Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your <u>name</u> and <u>study number</u> at the top right-hand of your work.

Exam CT5141 Theory of Elasticity

Wednesday 25 January 2006 at 9:00 – 12:00 hours

Problem 1 (4 points)

Consider a spherical dome (Fig 1.) The radius is *a* [m]. The thickness *t* [m] is everywhere the same. The dome is loaded by self-weight *p* [kN/m²], which is evenly distributed over the dome surface. We will derive a formula for the deflection of the dome using potential energy. Therefore, we assume the following displacement.

$$u_z = C_1 \cos(2\phi) + C_2 \cos^2(\phi)$$
$$u_\phi = C_3 \sin(2\phi) + C_4 \sin(4\phi)$$

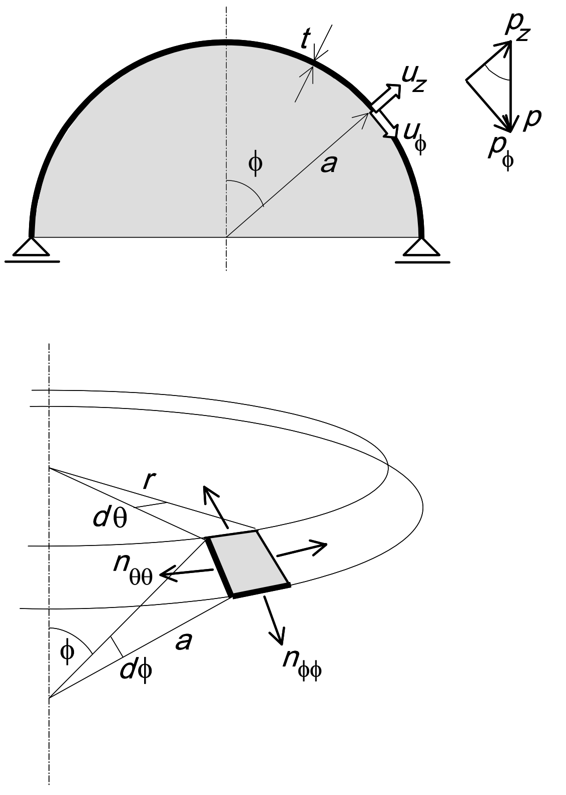

Figure 1. Spherical dome

a Is the displacement kinematically admissible? Explain your answer.

b Give the expression of the potential energy of the dome. (You do not need to evaluate the expression). Note that the dome has only membrane forces and no bending moments.

c The potential energy is evaluated by Maple with the following result. For simplification we used $\nu = 0$.

$$E_{\text{pot}} = \frac{\pi E t}{315}(126C_1^2 + 84C_1C_2 + 294C_2^2 + 336C_1C_3 + 672C_2C_3 + 840C_3^2 - 96C_1C_4 - \\ -192C_2C_4 - 480C_3C_4 + 2848C_4^2) + \pi p a^2\left(\tfrac{1}{2}C_1 - C_3 + \tfrac{2}{3}C_4\right).$$

Give the equations from which the coefficients C_1, C_2, C_3 and C_4 can be calculated.

d The equations have been solved by Maple with the following result.

$$C_1 = -1.36\frac{pa^2}{Et}, C_2 = -0.609\frac{pa^2}{Et},$$
$$C_3 = 0.697\frac{pa^2}{Et}, C_4 = -0.0216\frac{pa^2}{Et}$$

Derive a formula for the deflection of the

Formulas for spherical domes

$$\varepsilon_{\phi\phi} = \frac{1}{a}\left(\frac{du_\phi}{d\phi} + u_z\right)$$
$$\varepsilon_{\theta\theta} = \frac{1}{a}\left(u_\phi \cot\phi + u_z\right)$$
$$n_{\phi\phi} = \frac{Et}{1-\nu^2}\left(\varepsilon_{\phi\phi} + \nu\varepsilon_{\theta\theta}\right)$$
$$n_{\theta\theta} = \frac{Et}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{\phi\phi}\right)$$
$$n_{\phi\theta} = 0$$

dome top.

e How can this formula be checked?

Problem 2 (3 points)

A round high-rise building has an outside truss structure (Fig. 2). The radius of the building is a. The distance between the floors is h. The truss members have a cross-section area A, a Young's modulus E and a length l. The number of members in a cross-section of the building is n. The angle of a member with the building axis is α.

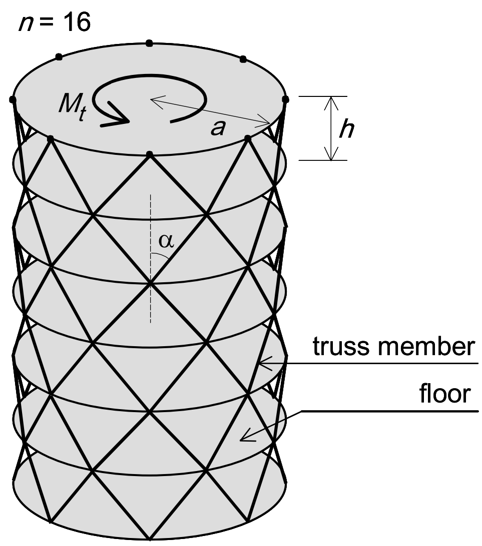

Figure 2. Round truss structure

a Show that for large n the normal force N in a truss member due a torsion moment M_t is

$$N = \frac{M_t}{n\, a \sin\alpha}.$$

b Derive a formula for the torsion stiffness $G\boldsymbol{I}_t$ of the building. Use either a direct method or complementary energy. The floors can be considered as infinitely stiff.

c Derive the angle α for which the torsion stiffness per amount of material $\dfrac{G\boldsymbol{I}_t}{nAl}$ is maximal.

Problem 3 (2 points)

A semi infinite linear elastic solid is loaded by a rigid circular plate (Fig. 3). The stress under the plate is found in the book of Timoshenko[1].

$$\sigma = \frac{F}{2\pi a\sqrt{a^2 - r^2}}$$

where F is the weight of the plate, a is its radius and r is the coordinate in the radial direction. The displacement u of the plate is

$$u = \frac{F(1-\nu^2)}{2aE}$$

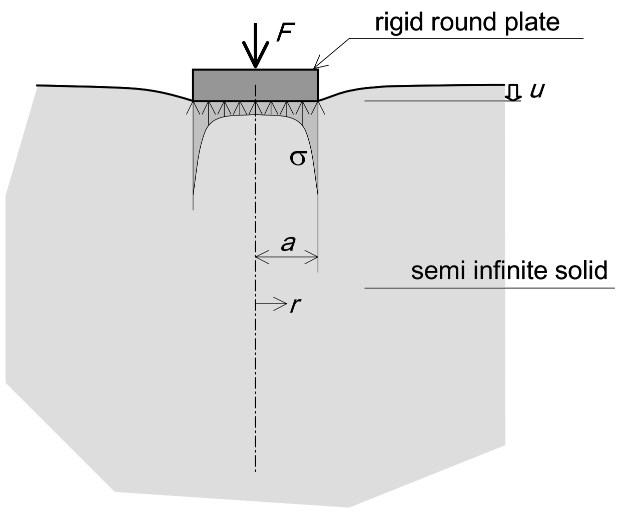

Figure 3. Solid loaded by a round plate

where E is Young's modulus and ν is Poisson's ratio.

[1] S.P. Timoshenko and J.N Goodier, Theory of Elasticity, McGraw-Hill, New York, third edition, p. 406.

a The semi infinite solid is replaced by distributed springs. What spring stiffness *k* [N/mm³] needs to be used to obtain the same deflection *u* and the same stress distribution σ? Why is the result not practical?

b The semi infinite solid is replaced by distributed springs. The spring stiffness *k* is independent of *r*. What spring stiffness needs to be used to obtain the same deflection *u*?

c The spring stiffness appears to depend on the radius *a* of the plate. Explain this. Often the soil on which a structure rests is modelled by distributed springs. Is this an accurate representation of the real situation?

Problem 4 (2 points)

A spring is made of a wire that is formed in a helix shape (Fig. 4). The radius of the spring is *a*, the thickness of the wire is *t* and the number of turns of the helix is *n*. The spring is loaded by a force *F* which causes an elongation *u*.

a What section property of the wire is important for the spring stiffness? (For example; axial stiffness or bending stiffness or torsion stiffness).

b Show that the spring stiffness is

$$k = \frac{G t^4}{64 n a^3}.$$

(Note that the length of the helix can be approximated by $l = n 2\pi a$.)

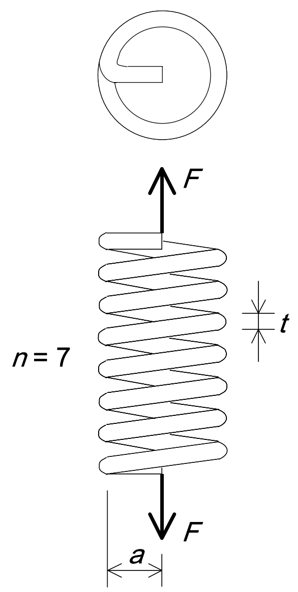

Figure 4. Helix spring

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Answers to Problem 1

a Admissible

Yes, the deflection is kinematically admissible because

$$\phi = 0 \quad \to \quad u_\phi = 0, \quad \frac{\partial u_z}{\partial \phi} = 0 \ \text{(symmetry)}$$

$$\phi = \tfrac{\pi}{2} \quad \to \quad u_\phi = 0 \qquad \text{(boundary condition)}$$

b Energy

$$E_{\text{pot}} = \int_{\phi=0}^{\frac{1}{2}\pi} \int_{\theta=0}^{2\pi} \left(\tfrac{1}{2} n_{\phi\phi}\varepsilon_{\phi\phi} + \tfrac{1}{2} n_{\theta\theta}\varepsilon_{\theta\theta} + u_z p \cos\phi - u_\phi p \sin\phi \right) a \sin\phi \, d\theta \, a \, d\phi$$

c Equations

$$\frac{\partial E_{\text{pot}}}{\partial C_1} = \frac{\pi E t}{315}(252C_1 + 84C_2 + 336C_3 - 96C_4) + \pi p a^2 \tfrac{1}{2} = 0$$

$$\frac{\partial E_{\text{pot}}}{\partial C_2} = \frac{\pi E t}{315}(84C_1 + 588C_2 + 672C_3 - 192C_4) = 0$$

$$\frac{\partial E_{\text{pot}}}{\partial C_3} = \frac{\pi E t}{315}(336C_1 + 672C_2 + 1680C_3 - 480C_4) - \pi p a^2 = 0$$

$$\frac{\partial E_{\text{pot}}}{\partial C_4} = \frac{\pi E t}{315}(-96C_1 - 192C_2 - 480C_3 + 5696C_4) + \pi p a^2 \tfrac{2}{3} = 0$$

d Deflection

$$\phi = 0 \quad \to \quad u_{z\,\max} = C_1 + C_2 \qquad \boxed{u_{z\,\max} = -1.97 \frac{p a^2}{E t}}$$

e Check

We can check the formula by a finite element analysis of a dome with specific dimensions and material properties.

Alternatively, we can check the formula by including more terms and coefficients in the functions u_z and u_ϕ.

Alternatively, we can calculate the membrane forces $n_{\phi\phi}, n_{\theta\theta}$ and check whether they are approximately in equilibrium with the load p.

Encore (not an exam problem)

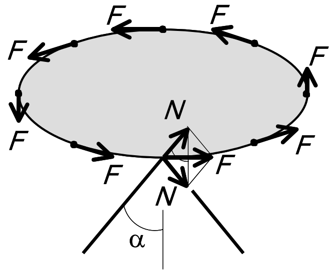
In a finite element analysis we have found

$$u_{z\,\max} = -1.73 \frac{p a^2}{E t}.$$

In this the following quantities have been used. p = 2500 N/m², a = 10 m, E = 29000e6 N/m², t = 0.1 m, ν = 0.2.

Answers to Problem 2

a Normal force



$$\left.\begin{aligned} M_t &= \frac{n}{2} F a \\ \sin\alpha &= \frac{\frac{1}{2}F}{N} \end{aligned}\right\} N = \frac{M_t}{n a \sin\alpha}$$

b Complementary Energy

truss $\quad E_c = n\frac{1}{2}\frac{N^2}{EA} l = n\frac{1}{2}\frac{1}{EA}\left(\frac{M_t}{n a \sin\alpha}\right)^2 \frac{h}{\cos\alpha}$

beam model $\quad E_c = \frac{1}{2}\frac{M_t^{\,2}}{G\boldsymbol{I}_t} h$

$$n\frac{1}{2}\frac{1}{EA}\left(\frac{M_t}{n a \sin\alpha}\right)^2 \frac{h}{\cos\alpha} = \frac{1}{2}\frac{M_t^{\,2}}{G\boldsymbol{I}_t} h$$

$$n\frac{1}{EA}\left(\frac{1}{n a \sin\alpha}\right)^2 \frac{1}{\cos\alpha} = \frac{1}{G\boldsymbol{I}_t}$$

$$\frac{EA\left(n a \sin\alpha\right)^2 \cos\alpha}{n} = G\boldsymbol{I}_t$$

$$\boxed{G\boldsymbol{I}_t = n a^2 EA \sin^2\alpha \cos\alpha}$$

c Angle

$$\frac{G\boldsymbol{I}_t}{n A l} = \frac{n a^2 EA \sin^2\alpha\cos\alpha}{n A l} = \frac{a^2 E \sin^2\alpha\cos^2\alpha}{h} =$$

$$\frac{d \sin^2\alpha\cos^2\alpha}{d\alpha} = 2\sin\alpha\cos\alpha\cos^2\alpha + \sin^2\alpha\, 2\cos\alpha\left(-\sin\alpha\right) = 0$$

$$\cos\alpha = \sin\alpha$$

$$1 = \tan\alpha$$

$$\alpha = \arctan 1 = \tfrac{1}{4}\pi \text{ rad}$$

$$\boxed{\alpha = 45^\circ}$$

Answers to Problem 3

a Distributed Springs

$$k = \frac{\sigma}{u} = \frac{\dfrac{F}{2\pi a\sqrt{a^2 - r^2}}}{\dfrac{F(1-\nu^2)}{2aE}} = \frac{E}{\pi(1-\nu^2)\sqrt{a^2-r^2}}$$

At the edges $r = a$ the spring stiffness k is infinite.

b Evenly Distributed Springs

$$k = \frac{\sigma_{mean}}{u} = \frac{\dfrac{F}{\pi a^2}}{\dfrac{F(1-\nu^2)}{2aE}} = \frac{2E}{\pi a(1-\nu^2)}$$

c Radius

In a solid, part of the load is transferred by shear to the neighbouring material. If the area πa^2 becomes larger the circumference $2\pi a$ becomes relatively smaller $\frac{2\pi a}{\pi a^2} = \frac{2}{a}$. Therefore, the contribution of the neighbouring material in carrying the load becomes relatively smaller.

Distributed springs are a very crude model for soil that supports a structure. Not only the computed stress distribution will be completely wrong but also the spring stiffness depends on the compressed area, which is often not known in advance.

Answers to problem 4

a Stiffness

Most of the wire is loaded in torsion and shear. Probably the torsion deformation is the largest. So, the torsion stiffness determines the spring stiffness.

b Stiffness

The following equations describe the problem.

$$F = ku \qquad M_t = Fa \qquad u = a\varphi$$
$$M_t = G\boldsymbol{I}_t\theta \qquad \varphi = \theta l$$
$$\boldsymbol{I}_t = \tfrac{1}{2}\pi(\tfrac{1}{2}t)^4 \qquad l = n\,2\pi a$$

Substitution gives

$$k = \frac{F}{u} = \frac{\dfrac{M_t}{a}}{a\varphi} = \frac{G\boldsymbol{I}_t\theta}{a^2\theta l} = \frac{G\,\frac{1}{2}\pi(\frac{1}{2}t)^4}{a^2 n\,2\pi a} = \frac{G\,t^4}{64\,n\,a^3}$$

Q.E.D.

Encore (not an exam problem)

The largest torsion stress in the wire is

$$\tau_{\text{max torsion}} = \frac{M_t}{\frac{1}{2}\pi(\frac{1}{2}t)^3} = 16\frac{Fa}{\pi t^3}$$

The maximum shear stress in the wire is

$$\tau_{\text{max shear}} = \frac{4}{3}\frac{F}{A} = \frac{4}{3}\frac{F}{\pi(\frac{1}{2}t)^2} = \frac{16}{3}\frac{F}{\pi t^2}$$

At one point in the wire cross-section they occur together in the same direction.

$$\tau_{\max} = \tau_{\text{max torsion}} + \tau_{\text{max shear}} = 16\frac{F}{\pi t^2}(\frac{a}{t} + \frac{1}{3})$$

According to the Von Misses criterion, the shear stress must be smaller than $\tau_{\max} < \frac{f_y}{\sqrt{3}}$

The largest bending stress in the horizontal wire parts is $\sigma_{\text{max bending}} = \dfrac{M}{\frac{1}{4}\pi(\frac{1}{2}t)^3} = 32\dfrac{Fa}{\pi t^3}$, which should be smaller than the yield stress f_y. The latter is critical when $a > t$.