Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your <u>name</u> and <u>study number</u> at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Wednesday 24 January 2007 at 9:00 – 12:00 hours

Problem 1 (5 points)

Consider an initially curved beam of an I shaped cross-section (Fig. 1). The objective is to calculate the moment of inertia I and section modulus W. The following equations describe the web behaviour.

$$\sigma_{\theta\theta} = \sigma_{rr} + r\,\frac{d\sigma_{rr}}{dr}$$

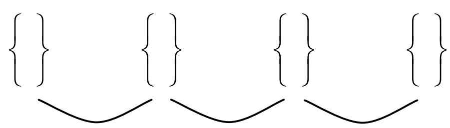
$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) + \varepsilon_i$$

$$\varepsilon_{\theta\theta} = \frac{u}{r}$$

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right)$$

$$\varepsilon_{rr} = \frac{du}{dr}$$

a Complete the following frame work.



What is ε_i in the equations?

b Derive the differential equation for the web part. Use the force method.

c Formulate the boundary conditions of the web part such that the differential equation can be solved (Fig. 2).

d Give the equations for deriving the normal force N, moment M, moment of inertia I and section modulus W (You do not need to evaluate the equations).

e Maple has evaluated the equations and plotted the result (Fig. 3). In frame analysis often a curved beam is approximated by several short straight beams. For what radius R is this a good approximation? Explain your answer.

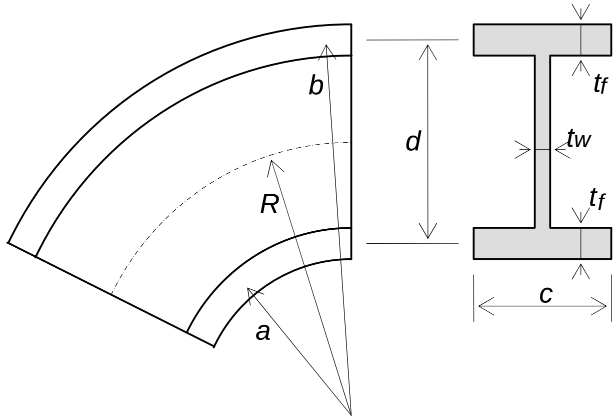


Figure 1. Curved I section

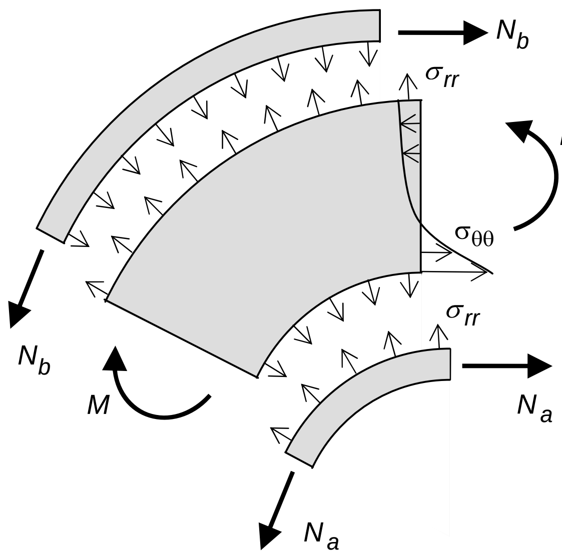


Figure 2. Force flow in the section parts

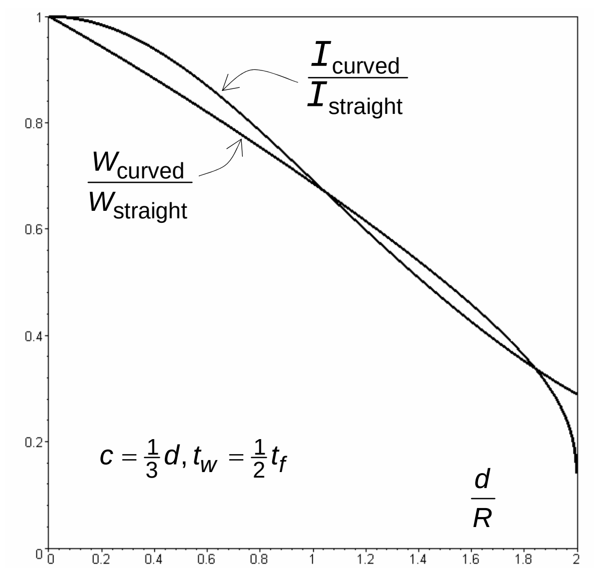


Figure 3. Moment of inertia I and section modulus W as function of the initial curvature

Problem 2 (3 points)

Consider a semi circular arch with a tension bar (Fig. 4). The materials are linear elastic. The bending stiffness of the arch is *EI* and the extension stiffness of the arch is infinitely large. The extension stiffness of the tension bar is *EA*. The evenly distributed loading is constant per unit of arch length (For example self weight).

The structure is calculated using complementary energy. We choose the force in the tension bar as redundant ϕ. The moment in the arch can be expressed in ϕ.

$$M(\varphi) = q r^2 (\tfrac{1}{2}\pi - \varphi \sin\varphi) - r(\phi + q r)\cos\varphi$$

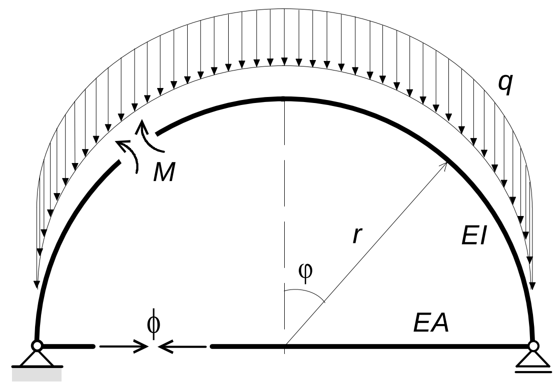


Figure 4. Arch with tension bar

a Show that the moment $M(0)$ in the top of the arch is in equilibrium with the loading.

b Give the formula for the complementary energy of the structure. (You do not need to evaluate the formula.)

c Maple has evaluated the complementary energy.

$$E_{\text{compl}} = \frac{\pi r^3}{48EI}[12\phi^2 - 12 r\,\phi q + r^2 q^2 (7\pi^2 - 66)] + \frac{\phi^2 r}{EA}$$

Calculate ϕ. Use the parameter $\beta = \dfrac{\pi r^2 EA}{EI}$

Problem 3 (2 points)

a Proof that the bi-moment distribution B of a prismatic beam does not depend on Young's modulus E.

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Answers to Problem 1

a Flange part

$$\{u\} \qquad \begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\end{Bmatrix} \qquad \begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\end{Bmatrix} \qquad \{p\}$$

kinematic equation — constitutive equations — equilibrium equation

ε_i is an artificial strain of the circumferential fibres. This causes a constant moment in the beam and no normal force or shear force.

b Differential equation

$$\varepsilon_{\theta\theta} = \frac{u}{r} \quad \rightarrow \quad u = r\,\varepsilon_{\theta\theta}$$

$$\varepsilon_{rr} = \frac{du}{dr} = \frac{d\,r\,\varepsilon_{\theta\theta}}{dr} = \varepsilon_{\theta\theta} + r\,\frac{d\varepsilon_{\theta\theta}}{dr}$$

$$\frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right) = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) + \varepsilon_i + r\,\frac{1}{E}\,\frac{d\sigma_{\theta\theta} - \nu\sigma_{rr}}{dr}$$

$$\sigma_{rr} - \nu\sigma_{\theta\theta} = \sigma_{\theta\theta} - \nu\sigma_{rr} + E\varepsilon_i + r\,\frac{d\sigma_{\theta\theta}}{dr} - \nu r\,\frac{d\sigma_{rr}}{dr}$$

$$(1+\nu)\sigma_{rr} - (1+\nu)\sigma_{\theta\theta} = E\varepsilon_i + r\,\frac{d\sigma_{\theta\theta}}{dr} - \nu r\,\frac{d\sigma_{rr}}{dr}$$

$$(1+\nu)\sigma_{rr} - (1+\nu)\left(\sigma_{rr} + r\,\frac{d\sigma_{rr}}{dr}\right) = E\varepsilon_i + r\left(\frac{d\sigma_{rr}}{dr} + \frac{d\sigma_{rr}}{dr} + r\,\frac{d^2\sigma_{rr}}{dr^2}\right) - \nu r\,\frac{d\sigma_{rr}}{dr}$$

$$+3r\,\frac{d\sigma_{rr}}{dr} + r^2\,\frac{d^2\sigma_{rr}}{dr^2} = -E\varepsilon_i$$

c Boundary conditions

$$r = a \quad \rightarrow \quad \sigma_{rr}(a)t_w a = N_a$$

$$r = b \quad \rightarrow \quad \sigma_{rr}(b)t_w b = -N_b$$

$$\sigma_{rr}(a)t_w a = E c t_f\left(\varepsilon_{\theta\theta}(a) - \varepsilon_i\right)$$

$$\sigma_{rr}(a)t_w a = c t_f\left(\sigma_{\theta\theta}(a) - \nu\sigma_{rr}(a)\right)$$

$$\sigma_{rr}(a)t_w a = c t_f\left(\sigma_{rr}(a) + a\,\frac{d\sigma_{rr}}{dr}\bigg|_{r=a} - \nu\sigma_{rr}(a)\right)$$

$$\sigma_{rr}(a)\left(t_w a - c t_f + c t_f \nu\right) = c t_f a\,\frac{d\sigma_{rr}}{dr}\bigg|_{r=a}$$

$$\sigma_{rr}(a)\left(\frac{t_w}{c t_f} - \frac{1-\nu}{a}\right) = \frac{d\sigma_{rr}}{dr}\bigg|_{r=a}$$

$$\sigma_{rr}(b) t_w b = -E c t_f (\varepsilon_{\theta\theta}(b) - \varepsilon_i)$$
$$\sigma_{rr}(b) t_w b = -c t_f \left(\sigma_{\theta\theta}(b) - \nu \sigma_{rr}(b)\right)$$
$$\sigma_{rr}(b) t_w b = -c t_f \left(\sigma_{rr}(b) + b \left.\frac{d\sigma_{rr}}{dr}\right|_{r=b} - \nu \sigma_{rr}(b)\right)$$
$$\sigma_{rr}(b)\left(t_w b + c t_f - c t_f \nu\right) = -c t_f b \left.\frac{d\sigma_{rr}}{dr}\right|_{r=b}$$
$$\sigma_{rr}(b)\left(-\frac{t_w}{c t_f} - \frac{1-\nu}{b}\right) = \left.\frac{d\sigma_{rr}}{dr}\right|_{r=b}$$

d Section quantities

$$N = t_w \int_{r=a}^{b} \sigma_{\theta\theta} dr + N_a + N_b$$
$$M = t_w \int_{r=a}^{b} \sigma_{\theta\theta}(R - r) dr + N_a \frac{d}{2} - N_b \frac{d}{2}$$
$$I = \frac{M}{E\kappa} \qquad \kappa = \frac{\varphi_i}{\varphi R} = \frac{\varepsilon_i}{R}$$
$$W = \frac{M}{\sigma_{\theta\theta,\max}} \qquad \sigma_{\theta\theta,\max} = \sigma_{\theta\theta}(a)$$

e Good approximation

In the graph we measure that if $\frac{d}{R} = 0.3$ a curved I section has approximately 3% less bending stiffness than a straight section. If $\frac{d}{R} = 0.1$ it has approximately a 3% less section modulus than a straight section. This 3% deviation is considered just acceptable therefore straight sections can be used to model a curved section if $\frac{d}{R} \le 0.1$.

Encore 1 (not an exam question)

Straight sections can still be used if $\frac{d}{R} > 0.1$, however, then the stresses need to be checked manually using the correct section modulus. If $\frac{d}{R} > 0.3$ then the correct moment of inertia needs to be entered manually in the frame program.

Encore 2 (not an exam question)

The exact solution is too large to be displayed here. With a Taylor expansion around $\frac{d}{R} = 0$ we find

$$I_{\text{curved}} \approx I_{\text{straight}} \left(1 - \frac{t_w^2 d^2 (t_w d + 12 t_f c) + 30 t_w d\, t_f^2 c^2 (5 + 4\nu) + 180 t_f^3 c^3 (1 - \nu^2)}{30 t_w d (t_w d + 2 t_f c)(t_w d + 6 t_f c)} \frac{d^2}{R^2}\right)$$

$$W_{\text{curved}} \approx W_{\text{straight}} \left(1 - \frac{t_w^2 d^2 + 6\nu t_f^2 c^2}{3 t_w d (2 t_f c + t_w d)} \frac{d}{R} \right)$$

where

$$I_{\text{straight}} = \tfrac{1}{12} t_w d^3 + 2 t_f c \left(\tfrac{1}{2} d\right)^2$$

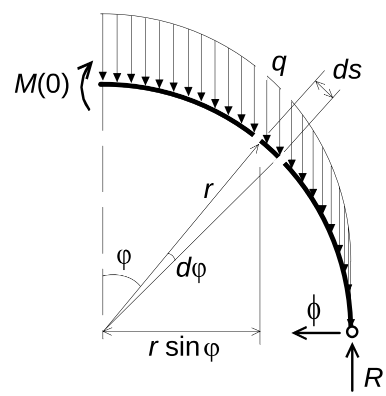
and

$$W_{\text{straight}} = \frac{I_{\text{straight}}}{\tfrac{1}{2} d}.$$

Answers to Problem 2

a <u>Equilibrium</u>

The loading on an arch part with a length *ds* is $q\,ds$. The lever arm of this loading is $r \sin\varphi$. With this we find for equilibrium



$$M(0) = R\,r - \phi r - \int_{\varphi=0}^{\frac{1}{2}\pi} q\,ds\,r \sin\varphi \;.$$

The support reaction *R* is equal to the loading *q* times the half the arch length.

$$R = q \tfrac{1}{2} \pi r$$

In the evaluation we use $ds = r\,d\varphi$.

$$\begin{aligned} M(0) &= q \tfrac{1}{2} \pi r^2 - \phi r - q r^2 \int_{\varphi=0}^{\frac{1}{2}\pi} \sin\varphi\, d\varphi \\ &= q \tfrac{1}{2} \pi r^2 - \phi r - q r^2 \left[-\cos\varphi \right]_0^{\frac{1}{2}\pi} \\ &= q \tfrac{1}{2} \pi r^2 - \phi r - q r^2 \left[-\cos \tfrac{1}{2}\pi + \cos 0 \right] \\ &= q \tfrac{1}{2} \pi r^2 - \phi r - q r^2 \left[-0 + 1 \right] \\ &= q \tfrac{1}{2} \pi r^2 - \phi r - q r^2 \end{aligned}$$

According to Problem 2 the moment in the top is equal to

$$\begin{aligned} M(0) &= q r^2 (\tfrac{1}{2}\pi - 0 \sin 0) - r(\phi + q r) \cos 0 \\ &= q r^2 \tfrac{1}{2}\pi - r(\phi + q r), \end{aligned}$$

which was to be shown.

b Complementary energy

$$E_{\text{compl}} = \int\limits_{\varphi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2} M\,\kappa\, ds + \tfrac{1}{2}\phi\,\varepsilon\, 2r$$

In this we substitute the constitutive equations and $ds = r\,d\varphi$

$$E_{\text{compl}} = \int\limits_{\varphi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2}\frac{M^2}{EI} r\,d\varphi + \tfrac{1}{2}\frac{\phi^2}{EA}\,2r\ .$$

c Bar force

$$E_{\text{compl}} = \frac{\pi r^3}{48EI}[12\phi^2 - 12r\,\phi\, q + r^2 q^2(7\pi^2 - 66)] + \frac{\phi^2 r}{EA}$$

$$\frac{\partial E_{\text{compl}}}{\partial\phi} = 0 \quad\rightarrow\quad \frac{\pi r^3}{48EI}[24\phi - 12r\,q] + \frac{2\phi r}{EA} = 0$$

$$\frac{\pi r^3 EA}{48EI}[24\phi - 12r\,q] + 2\phi r = 0$$

$$\frac{\beta r}{48}[24\phi - 12r\,q] + 2\phi r = 0$$

$$\tfrac{24}{48}\beta\phi - \tfrac{12}{48}\beta r\,q + 2\phi = 0$$

$$2\beta\phi + 8\phi = \beta r\,q$$

$$2\phi(\beta + 4) = \beta r\,q$$

$$\boxed{\phi = \tfrac{1}{2} q\, r\,\frac{\beta}{\beta + 4}}$$

Answers to problem 3

The differential equation for torsion including restrained warping is

$$EC_w\,\frac{d^4\varphi}{dx^4} - GI_t\,\frac{d^2\varphi}{dx^2} = m_x\ .$$

The bi-moment is calculated with

$$B = -EC_w\,\frac{d^2\varphi}{dx^2}\,.$$

Therefore,

$$-\frac{d^2B}{dx^2} + \frac{GI_t}{EC_w}B = m_x$$

provided that EC_w is not a function of x.

Suppose that Young's modulus is increased by a factor α in the latter equation. Then also the shear modulus G is increased by α and the equation – from which B can be solved – remains unchanged. Q.E.D.

Encore (not an exam question)
However, if the boundary conditions include an imposed rotation or an imposed warping that is not proportional to $\frac{1}{\alpha}$ then changes in B nonetheless occur. Examples are foundation settlements or temperature loading.