Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your <u>name</u> and <u>study number</u> at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Wednesday 31 October 2006 at 14:00 – 17:00 hours

Problem 1 (5 points)

A steel girder is loaded by forces *F* (Fig. 1). The flanges and stiffeners have a cross-section area *d* x *t*. The webs have a thickness *t*. We want to calculate the deflection *u* of the girder and use complementary energy. The force flow is drawn in Figure 2. Each web panel has a homogeneous stress state. The forces in flanges and stiffeners vary linearly over the lengths.

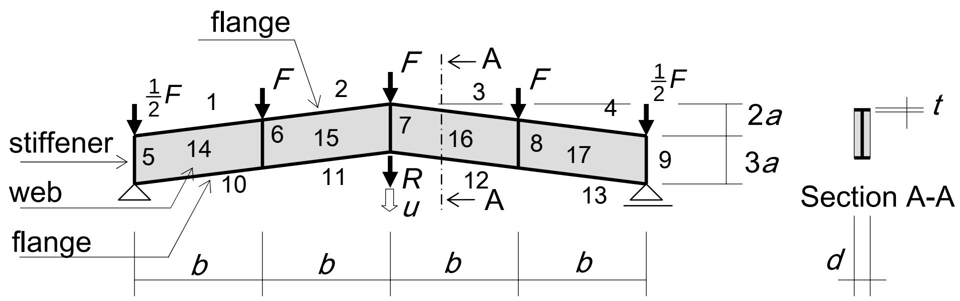


Figure 1. Steel plate girder

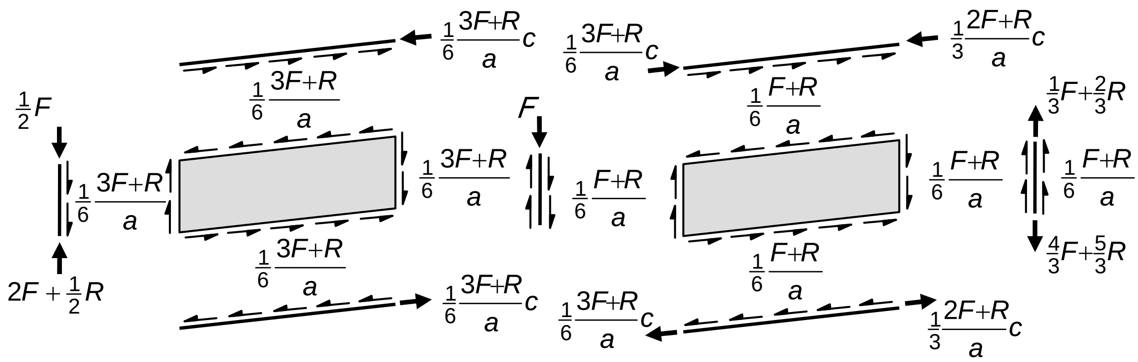


Figure 2. Forces in the components of the left part of the girder. $c = \sqrt{a^2 + b^2}$

a Show that the complementary energy in each flange part en stiffener can be calculated by

$$E_c = \frac{l}{6Edt}(N_1^2 + N_1 N_2 + N_2^2)\,,$$

where N_1, N_2 are the forces at each end of the part and l is the part length.

b Show that the complementary energy in each web part can be calculated by

$$E_c = \frac{n^2 A}{Et}(1 + \nu + 2\frac{a^2}{b^2})\,,$$

where n [N/m] is the edge shear force and A [m²] is the area of a web part.

c The complementary energy of the girder can be derived as

$$E_c = \frac{F^2}{Et}\left(\tfrac{23}{27}\frac{c^3}{da^2} + \tfrac{89}{12}\frac{a}{d} + \tfrac{5}{3}(1+\nu)\frac{b}{a} + \tfrac{10}{3}\frac{a}{b}\right) +$$
$$+ \frac{FR}{Et}\left(\tfrac{19}{27}\frac{c^3}{da^2} + \tfrac{65}{12}\frac{a}{d} + \tfrac{4}{3}(1+\nu)\frac{b}{a} + \tfrac{8}{3}\frac{a}{b}\right) +$$
$$+ \frac{R^2}{Et}\left(\tfrac{4}{27}\frac{c^3}{da^2} + \tfrac{29}{12}\frac{a}{d} + \tfrac{1}{3}(1+\nu)\frac{b}{a} + \tfrac{2}{3}\frac{a}{b}\right).$$

Explain how this result can be obtained from Table 1.

d Assume that R = 0. Derive a formula for the deflection u of the girder.

e Suppose we want to check this formula without using complementary energy. What would be a good way to do so? Explain your answer.

element	N_1	N_2	l	E_c
1	0	$-\tfrac{1}{6}\frac{3F+R}{a}c$	c	$\frac{9F^2+6FR+R^2}{216Et}\frac{c^3}{da^2}$
2	$-\tfrac{1}{6}\frac{3F+R}{a}c$	$-\tfrac{1}{3}\frac{2F+R}{a}c$	c	$\frac{37F^2+32FR+7R^2}{216Et}\frac{c^3}{da^2}$
3	$-\tfrac{1}{3}\frac{2F+R}{a}c$	$-\tfrac{1}{6}\frac{3F+R}{a}c$	c	$\frac{37F^2+32FR+7R^2}{216Et}\frac{c^3}{da^2}$
4	$-\tfrac{1}{6}\frac{3F+R}{a}c$	0	c	$\frac{9F^2+6FR+R^2}{216Et}\frac{c^3}{da^2}$
5	$-2F-\tfrac{1}{2}R$	$-\tfrac{1}{2}F$	$3a$	$\frac{21F^2+9FR+R^2}{8Et}\frac{a}{d}$
6	0	$-F$	$3a$	$\tfrac{1}{2}\frac{F^2}{Et}\frac{a}{d}$
7	$\tfrac{4}{3}F+\tfrac{5}{3}R$	$\tfrac{1}{3}F+\tfrac{2}{3}R$	$3a$	$\frac{7F^2+19FR+13R^2}{6Et}\frac{a}{d}$
8	0	$-F$	$3a$	$\tfrac{1}{2}\frac{F^2}{Et}\frac{a}{d}$
9	$-2F-\tfrac{1}{2}R$	$-\tfrac{1}{2}F$	$3a$	$\frac{21F^2+9FR+R^2}{8Et}\frac{a}{d}$
10	0	$\tfrac{1}{6}\frac{3F+R}{a}c$	c	$\frac{9F^2+6FR+R^2}{216Et}\frac{c^3}{da^2}$

11	$\frac{1}{6}\frac{3F+R}{a}c$	$\frac{1}{3}\frac{2F+R}{a}c$	c	$\frac{37F^2+32FR+7R^2}{216Et}\frac{c^3}{da^2}$
12	$\frac{1}{3}\frac{2F+R}{a}c$	$\frac{1}{6}\frac{3F+R}{a}c$	c	$\frac{37F^2+32FR+7R^2}{216Et}\frac{c^3}{da^2}$
13	$\frac{1}{6}\frac{3F+R}{a}c$	0	c	$\frac{9F^2+6FR+R^2}{216Et}\frac{c^3}{da^2}$
element	n		A	E_c
14	$-\frac{1}{6}\frac{3F+R}{a}$		$3ab$	$\frac{9F^2+6FR+R^2}{12Et}\left((1+\nu)\frac{b}{a}+2\frac{a}{b}\right)$
15	$-\frac{1}{6}\frac{F+R}{a}$		$3ab$	$\frac{F^2+2FR+R^2}{12Et}\left((1+\nu)\frac{b}{a}+2\frac{a}{b}\right)$
16	$\frac{1}{6}\frac{F+R}{a}$		$3ab$	$\frac{F^2+2FR+R^2}{12Et}\left((1+\nu)\frac{b}{a}+2\frac{a}{b}\right)$
17	$\frac{1}{6}\frac{3F+R}{a}$		$3ab$	$\frac{9F^2+6FR+R^2}{12Et}\left((1+\nu)\frac{b}{a}+2\frac{a}{b}\right)$

Table 1. Complementary energy in the elements of the girder. $c=\sqrt{a^2+b^2}$

Problem 2 (2 points)

a What is a bi-moment and how is it defined?

b A typical bi-moment distribution is locally very large and small in other sections of a beam. What causes the large value?

c Suppose that the bi-moment in a section is known (for example 927 Nm2). How can we calculate the largest stress in that cross-section?

d Commonly used frame analysis programs (for example Matrix Frame, ESA PT, STAAD Pro) do not calculate bi-moments. Apparently the software developers think it is not important. In what situation the bi-moment cannot be ignored?

Problem 3 (3 points)

a “The bending moments in a plate under a point load are infinitely large.” Is this statement true or false? Explain your answer.

b The stiffness matrix of any linear elastic system is symmetrical (Fig. 8). Who was the first to proof this? Give an outline of the proof.

c A program for bridge design can compute the deformation of the bridge middle due to all possible positions of the load. The implemented algorithm uses reciprocity of load and displacement. What are limitations of this program on bridge models and loading?

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} 8417 & 408 & 524 & 0 & 0 & 0 \\ 411 & 496 & 313 & 0 & 0 & 0 \\ 518 & 304 & 1089 & 0 & 0 & 0 \\ 0 & 0 & 0 & 168 & 0 & 0 \\ 0 & 0 & 0 & 0 & 704 & 0 \\ 0 & 0 & 0 & 0 & 0 & 472 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}
$$

Figure 8. Experimentally determined stiffness matrix of a wood cube (African Mahogany). [N/mm²] Calculated from data in [1] pages 4-2 and 4-3. *x* Is the fibre direction, *y* is the tangential direction and *z* is the radial direction.

References

[1] D.W. Green, J.E. Winandy, D.E. Kretschmann, Mechanical Properties of Wood, Chapt. 4 in *Wood Handbook*, U.S. Department of Agriculture, online www.fpl.fs.fed.us

Exam CT5141, 31 October 2006
Answers to Problem 1

a Flange part

The axial force varies linear between N_1 and N_2.

$$N = (1-\frac{x}{l})N_1 + \frac{x}{l}N_2$$

The constitutive equation is

$$N = E\,d\,t\,\varepsilon \quad \text{or} \quad \varepsilon = \frac{N}{E\,d\,t}$$

The internal complementary energy is

$$E_c = \int_{x=0}^{l} \tfrac{1}{2} N\,\varepsilon\,dx = \tfrac{1}{2}\int_{x=0}^{l} \frac{N^2}{E\,d\,t}\,dx = \tfrac{1}{2}\frac{1}{E\,d\,t}\int_{x=0}^{l}\left((1-\frac{x}{l})N_1 + \frac{x}{l}N_2\right)^2 dx =$$

$$= \tfrac{1}{2}\frac{1}{E\,d\,t}\int_{x=0}^{l} (1-\frac{x}{l})^2 N_1^2 + 2(1-\frac{x}{l})\frac{x}{l}N_1N_2 + (\frac{x}{l})^2 N_2^2\,dx =$$

$$= \tfrac{1}{2}\frac{1}{E\,d\,t}\int_{x=0}^{l}\left(1-2\frac{x}{l}+\frac{x^2}{l^2}\right)N_1^2 + 2\left(\frac{x}{l}-\frac{x^2}{l^2}\right)N_1N_2 + \frac{x^2}{l^2}N_2^2\,dx =$$

$$= \tfrac{1}{2}\frac{1}{E\,d\,t}\left[\left(x-\frac{x^2}{l}+\tfrac{1}{3}\frac{x^3}{l^2}\right)N_1^2 + 2\left(\tfrac{1}{2}\frac{x^2}{l}-\tfrac{1}{3}\frac{x^3}{l^2}\right)N_1N_2 + \tfrac{1}{3}\frac{x^3}{l^2}N_2^2\right]_0^l =$$

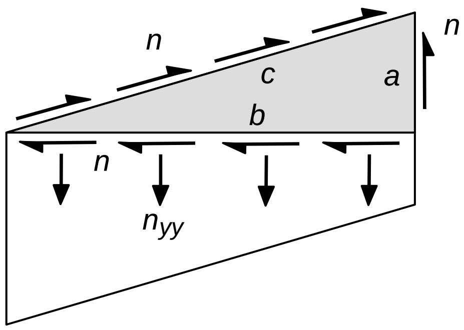
$$= \tfrac{1}{2}\frac{1}{E\,d\,t}\left[\left(l-l+\tfrac{1}{3}l\right)N_1^2 + 2\left(\tfrac{1}{2}l-\tfrac{1}{3}l\right)N_1N_2 + \tfrac{1}{3}lN_2^2\right] =$$

$$= \tfrac{1}{2}\frac{1}{E\,d\,t}\left[\tfrac{1}{3}lN_1^2 + \tfrac{1}{3}lN_1N_2 + \tfrac{1}{3}lN_2^2\right] =$$

$$= \tfrac{l}{6}\frac{1}{E\,d\,t}\left[N_1^2 + N_1N_2 + N_2^2\right]$$

Q.E.D

b Web part



The shear force n_{xy} per unit length in a web part is

$$n_{xy} = n$$

There is also a normal force in a web part

$$n_{yy} = 2n\frac{a}{b}.$$

The latter can be derived from equilibrium (see Figure) $na + nc\dfrac{a}{c} - n_{yy}b = 0$.

The constitutive relations are

$$\varepsilon_{yy} = \frac{n_{yy}}{Et}, \qquad \gamma_{xy} = \frac{n_{xy}}{Gt}$$

Where $G = \dfrac{E}{2(1+\nu)}$. The internal complementary energy is

$$E_c = \left(\tfrac{1}{2} n_{xy} \gamma_{xy} + \tfrac{1}{2} n_{yy} \varepsilon_{yy}\right) A =$$

$$= \left(\tfrac{1}{2} \frac{n_{xy}^2}{Gt} + \tfrac{1}{2} \frac{n_{yy}^2}{Et}\right) A =$$

$$= \left((1+\nu) \frac{n_{xy}^2}{Et} + \tfrac{1}{2} \frac{4 n_{xy}^2 \dfrac{a^2}{b^2}}{Et} \right) A =$$

$$= \frac{A\, n_{xy}^2}{Et} \left((1+\nu) + 2 \frac{a^2}{b^2} \right)$$

Q.E.D.

c Complementary energy
The complementary energy of the girder can be obtained by adding the complementary energy of all elements in Table 1.

d Deflection
When we assume that the displacement *u* is imposed the complementary energy is

$$E_{compl} = E_c - R\, u$$

where *R* is a support reaction. The girder is now indetermined to the first degree and we select *R* as redundant. The complementary energy needs to be minimal with respect to *R*.

$$\frac{\partial E_{compl}}{\partial R} = \frac{\partial E_c}{\partial R} - u = 0 \,.$$

So

$$u = \frac{\partial E_c}{\partial R}$$

$$u = \frac{F}{E\,t} \left(\tfrac{19}{27} \frac{c^3}{da^2} + \tfrac{65}{12} \frac{a}{d} + \tfrac{4}{3} (1+\nu) \frac{b}{a} + \tfrac{8}{3} \frac{a}{b} \right) + 2 \frac{R}{E\,t} \left(\tfrac{4}{27} \frac{c^3}{da^2} + \tfrac{29}{12} \frac{a}{d} + \tfrac{1}{3} (1+\nu) \frac{b}{a} + \tfrac{2}{3} \frac{a}{b} \right).$$

Thus, we know the relation between *u* and *R*. It does not matter any more which of them was imposed. When *R* = 0 this simplifies to

$$u = \frac{F}{E\,t} \left(\tfrac{19}{27} \frac{c^3}{da^2} + \tfrac{65}{12} \frac{a}{d} + \tfrac{4}{3} (1+\nu) \frac{b}{a} + \tfrac{8}{3} \frac{a}{b} \right).$$

e Check

1) We can check the formula by comparing it to the formula for deflection of a slender beam. This would only be a rough check because the effect of the camber and shear deformation would not been included in the slender beam formula.
2) We can check one prediction of the formula by comparing it to the results of a finite element analysis (plate elements, linear elastic analysis)
3) We can derive exactly the same formula by the direct method, writing down and solving the kinematic, constitutive and equilibrium equations for all 17 elements.

Answers to Problem 2

a Bi-moment

A bi-moment is the distribution of stresses that pulls torsion warping out of a cross-section. It is defined as

$$B = -\int_A \sigma_{xx} \psi \, dA$$

where $\psi(y,z)$ is the warping function and A is the cross-section area.

b Large value

A peak in the bi-moment distribution is caused by 1) a joint that restrains warping, 2) a change in the beam cross-section (non-prismatic) or 3) a concentrated torsion moment loading.

c Largest stress

The stresses in a cross-section are calculated by a cross-section analysis program. Alternatively, formulas exist for some cross-sections.

d Important

For thin-wall open cross-sections (for example cold formed steel sections) the bi-moment can be important. In these sections the peak stresses due to a bi-moment can be very large. Also, most of the torsion stiffness of these beams comes from restraint warping. This cannot be ignored when their torsion stiffness is important.

Answers to problem 3

a Point load

True, according to the plate theory the moments under a point load are infinitely large. (This is true for both thin plates (Kirchhoff theory) and thick plates (Reissner theory).)
False, in reality they do not go to infinity. The plate theory is not valid in an area with a radius of approximately 1.5 times the plate thickness around a point load.

b Symmetry

James Clark Maxwell 1831- 1879

$$k_{ij} = \frac{\partial E_s}{\partial u_j \partial u_i} = \frac{\partial E_s}{\partial u_i \partial u_j} = k_{ji} \,,$$

where k_{ij} is an element of the stiffness matrix, u_i is a displacement and E_s is the strain energy in the system.

c Reciprocity

Reciprocity occurs only when the model is linear-elastic – which means small strains, small displacements, no yielding and no cracking.