\,,$$

in which the constitutive equation of a plain strain state has been used

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \dfrac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ \dfrac{E\nu}{(1+\nu)(1-2\nu)} & \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}.$$

Derive the formula for the deflection *u.*

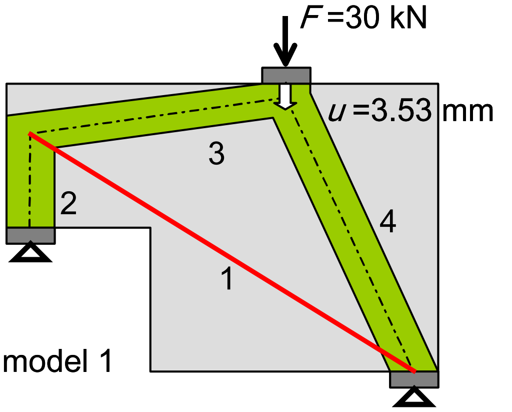
d Is this formula exact or an approximation? Explain your answer.

Problem 3 (3 points)

A structural engineer designs a reinforced concrete deep beam using the strut-and-tie method (Dutch; vakwerkanalogie). He comes up with four equilibrium systems to carry the load F (Fig. 4, 5, 6, 7). The force flow and deformation are computed by a frame analysis program. The struts have a stiffness EA that is thirty times larger than the ties.

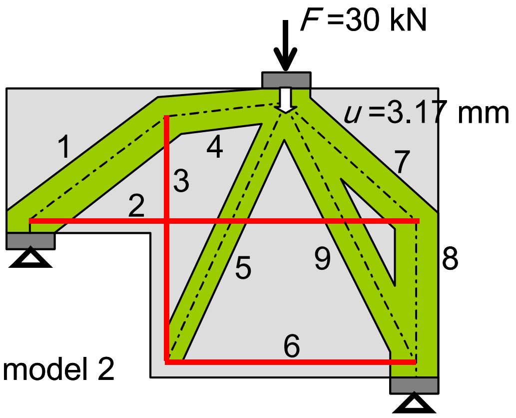
a Check the displacement of the first model using minimum complementary energy or Castigliano's theorem.

b Which of these strut-and-tie models is the best? Explain your answer.



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	0.517 F	5.97 m	15000 kN
2	-0.333 F	1.20	450000
3	-0.446 F	3.44	450000
4	-1.034 F	3.98	450000

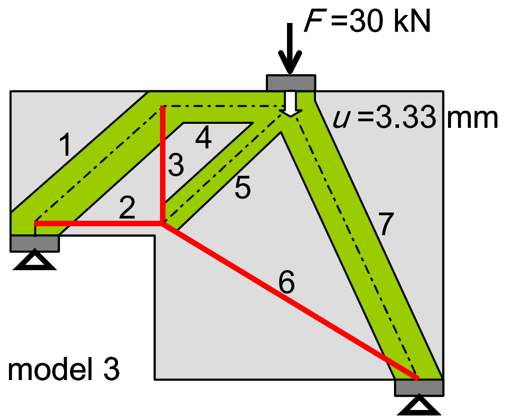
Figure 4. Strut-and-tie model 1



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	-0.564 F	2.28 m	450000 kN
2	0.445 F	5.20	15000
3	0.318 F	3.30	15000
4	-0.446 F	1.60	450000
5	-0.352 F	3.76	450000

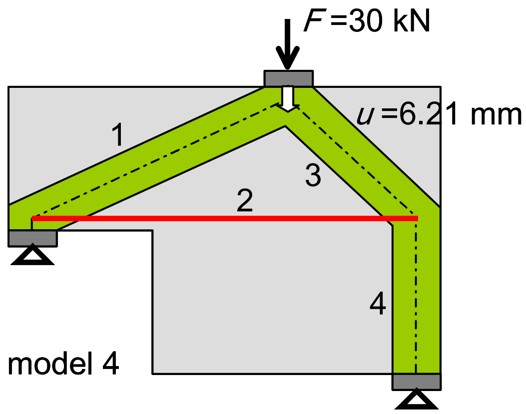
6	0.150 *F*	3.40	15000
7	-0.579 *F*	2.34	450000
8	-0.371 *F*	1.90	450000
9	-0.320 *F*	3.85	450000

Figure 5. Strut-and-tie model 2



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	-0.504 *F*	2.27 m	450000 kN
2	0.378 *F*	1.70	15000
3	0.333 *F*	1.50	15000
4	-0.378 *F*	1.70	450000
5	-0.089 *F*	2.27	450000
6	0.522 *F*	4.00	15000
7	-1.041 *F*	3.98	450000

Figure 6. Strut-and-tie model 3



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	-0.826 *F*	3.72 m	450000 kN
2	0.756 *F*	5.10	15000
3	-1.008 *F*	2.27	450000
4	-0.667 *F*	2.10	450000

Figure 7. Strut-and-tie model 4

Exam CT5141, 4 November 2005
Answers to Problem 1

a Differental Equations

The differential equation for the middle of the plate is $w'''' + \dfrac{2w'''}{r} - \dfrac{w''}{r^2} + \dfrac{w'}{r^3} = \dfrac{p}{D}$.

The differential equation for the outside of the plate is $w'''' + \dfrac{2w'''}{r} - \dfrac{w''}{r^2} + \dfrac{w'}{r^3} = 0$.

b Boundary conditions

$$r = 0 \quad \rightarrow \quad \varphi = 0,\ v_r = 0$$
$$r = a \quad \rightarrow \quad w^- = w^+,\ \varphi^- = \varphi^+,\ m_{rr}^- = m_{rr}^+,\ v_r^- = v_r^+$$
$$r = b \quad \rightarrow \quad w = 0,\ \varphi = 0$$

c Deflection

$$w = \frac{Fb^2}{64\pi D}\left(4 - 3\frac{a^2}{b^2} - 4\left(\frac{a^2}{b^2} + 2\frac{r^2}{b^2}\right) ln\frac{b}{a} - \frac{r^2}{b^2}\left(2\frac{a^2}{b^2} - \frac{r^2}{a^2}\right)\right) \quad \text{if } 0 \le r \le a$$

$$r = 0$$

$$w_{\max} = \frac{Fb^2}{64\pi D}\left(4 - 3\frac{a^2}{b^2} - 4\frac{a^2}{b^2} ln\frac{b}{a}\right)$$

$$w_{\max} = \frac{Fb^2}{16\pi D}\left(1 - \frac{a^2}{b^2}\left(\tfrac{3}{4} + ln\frac{b}{a}\right)\right)$$

d Moments

$$m_{rr} = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

$$m_{\theta\theta} = -D\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$$

e De Saint Venant

De principle of De Saint Venant is valid in this situation. The moment at some distance of the plate middle is almost not influenced by the distribution of the resultant load *F*.

> Principle of De Saint Venant
> When the forces that are acting on a small part of the surface of an elastic body are replaced by another statically equivalent system of forces, locally large changes in the stresses occur, however, at a distance – which is large compared to the length over which the forces are changed – the influence is negligible.

Answers to Problem 2

a Kinematically Admissible

The kinematic boundary conditions are

$$x = -\tfrac{1}{2}a \quad \rightarrow \quad u_x = 0, u_y = 0$$
$$x = \tfrac{1}{2}a \quad \rightarrow \quad u_x = 0, u_y = 0$$

When we substitute $x = -\tfrac{1}{2}a$ and $x = \tfrac{1}{2}a$ in the displacement field we observe that the displacements are indeed 0. So, the displacement field is kinematically admissible.

b Potential Energy

$$E_{pot} = c \int_{y=-\frac{1}{2}b}^{\frac{1}{2}b} \int_{x=-\frac{1}{2}a}^{\frac{1}{2}a} \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy} + \tfrac{1}{2}\sigma_{xy}\gamma_{xy}\, dx\, dy - c \int_{x=-\frac{1}{2}a}^{\frac{1}{2}a} p u_y dx$$

c Deflection

$$\frac{dE_{pot}}{du} = \tfrac{16}{3}\frac{c\, b\, G\, u}{a} - \tfrac{2}{3} p\, a\, c = 0$$
$$u = \frac{p\, a^2}{8 G\, b}$$

d Exact or Approximation

The solution is exact if equilibrium is fulfilled.

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ G\gamma_{xy} \end{bmatrix}$$

At $y = -\tfrac{1}{2}b$ we have $\sigma_{yy} = 0 \neq -p$. So, the stress is not in equilibrium with the load. Therefore, the formula is an approximation.

Answers to Problem 3

a Castigliano

$$E_c = \tfrac{1}{2}\sum_i \frac{N_i^2 l_i}{EA_i} = \tfrac{1}{2}\left(\frac{(0.517\,F)^2 5.97}{15000} + \frac{(-0.333\,F)^2 1.20}{450000} + \frac{(-0.446\,F)^2 3.44}{450000} + \frac{(-1.034\,F)^2 3.98}{450000}\right)$$

$$E_c = 0.0000588\,F^2 \text{ m/kN}$$

$$u = \frac{\partial E_c}{\partial F} = 2 \times 0.0000588\,F = 2 \times 0.0000588 \times 30 = 0.00353\,\text{m},$$

which is correct.

b <u>The Best</u>

Any of the following four answers is correct.

A Each strut-and-tie model is equally suitable. Plasticity theory states that any equilibrium system that does not violate the yield conditions gives a lower bound for the strength of the structure. Experiments have shown that reinforced concrete is sufficiently ductile to apply plasticity theory.

B Strut-and-tie model 1 is the best because the value $\sum N_j l_j$ is the smallest, where j obtains the number of all ties. This leads to the least amount of reinforcement.

Model 1: $\sum N_j l_j$ = 93 kNm

Model 2: $\sum N_j l_j$ = 116 kNm

Model 3: $\sum N_j l_j$ = 97 kNm

Model 4: $\sum N_j l_j$ = 116 kNm

C Strut-and-tie model 2 is the best because its complementary energy is the smallest.

$E_{\text{compl}} = \tfrac{1}{2}\sum \frac{N_i^2 l_i}{EA_i}$ where i obtains the numbers of all struts and all ties.

Model 1: E_{compl} = 0.053 kNm

Model 2: E_{compl} = 0.046 kNm

Model 3: E_{compl} = 0.050 kNm

Model 4: E_{compl} = 0.093 kNm

D Strut-and-tie model 2 is the best because its deformation is the smallest. After all, the deformation calculated with minimum complementary energy decreases when the model is improved.

Note that that we cannot use potential energy in this problem because we are not looking for the correct displacements. We use complementary energy because we are looking for the correct force flow.

Note that if you calculate the complementary energy as $E_{\text{compl}} = \frac{1}{2}\sum \frac{N_i^2 l_i}{EA_i} - Fu$ you assume that the displacement u is imposed. Clearly, u needs to be the same for all strut-and-tie models otherwise they cannot be compared. F can be solved from minimum complementary energy and subsequently $E_{\text{compl,min}}$ can be calculated. This approach is possible but not practical.