Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Wednesday 23 January 2008 at 9:00 – 12:00 hours

Problem 1 (1 points)

When we analyse structural details with linear elastic material behaviour we often find infinitely large stresses in some points.

a Show in each situation of Figure 1 in which points we will find infinitely large stresses (or infinitely small stresses).

b Clearly, in reality infinitely large stresses do not occur. How do we interpret these stresses in structural design?

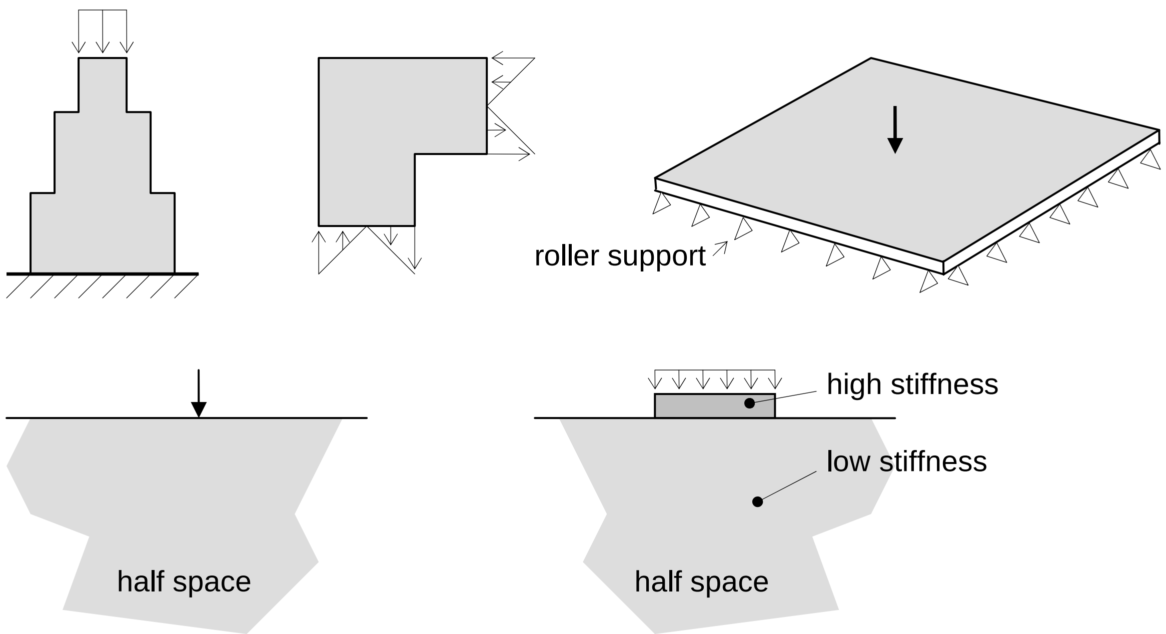


Figure 1. 5 Situations that can be analysed with a linear elastic model

Problem 2 (2 points).

Consider a point load *F* on the edge of a semi infinite plate with a thickness *t*. Often the stresses in the plate are approximated by a uniform distribution over an angle of 45 degrees (Fig. 2). Compare this approximate solution to the exact solution according to the theory of elasticity.

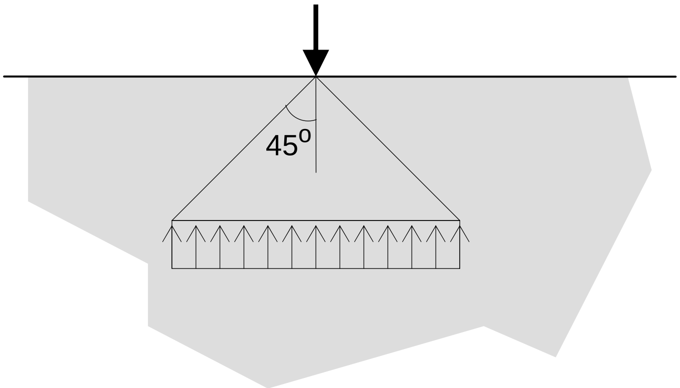


Figure 2. Point load on the edge of a large plate

Problem 3 (4 points)

A simply supported beam is loaded by a transverse force Q and an axial force F (Fig. 3). The beam has a flexural stiffness EI. The axial stiffness EA is very large and therefore not included in the analysis. We want to derive a formula for the deflection of this beam. We apply the principle of minimum potential energy with the deflection function

$$w = \bar{w} \sin \frac{\pi x}{l} .$$

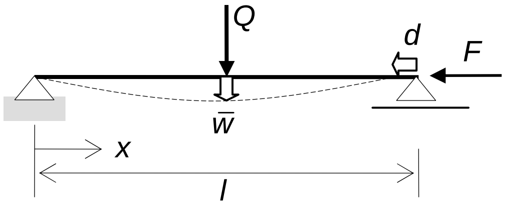


Figure 3. Simply supported beam

The right-hand support of the beam has a horizontal displacement d (Fig. 4). This can be approximated as

$$d = \frac{\pi^2 \bar{w}^2}{4 l} .$$

The beam kinematic equation is

$$\kappa = -\frac{d^2 w}{dx^2}$$

and its constitutive equation is

$$M = EI \kappa .$$

a What is the advantage of using potential energy compared to using a direct method?

b Write down the potential energy equation of this structure.

c Derive a formula for the deflection $\bar{w}$ of the beam. (Use $\int_0^l \sin^2 \frac{\pi x}{l} dx = \frac{1}{2} l$)

Subsequently, two changes are proposed. The beam has nonlinear material behaviour

$$M = EI\,\kappa(1 - h\,\kappa) \qquad \kappa < \frac{1}{2h}$$

and a small initial deformation

$$w_o = \bar{w}_o \sin\frac{\pi x}{l}.$$

Therefore, the total deflection becomes $w_o + w$.

d Write down the potential energy equation including the latter changes (You do not need to evaluate the equation).

Problem 4 (2 points)

A high building has a reinforced concrete stability core that is clamped in a thick foundation plate (Fig. 4). The dimensions a = 1 m and h = 0.2 m. The torsion constant I_w = 72 m^3 and the warping constant C_w = 60 m^6. The height of the building is 100 m. Wind causes an evenly distributed torsion loading of 2000 kNm/m. In Figure 5 the torsion response of the building is plotted.

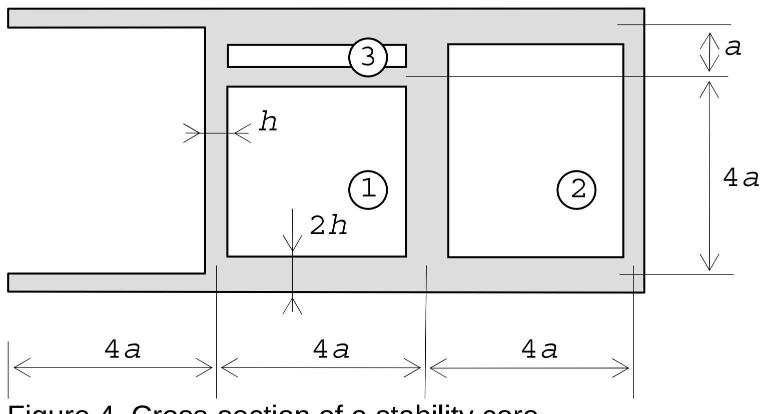
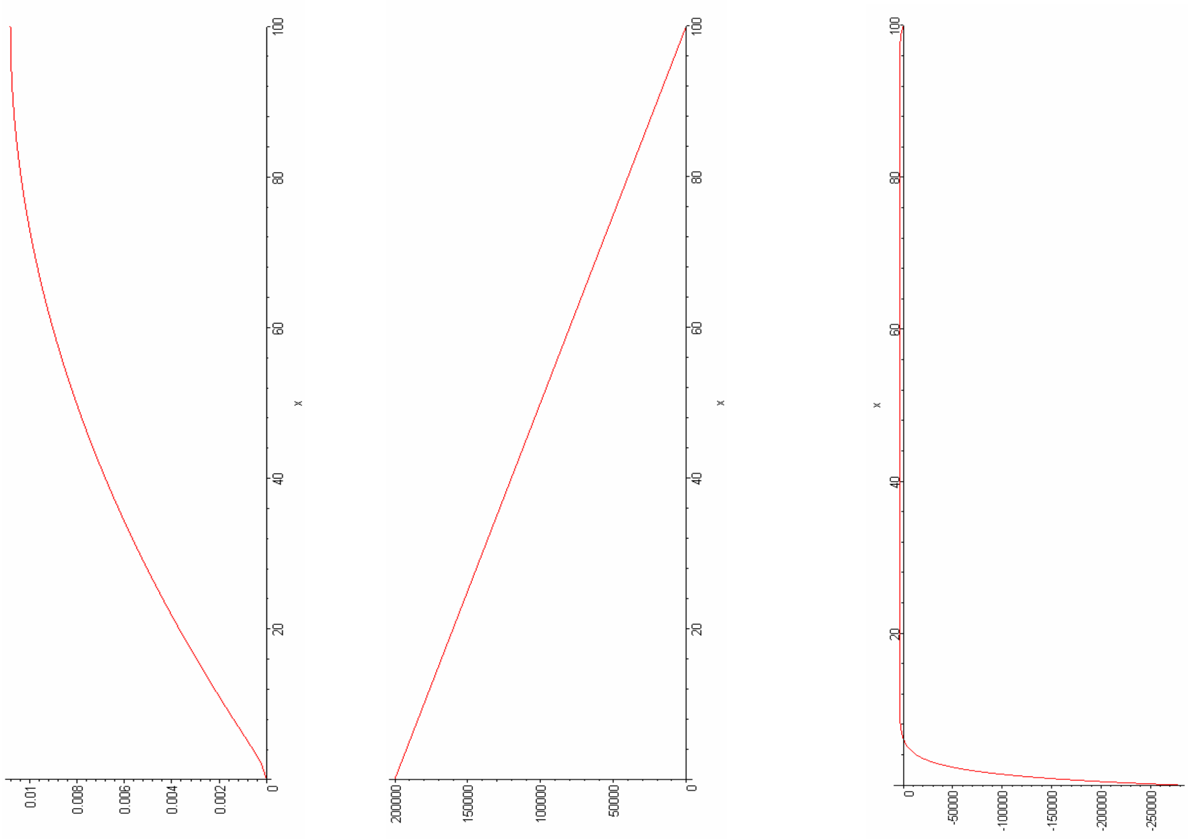


Figure 4. Cross-section of a stability core

a How can we calculate the torsion constant of the cross-section? How can we calculate the warping constant of the cross-section?

b Why does a bi moment occur in the cross-section?

c Is the warping stiffness important for this stability core?



Angular rotation φ

Torsion moment distribution M_w

Bi moment distribution B

Figure 5. Torsion response of a 100 m high building

Problem 5 (1 point)

An asphalt pavement is loaded by a car wheel (Figure 6). This situation is modelled as a point load on a half-space. Clearly the calculated stresses are realistic only at some distance of the point load. Which principle describes this property of the model?

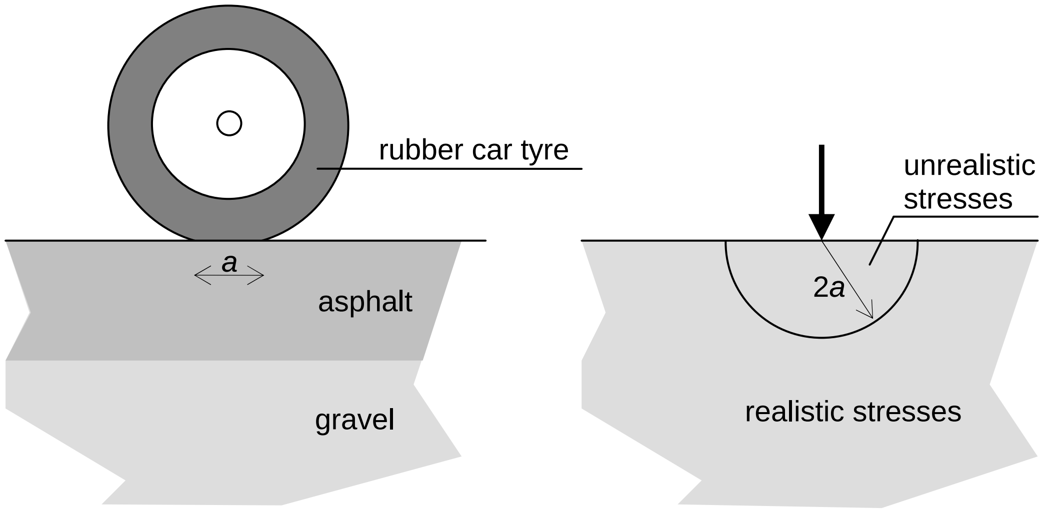


Figure 6. Car wheel on asphalt pavement   Linear elastic model

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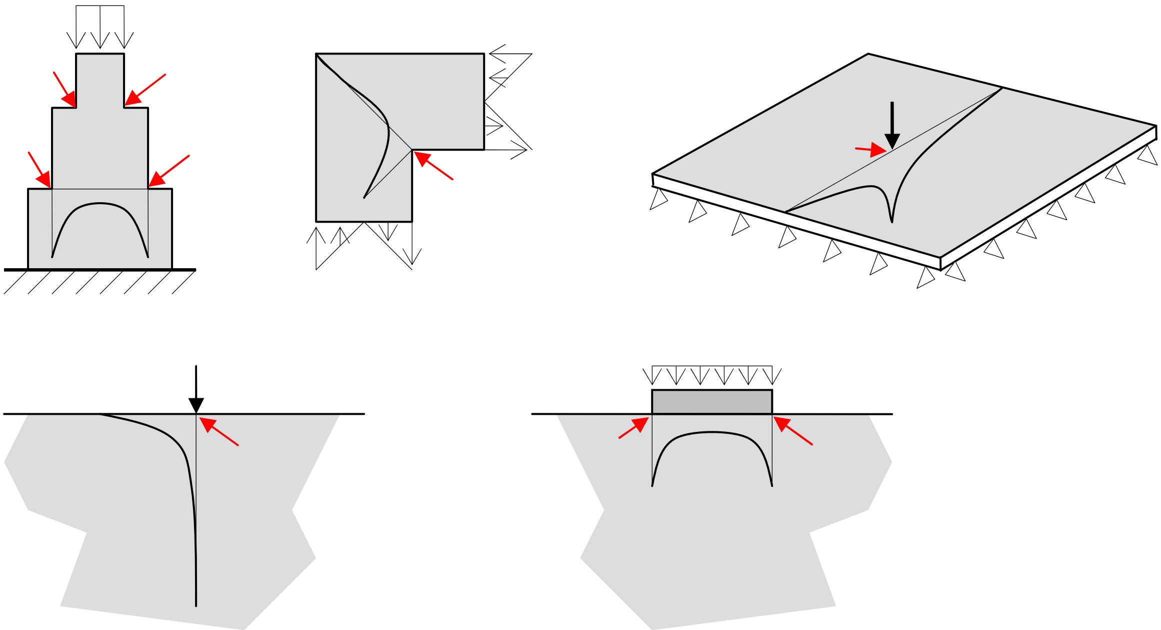
Answers to Problem 1

a Infinity



b Interpretation
In reality the sharp edges have some radius which reduces the peak stresses. Also a point load is distributed over a small area. In addition, the material will yield, crack or crush locally which redistributes the force flow such that the peak stresses are much reduced.

Answer to Problem 2

The stress in the approximate distribution is

$$\sigma_{yy} = \frac{-F}{2d\,t}$$

Where *d* is the depth at which the stress is calculated. In the linear-elastic solution the stress under the point load is ($\vartheta = 0$)

$$\sigma_{yy} = \sigma_{rr} = \frac{-2F}{\pi\, d\, t}$$

(lecture book p. 109) which is $\dfrac{\frac{2}{\pi} - \frac{1}{2}}{\frac{1}{2}} = 27\%$ larger than the approximated stress.

Encore (not an exam problem)
Using the figure we derive the following equations.

$$\sigma_{rr} a \cos\vartheta = \sigma_{yy} b$$

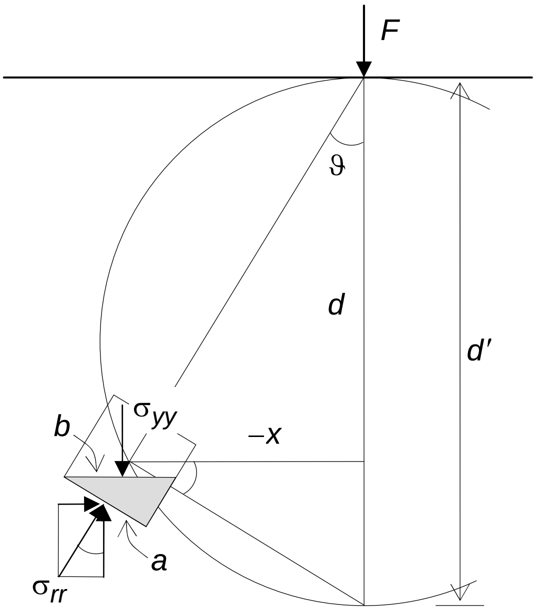
$$\cos\vartheta = \frac{a}{b}$$

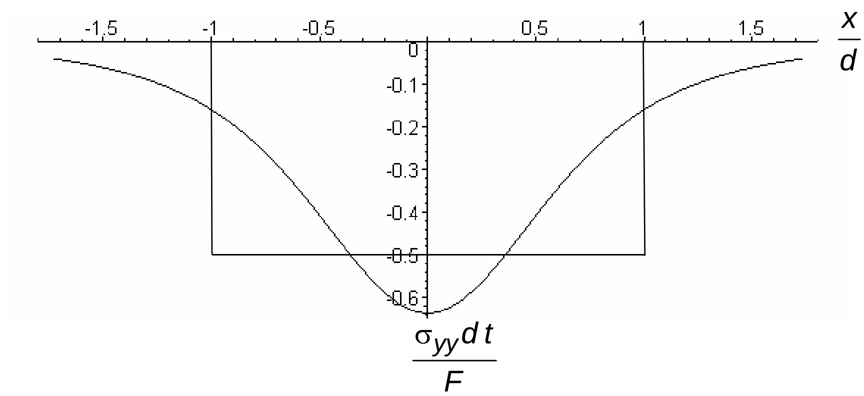
$$\tan\vartheta = \frac{-x}{d}$$

$$\tan\vartheta = \frac{d' - d}{x}$$

$$\sigma_{rr} = \frac{-2F}{\pi d' t}$$

These are applied to plot the following graph which shows the approximated and the linear elastic stress distribution.





Answers to Problem 3

a Advantage

With potential energy we can derive an approximate solution by starting with an estimated deformation function. This is useful when a formula is needed and the equations cannot be solved exactly.

b Potential energy

$$E_{\text{pot}} = \int_0^l \tfrac{1}{2} M \kappa \, dx - Q\bar{w} - F d$$

c Deflection

$$\kappa = -\frac{\pi^2}{l^2} \bar{w} \sin\frac{\pi x}{l}$$

$$E_{\text{pot}} = \tfrac{1}{2} EI \int_0^l \kappa^2 \, dx - Q\bar{w} - F \frac{\pi^2 \bar{w}^2}{4 l}$$

$$E_{\text{pot}} = \tfrac{1}{2} EI \frac{\pi^4}{l^4} \bar{w}^2 \int_0^l \sin^2 \frac{\pi x}{l} dx - Q\bar{w} - F \frac{\pi^2 \bar{w}^2}{4l}$$

$$E_{\text{pot}} = \tfrac{1}{2} EI \frac{\pi^4}{l^4} \bar{w}^2 \tfrac{1}{2} l - Q\bar{w} - F \frac{\pi^2 \bar{w}^2}{4l}$$

$$E_{\text{pot}} = \frac{\pi^4 EI \bar{w}^2}{4l^3} - Q\bar{w} - F \frac{\pi^2 \bar{w}^2}{4l}$$

$$\frac{dE_{pot}}{d\bar{w}} = 2\frac{\pi^4 EI \bar{w}}{4l^3} - Q - 2F \frac{\pi^2 \bar{w}}{4l} = 0$$

$$\left[2\frac{\pi^4 EI}{4l^3} - 2F \frac{\pi^2}{4l}\right] \bar{w} = Q$$

$$\bar{w} = \frac{Q}{\dfrac{\pi^4 EI}{2l^3} - \dfrac{\pi^2 l^2 F}{2l^3}}$$

$$\bar{w} = \frac{2l^3 Q}{\pi^2(\pi^2 EI - l^2 F)}$$

d Potential energy including changes

$$E_{\text{pot}} = \int_0^l \int_0^\kappa M \, d\xi - Q\bar{w} - F d$$

$$E_{\text{pot}} = \int_{x=0}^{l} \int_{\xi=0}^{\kappa} EI \, \xi(1 - h\xi) d\xi \, dx - Q\bar{w} - F\left[\frac{\pi^2(\bar{w} + \bar{w}_o)^2}{4l} - \frac{\pi^2 \bar{w}_o^{\,2}}{4l}\right]$$

Encore (not and exam problem)
Maple gives the solution of

$$\frac{dE_{\text{pot}}}{d\bar{w}} = 0,$$

which is

$$\bar{w} = \frac{l^2}{16\pi^3}\left[\frac{3\pi^2}{h} - \frac{3Fl^2}{EI\,h} - \sqrt{\frac{9\pi^4}{h^2} + \frac{9F^2 l^4}{EI^2 h^2} - \frac{18\pi^2 F l^2}{EI\,h^2} - \frac{96\pi^3 \bar{w}_o F}{EI\,h} - \frac{192\pi Q l}{EI\,h}}\right]$$

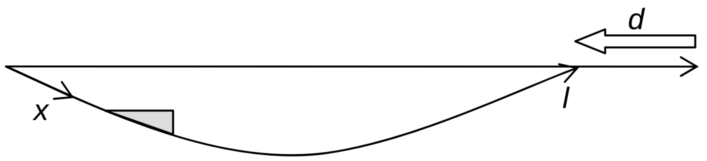
In the derivation is used that $h > 0$.

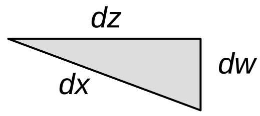
Encore (not an exam problem)
The provided formula for *d* is derived below.

$$dz^2 + dw^2 = dx^2$$

$$dz = \sqrt{dx^2 - dw^2} = \sqrt{1 - \left(\frac{dw}{dx}\right)^2}\, dx \approx 1 - \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 - \tfrac{1}{8}\left(\frac{dw}{dx}\right)^4 - \ldots$$

$$d = l - \int_{x=0}^{l} dz \approx l - \int_{x=0}^{l} 1 - \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 dx = l - \int_{x=0}^{l} 1 - \tfrac{1}{2}\left(\frac{d}{dx}\bar{w}\sin\frac{\pi x}{l}\right)^2 dx = \frac{\pi^2 \bar{w}^2}{4l}$$



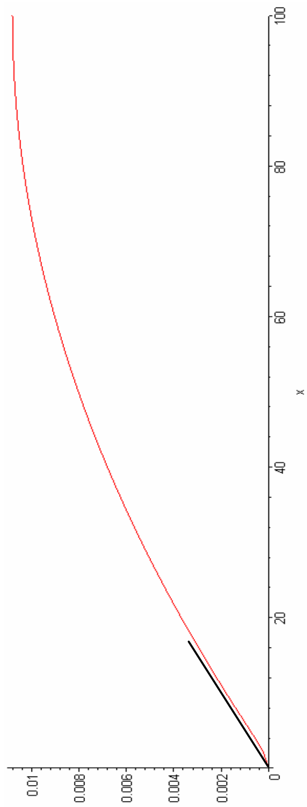


Answers to problem 4

a <u>Calculation</u>
The torsion constant can be calculated by hand using the membrane analogy. Alternatively, a computer program can be used. This program solves the differential equation of Laplace using the finite element method. The warping constant can be calculated by a computer program only.

b <u>Bi moment</u>
The bi moment occurs where warping is restrained. In the stability core warping cannot occur at the bottom where it is fixed in the thick foundation slab.

c <u>Important</u>
In this case restrained warping gives only a small reduction of the rotation at the top of the building. In the right-hand figure the black line shows the start of the deformation line if warping would not have been included in the analysis. Therefore warping could have been neglected.



Answer to problem 5

Principle of De Saint Venant