Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number
at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Wednesday 3 November 2006 at 9:00 – 12:00 hours

Problem 1 (3 points)

A program for computation of beams needs to be extended with a non-prismatic element (Fig. 1). The thickness *t* and Young's modulus *E* are constant over the length *L*. The depth varies linearly between d_1 at the left end to d_2 at the right end. In this problem the stiffness matrix of the element is derived. We choose using complementary energy and select the following moment distribution

$$M = M_1(1-\frac{x}{L}) - M_2\frac{x}{L}.$$

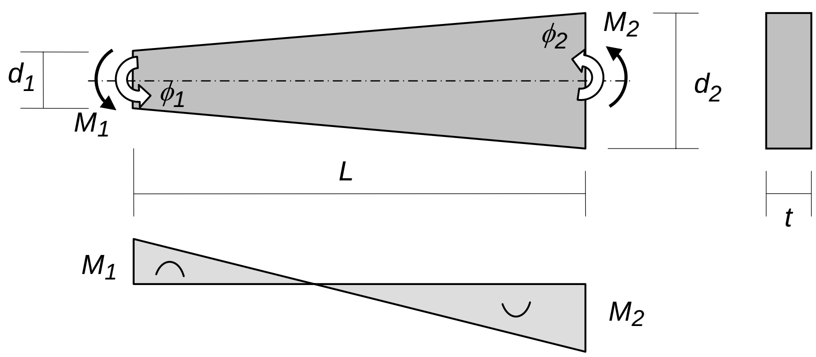

Figure 1. Non-prismatic beam element and moment distribution

a Could we also choose another moment distribution?

b Give the formula for the complementary energy of the beam. (You do not need to evaluate the formula.)

c Maple evaluates the internal complementary energy as

$$E_c = \frac{L}{E\,t}\left(M_1^2 c_1 + M_1 M_2 c_2 + M_2^2 c_3\right)$$

where,

$$c_1 = \frac{3}{(d_2-d_1)^3}\left(3 - 4\frac{d_1}{d_2} + \frac{d_2^2}{d_1^2} + 2\ln\frac{d_2}{d_1}\right)$$

$$c_2 = \frac{3}{(d_2-d_1)^3}2\left(\frac{d_1}{d_2} - \frac{d_2}{d_1} + 2\ln\frac{d_2}{d_1}\right)$$

$$c_3 = \frac{3}{(d_2-d_1)^3}\left(-3 + 4\frac{d_1}{d_2} - \frac{d_1^2}{d_2^2} - 2\ln\frac{d_1}{d_2}\right)$$

How can we use this to derive the stiffness matrix?

Problem 2 (2 points)

Calculate the shear stiffness GA_r of a rectangular tubular section (Fig. 2 and 3).

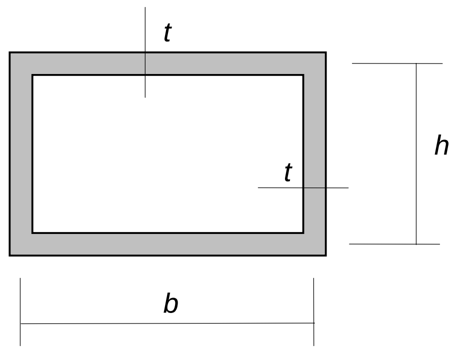

Figure 2. Cross-section of a rectangular tube

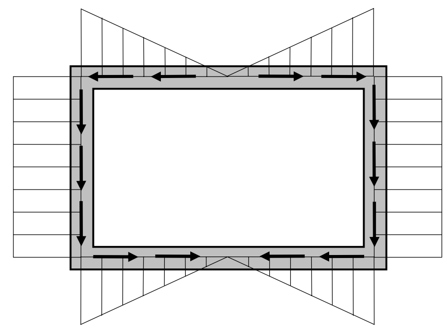

Figure 3. Shear stress distribution

Problem 3 (3 points)

The core of a high-rise building has a triangular cross-section shape (Fig. 4). The shear modulus of the wall material is *G*. We will calculate the torsion properties.

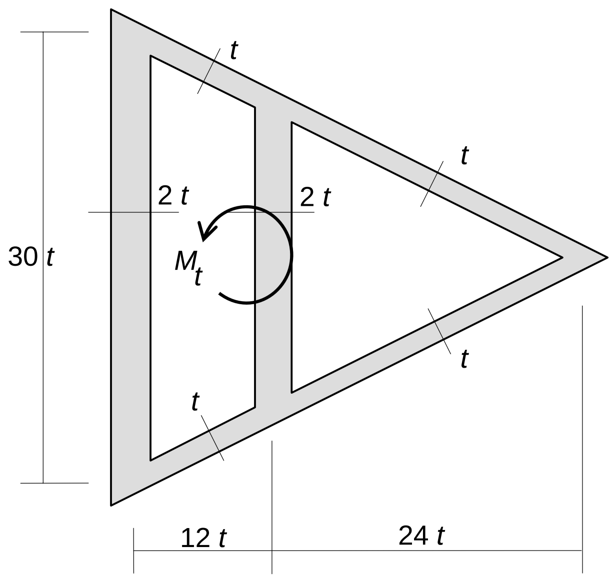

Figure 4. Cross-section of the core of a high-rise building

a Formulate the equilibrium equations of the weight less plates. (You do not need to solve the equations.)

b The solution of the equations is $w_1 = \frac{10500}{1531}\frac{pt^2}{S}$, $w_2 = \frac{7620}{1531}\frac{pt^2}{S}$.

Calculate the torsion stiffness.

c Calculate the shear stresses in the walls. Draw a picture of the cross-section with the stresses in the correct directions.

Problem 4 (2 points)

a A deep beam is analysed twice using a finite element program (Fig. 5). The first time normal elements were used. The second time high accuracy elements were used. Unfortunately, the program output does not mention the used element type. However, it is known that both element types were derived using potential energy.

Which of the two analyses will give the largest deflection? Explain your answer.

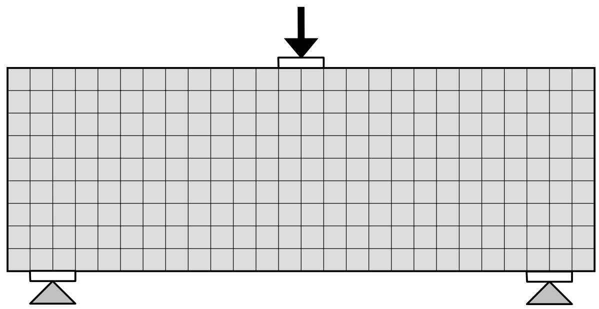

Figure 5. Finite element model of a deep beam

b A scientist published a model for a glued timber beam [1] (Fig. 6). The beam is simply supported and carries an evenly distributed load. The potential energy of the beam consist of three contributions

$$E_{pot} = U_B + U_C + U_L$$

where U_B is due to the wood, U_C is due to the glue and U_L is due to the loading.

The relation between shear stress τ and displacement Δ of the glue layer is experimentally determined (Fig. 7). Subsequently, the scientist makes a mistake by calculating the strain energy in the glue layer as

$$U_C = t \int_0^L \tau \Delta dx \, .$$

What is the correct formula for the potential energy of the glue layer?

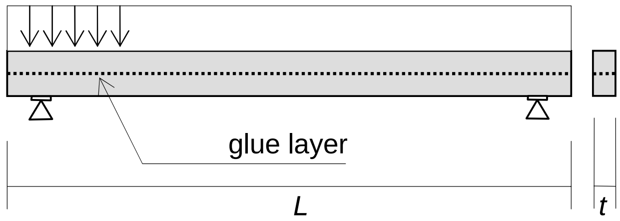

Figure 6. Glued timber beam

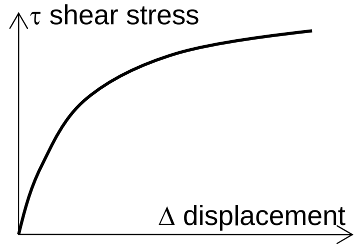

Figure 7. Shear stress-displacement diagram of a glue layer

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Answers to Problem 1

a Another moment distribution

When using complementary energy the moment distribution must be in equilibrium with the loading. So, $M_1 = M(0)$, $M_2 = M(L)$ and $\frac{d^2M}{dx^2} = q = 0$. This, is the case only for the proposed moment distribution, therefore, the distribution cannot be different.

b Complementary energy

$$E_{compl} = E_c = \int_0^L \tfrac{1}{2} \frac{M^2}{EI} dx$$

c Stifness matrix

The flexibility matrix can be derived as $C_{ij} = \frac{\partial^2 E_c}{\partial M_i \partial M_j}$. Therefore

$$C = \begin{bmatrix} \frac{\partial^2 E_c}{\partial M_1^2} & \frac{\partial^2 E_c}{\partial M_1 \partial M_2} \\ \frac{\partial^2 E_c}{\partial M_2 \partial M_1} & \frac{\partial^2 E_c}{\partial M_2^2} \end{bmatrix} = \frac{L}{E\,t} \begin{bmatrix} 2c_1 & c_2 \\ c_2 & 2c_3 \end{bmatrix}.$$

The stiffness matrix is the inverse of the flexibility matrix

$$K = C^{-1}.$$

Encore (Not an exam question)

$$K = \frac{E\,t}{L(4c_1c_3 - c_2^2)} \begin{bmatrix} 2c_3 & -c_2 \\ -c_2 & 2c_1 \end{bmatrix}.$$

Answers to Problem 2

Complementary energy of the cross-section

$$E_c = \int_A \tfrac{1}{2} \tau \gamma \, dA \, \Delta x = \tfrac{1}{2} \frac{1}{G} \int_A \tau^2 \, dA \, \Delta x = \tfrac{1}{2} \frac{1}{G} \left(4 \int_{y=0}^{\frac{b}{2}} \left(\tau_{max} \frac{2y}{b} \right)^2 dy\, t + 2 \int_{x=0}^{h} \tau_{max}^2 dx\, t \right) \Delta x =$$

$$= \tfrac{1}{2} \frac{1}{G} \left(4 \frac{4\tau_{max}^2}{b^2} \tfrac{1}{3} (\frac{b}{2})^3 t + 2\tau_{max}^2 h\, t \right) \Delta x = \frac{\tau_{max}^2}{G} t \left(\tfrac{1}{3} b + h \right) \Delta x$$

Complementary energy of the wire-frame model

$$E_c = \tfrac{1}{2} \frac{V^2}{GA_r} \Delta x = \tfrac{1}{2} \frac{(2\tau_{max} h\, t)^2}{GA_r} \Delta x = 2 \frac{\tau_{max}^2 h^2 t^2}{GA_r} \Delta x$$

Equal

$$\frac{\tau_{max}^2}{G} t \left(\tfrac{1}{3} b + h \right) \Delta x = 2 \frac{\tau_{max}^2 h^2 t^2}{G A_r} \Delta x$$

$$G A_r = G \frac{6 h^2 t}{3h + b}$$

Encore (Not an exam question)

If we want to include the outer dimensions in the formula we substitute $h = h - t$ and $b = b - t$.

$$G A_r = G \frac{6(h-t)^2 t}{3h + b - 4t}$$

$$\tau_{max} = \frac{V}{2(h-t)t}$$

Mr. J.M. de Wit has checked these formulas with ANSYS (FEM, Volume elements) [2]. He proposed the following improved formulas.

$$G A_r = G t (2.17 h + 1.89 t - 0.37 b)$$

$$\tau_{max} = \frac{V}{t (1.79 h - 1.31 t)(1 - 0.0420 \frac{h}{b})}$$

Mr. R. Velner has shown that round corners have a large influence [3]. He proposed

$$G A_r = G t (2.26 h + 0.74 t + 0.42 r - 0.68 b)$$

$$\tau_{max} = \frac{V}{t (1.88 h - 1.44 t - 0.43 r)(1 - 0.066 \frac{h}{b} - 0.01 \frac{t}{r})}$$

where r is the external radius of the corners.

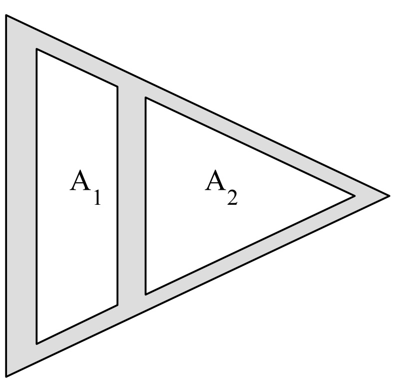
Answers to problem 3

a Weight less plates

We assume that plate 2 is higher than plate 1.

$$S \left(\frac{w_1}{2t} 30t - \frac{w_2 - w_1}{2t} 20t + 2 \frac{w_1}{t} 13t \right) = p A_1$$

$$S \left(\frac{w_2 - w_1}{2t} 20t + \frac{w_2}{t} 26t + \frac{w_2}{t} 26t \right) = p A_2$$



b Torsion stiffness

$$w_1 = \tfrac{10500}{1531}\frac{pt^2}{S}, \quad w_2 = \tfrac{7620}{1531}\frac{pt^2}{S}$$

So

$$\phi_1 = \tfrac{10500}{1531}2\theta G t^2, \quad \phi_2 = \tfrac{7620}{1531}2\theta G t^2$$

$$M_t = 2(\phi_1 A_1 + \phi_2 A_2) = 2\left(\tfrac{10500}{1531}2\theta G t^2 12t\frac{30t+20t}{2} + \tfrac{7620}{1531}2\theta G t^2 \tfrac{1}{2}20t\,24t\right) =$$

$$= \frac{19915200}{1531}\theta G t^4 \approx 13000\,\theta G t^4$$

So,

$$GI_t = 13000\,G t^4$$

c Stresses

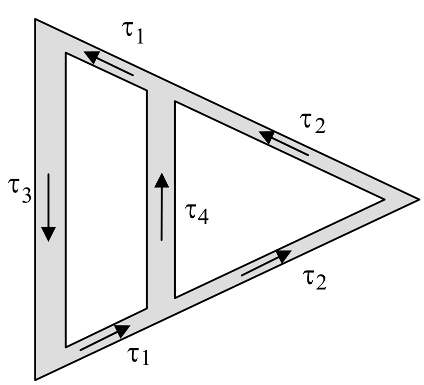
$$\theta G t = \frac{M_t}{13000t^3}$$

$$\tau_1 = \frac{\phi_1}{t} = \frac{10500}{1531}2\theta G t = \frac{10500}{1531}2\frac{M_t}{13000t^3} = \frac{21}{19903}\frac{M_t}{t^3}$$

$$\tau_2 = \frac{\phi_2}{t} = \frac{7620}{1531}2\theta G t = \frac{7620}{1531}2\frac{M_t}{13000t^3} = \frac{381}{497575}\frac{M_t}{t^3}$$

$$\tau_3 = \frac{\phi_1}{2t} = \frac{10500}{1531}\theta G t = \frac{10500}{1531}\frac{M_t}{13000t^3} = \frac{21}{39806}\frac{M_t}{t^3}$$

$$\tau_4 = \frac{\phi_1 - \phi_2}{2t} = \frac{10500-7620}{1531}\theta G t = \frac{10500-7620}{1531}\frac{M_t}{13000t^3} = \frac{72}{497575}\frac{M_t}{t^3}$$

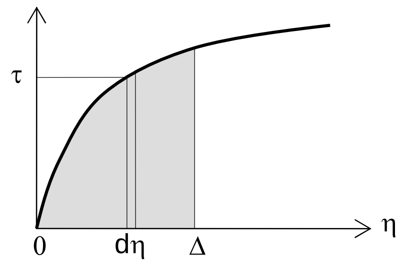


Answers to problem 4

a Deflection
The high accuracy elements will give a larger deflection because displacements approximated with potential energy are always too small.

b Strain energy

$$U_C = t\int_0^L\int_0^\Delta \tau(\eta)\,d\eta\,dx$$



References

1 Journal of Structural Engineering Vol. 120, No. 6. June 1994, pp. 1909-1929

2 J.M. de Wit (2005) “Afschuifstijfheid en maximale schuifspanning van rechthoekige kokerprofielen”, B.Sc. project report, Delft University of Technology, (in Dutch), online http://www.mechanics.citg.tudelft.nl/graduation/BSc.projects.htm

3 R. Velner (2005) “Afschuifstijfheid van kokers met ronde hoeken”, B.Sc. project report, Delft University of Technology, (in Dutch), online http://www.mechanics.citg.tudelft.nl/graduation/BSc.projects.htm