Theory of Elasticity

Exam Problems and Answers
Lecture CT5141 (Previously B16)

Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

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Preface

This lecture book contains the problems and answers of the exams elasticity theory from June 1997 until January 2003. It has been assembled with care. If nevertheless a mistake is found it would be appreciated if this is reported to the instructor.

Contents

Technische Universiteit Delft
Faculteit der Civiele Techniek
Mechanica & Constructies

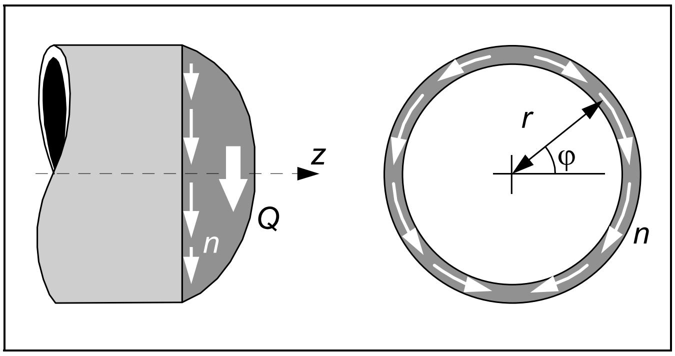
Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie b16
19 juni 1997 van 9.00 -12.00 uur

Problem 1 (3 points)

A hollow tube is loaded by a shear force. The wall thickness t is small compared to the radius r of the tube. In the cross-section a shear force n per unit of circumference occurs due to the resulting shear force Q. We approximate the force n with the following function.

$$n = \hat{n} \cos \varphi$$



The tube material can be considered as linear elastic with a shear modulus G.

a Calculate the resulting shear force Q due to the shear force n. (The positive direction of Q is $\varphi = -\frac{1}{2}\pi$.)

b Give the expression of the complementary energy of a slice of the tube. The slice has a length Δz.

c Show that the expression of the complementary energy can be reworked to the following result.

$$\frac{E_{compl}}{\Delta z} = \frac{\pi r \hat{n}^2}{2Gt} .$$

d Derive the formula for the shear stiffness GA_d.

e What is the quantity of the shape factor η in $A_d = \frac{A}{\eta}$.

Formulae

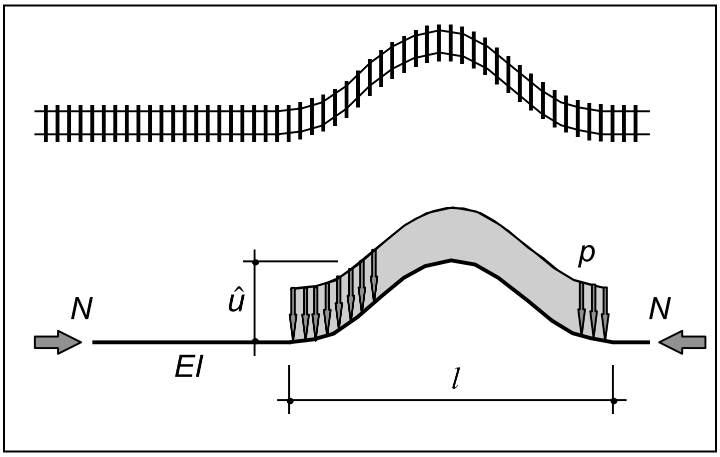
$$\int_0^{2\pi} \cos \varphi \, d\varphi = 0$$

$$\int_0^{2\pi} \cos^2 \varphi \, d\varphi = \pi$$

$$\int_0^{2\pi} \cos^3 \varphi \, d\varphi = 0$$

Problem 2 (2 points)

A straight track is loaded in compression because the rails have expanded on a hot day. The track might buckle as shown in the figure. To analyse this situation an assumption is made on the displacement field and the potential energy of the buckled track is determined. The result is



$$E_{pot} = 4\pi \frac{\hat{u} EI}{l^3} + \tfrac{1}{2} \hat{u}\, l\, p - \frac{\pi^2}{4} \frac{\hat{u}^2 N}{l}$$

N	normal force in the track
l	buckling length
p	friction force between the track and the ballast bed
EI	bending stiffness of the track
$\hat{u}$	largest deflection of the track

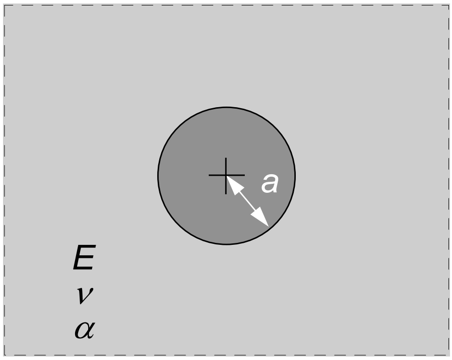
a In relation to what parameter of parameters should the potential energy of the track be minimal? Explain your answer. (You do not need to calculate something.)

A statically indetermined system can be analysed by the force method or the displacement method. In the force method we choose a number of redundants as fundamental unknowns. In the displacement method the displacements are the fundamental unknowns. The force method is convenient if the problem is statically indetermined to a small degree. However, modern computer programs use the displacement method without exception.

b Why is the displacement method used in computer programs for structural analysis?

Problem 3 (5 points)

A square plate of room temperature has a round hole in it. The hole is filled with a plug of the same material and the same thickness as the square plate. The plug exactly fits the hole if it is cooled down *T* degrees from room temperature. After a while the plug assumes the room temperature again and is stuck in the hole.



Data

Radius *a* of the plug is small compared to the dimensions of the plate.
The thickness of both the plate as the plug is *t*.
The elasticity modulus of the material is *E*.
The Poison's coefficient of the material is ν.
The linear expansion coefficient of the material is α.

Questions

a Derive an expression for the *plate* which relates the stress on the edge of the hole to the displacement of this edge. (So, the plug is replaced by a stress on to the edge of the hole.)

b Derive an expression for the *plug* which relates the stress on the edge to the displacement of the edge. (So, the plate is replaced by a stress on to the edge of the plug.)

c Formulate the transition conditions between the plug and the plate.

d Calculate the stress between the plate and the plug.

e Calculate the stress distribution in the plug and the plate.

Suggestions

The general solution of the radial displacement in an axial symmetrical thin plate is

$$u(r) = A\frac{a}{r} + B\frac{r}{a}$$

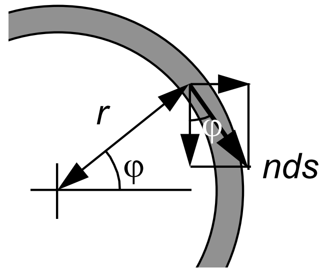
The constitutive relation of this plate is

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{Bmatrix}.$$

Tentamen b16, 19 juni 1997
Answers to Problem 1

a Shear Force

The resulting shear force Q is the integral of the vertical component of the shear force $n\,ds$ over the circumference s of the tube.



$$Q = \int_{s=0}^{2\pi r} n\,ds\,\cos\varphi$$

Evaluation gives

$$Q = \int_{\varphi=0}^{2\pi} \hat{n}\cos\varphi\, r\,d\varphi \cos\varphi$$

$$Q = \hat{n}\,r \int_{\varphi=0}^{2\pi} \cos^2\varphi\;\, d\varphi$$

$$\boxed{Q = \hat{n}\,r\,\pi}$$

b Complementary Energy

The complementary energy is the shear force $n\,ds$ times the displacement $\gamma\,\Delta z$ over 2, integrated over the circumference s of the tube.

$$\boxed{E_{compl} = \int_{s=0}^{2\pi r} \tfrac{1}{2}\, n\,ds\,\gamma\,\Delta z}$$

c Evaluation of the complementary energy gives

$$E_{compl} = \int_{\varphi=0}^{2\pi} \tfrac{1}{2}\, n\,r\,d\varphi\,\frac{n}{Gt}\Delta z = \frac{r\,\Delta z}{2Gt}\int_0^{2\pi} n^2\,d\varphi = \frac{r\,\Delta z}{2Gt}\hat{n}^2\int_0^{2\pi}\cos^2\varphi\,d\varphi\,.$$

$$\boxed{E_{compl} = \frac{r\,\Delta z}{2Gt}\hat{n}^2\pi}$$

d Shear Stiffness

The complementary energy due to the shear force Q is equal to the complementary energy due to the shear flow n. From this we derive the shear stiffness GA_d. (See lecture book Energy Principles, page 14.)

$$E_{compl,Q} = E_{compl,n}$$

$$\frac{1}{2}\frac{Q^2}{GA_d}\Delta z = \frac{r\Delta z}{2Gt}\hat{n}^2\pi$$

Or,

$$GA_d = \frac{G\,t}{r\,\hat{n}^2\pi} Q^2$$

$$\boxed{GA_d = G\,t\,r\,\pi}$$

e Shape Factor

The section area of a thin tube is

$$A = 2\pi r\,t$$

so that the shape factor η becomes

$$\eta = \frac{GA}{GA_d} = \frac{G2\pi r\,t}{G\,t\,r\,\pi}.$$

$$\boxed{\eta = 2}$$

Answers to Problem 2

a Potential energy should be minimised as to the parameters that describe the displacement field. The buckling shape of the track is described by $\hat{u}$ and l.

b In the first computers little memory was available, therefore the system of equations that had to be solved needed to be as small as possible. The force method often yields few unknown and equations so that this method was used in old computer programs. However, it proved complicated to automatically select the redundants. Many studies have been devoted to this subject but soon computers with more memory were developed so that the displacement method could be used. The displacement method often needs more memory but is easier to program than the force method.

Answers to Problem 3

Because the radius *a* is small compared to the plate dimensions the problem can be treated as an axial symmetric plate of which the outer edge is infinitely far from the hole.

a Analysis of the Plate

For the displacement method holds that

$$u(r) = A\frac{a}{r} + B\frac{r}{a}$$

so that

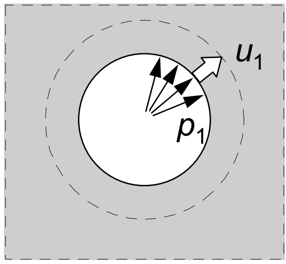
$$\begin{aligned} \varepsilon_{rr} &= \frac{du}{dr} = -A\frac{a}{r^2} + B\frac{1}{a} \\ \varepsilon_{\theta\theta} &= \frac{u}{r} = A\frac{a}{r^2} + B\frac{1}{a} \end{aligned}.$$

For very large r the stresses and strains are zero, therefore

$$B = 0.$$

When the plug is at room temperature it is compressed and exerts a force p_1 per unit of edge length into the direction of the radius r.

$$p_1 = -\sigma_{rr} t$$



where t is the plate thickness.

Because

$$\sigma_{rr} = \frac{E}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta})$$

we find for $r = a$

$$p_1 = \frac{Et}{1-\nu^2}\frac{A(1-\nu)}{a} = \frac{Et}{1+\nu}\frac{A}{a}.$$

We define $u(a) = u_1$. Therefore $u_1 = A$. So the relation between p_1 and u_1 is

$$p_1 = \frac{1}{1+\nu}\frac{Et}{a}u_1$$

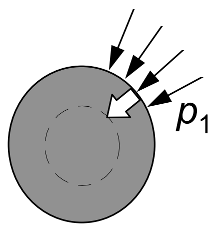
or

$$\boxed{u_1 = (1+\nu)\frac{a}{Et}p_1}. \qquad (1)$$

b Analysis of the Plug

The plug is compressed in all directions by a distributed force p_2 that is directed inwards. This gives a homogeneous stress distribution.

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{p_2}{t}.$$

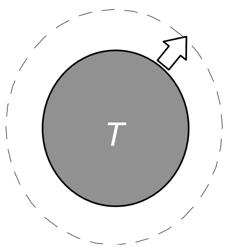


So

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = -\frac{1-\nu}{Et}p_2.$$

The radial displacement of edge of the plug is (directed inwards)

$$a\left|\varepsilon_{rr}\right| = (1-\nu)\frac{a}{Et}p_2$$



The plug becomes T degrees warmer. If expanding feely this would give an edge displacement of

$$\alpha Ta$$

so that the total outward directed displacement becomes

$$\boxed{u_2 = \alpha T a - (1-\nu)\frac{a}{Et} p_2}. \tag{2}$$

c Transition Conditions

The displacements of the edge of the plug and the edge of the hole need be equal.

$$\boxed{u_1 = u_2 = u} \tag{3}$$

In addition there needs to be equilibrium

$$\boxed{p_1 = p_2 = p}. \tag{4}$$

d Calculation of the Temperature Problem (Force Method)

When we substitute (1) and (2) into (3) we find.

$$a\alpha T - (1-\nu)\frac{a}{Et} p_2 = (1+\nu)\frac{a}{Et} p_1.$$

From this and (4) it follows that the force per unit of edge length is

$$\boxed{p = \tfrac{1}{2} Et\alpha T}.$$

Now u can be calculated with (1) or (2).

$$u = \tfrac{1}{2}(1+\nu) a\alpha T$$

e Stress Distribution

We found already for the strains in the plate

$$\begin{aligned} \varepsilon_{rr} &= -A\frac{a}{r^2} \\ \varepsilon_{\theta\theta} &= A\frac{a}{r^2} \end{aligned}.$$

Substitution in the constitutive relation gives the stresses

$$\begin{aligned} \sigma_{rr} &= \frac{-E}{1+\nu} A\frac{a}{r^2} \\ \sigma_{\theta\theta} &= \frac{E}{1+\nu} A\frac{a}{r^2} \end{aligned}.$$

The constant A is

$$A = u_1 = \tfrac{1}{2}(1+\nu) a\alpha T$$

so that

$$\boxed{\begin{aligned}\sigma_{rr}(r) &= -\tfrac{1}{2}E\alpha T\,\frac{a^2}{r^2}\\ \sigma_{\theta\theta}(r) &= \tfrac{1}{2}E\alpha T\,\frac{a^2}{r^2}\end{aligned}}.$$

On the edge of the hole the stress is

$$\sigma_{rr} = -\tfrac{1}{2}E\alpha T$$

so that the homogeneous and isotropic stress in the plug is

$$\boxed{-\tfrac{1}{2}E\alpha T}.$$

<u>Remarks</u>

The solution is independent of the thickness t and the Poison's ratio ν.

The stress in the plug is approximately halve the value that would occur in a completely restrained plug.

<u>Alternative Answer to Problem **3d**</u> (displacement method)
When we substitute (1) and (2) in (4) we find

$$\frac{u_1}{(1+\nu)\dfrac{a}{EI}} = \frac{u_2 - \alpha T a}{-(1-\nu)\dfrac{a}{EI}}.$$

From this and (3) it follows that the displacement of the edge is

$$u = \tfrac{1}{2}(1+\nu)a\alpha T\ .$$

Now p can be calculated with (1) or (2).

$$\boxed{p = \tfrac{1}{2}Et\alpha T}$$

Technische Universiteit Delft
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Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie b16
12 januari 1998 van 14.00 -17.00 uur

Problem 1 (3 points)

A box-girder beam is loaded at torsion. The thickness of all walls is *h* as shown in the figure.

We calculate the box-girder with the membrane analogy. The weightless plates in the corners of the box-girder will have the same displacement because of rotational symmetry

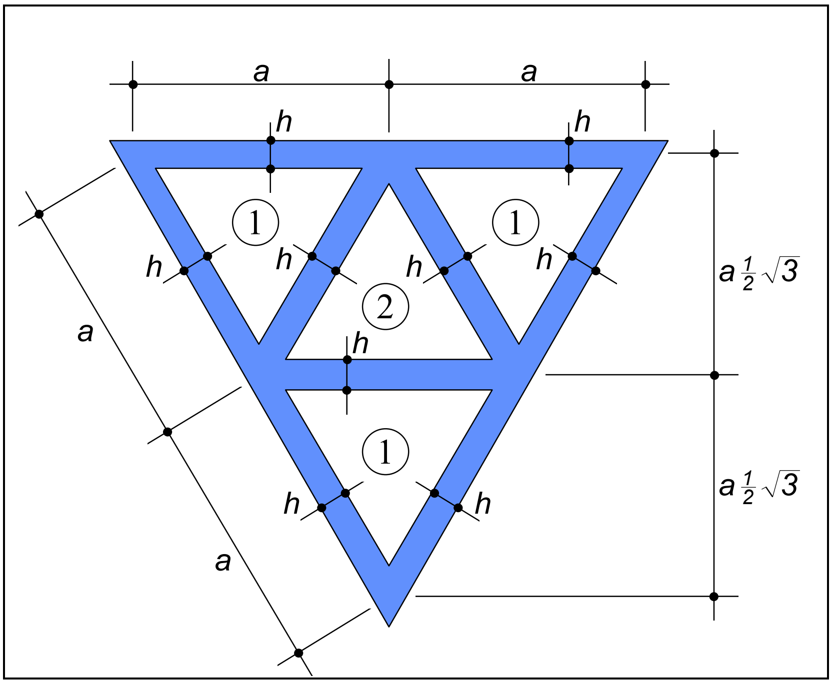
You can assume that the wall thickness *h* is small compared to the width of the box-girder.

a Calculate the displacements w_1 and w_2 of the weightless plates.

b Calculate the torsion stiffness GI_w of the cross-section.

c Calculate the shear stresses in the cross-section and draw them in the correct direction.

d Suppose that warping (Dutch: welving) of the box-girder is locally restrained by a clamped boundary condition. Will this cause the torsion stiffness to be larger, smaller or will it remain unchanged?



Problem 2 (2 points)

A structural engineer calculates the stresses in a reinforced concrete floor using the finite element method. He uses linear elastic elements. Underneath a concentrated load very large moments appear. What is your advise for this structural engineer? Choose from the following options and explain your answer.

A Calculate the moments again with a finer element mesh around the concentrated load.
B Replace the concentrated load by a distributed load over a small area.
C Use the moments at some distance of the concentrated load to calculate the reinforcement.
D Use the resultant of the moment over some width around the concentrated load to calculate the reinforcement.

Problem 3 (2 points)

An oil company drills a hole in deep rock layer. Due to the geological origination a pressure p is present in all horizontal directions of the material. The hole changes this stress distribution.

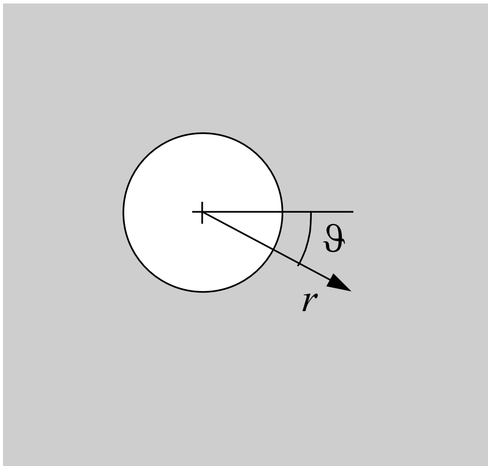
We consider the rock to be of a linear elastic material. The situation is axial symmetrical with a coordinate r in the radial direction and the angle ϑ in the horizontal plane.

The general solution of the stress distribution is

$$\sigma_{rr} = 2C_2 + C_3 \frac{1}{r^2} + C_4(1 + 2\ln r)$$

$$\sigma_{\vartheta\vartheta} = 2C_2 - C_3 \frac{1}{r^2} + C_4(3 + 2\ln r)$$

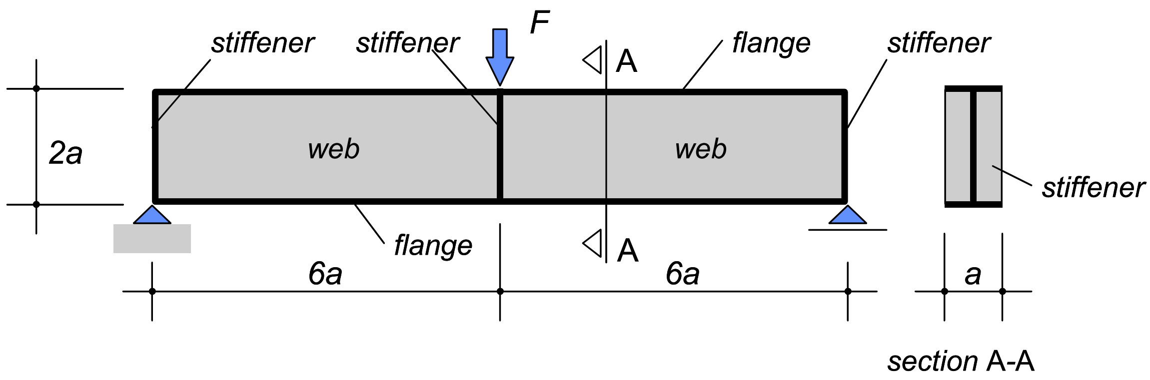
$$\sigma_{r\vartheta} = 0$$



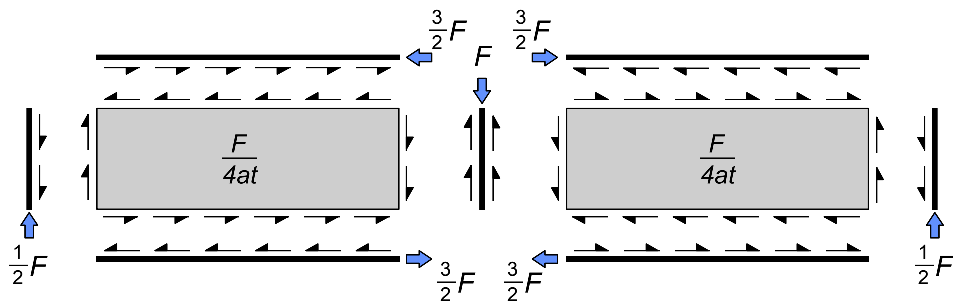
Calculate the tangential stress $\sigma_{\vartheta\vartheta}$ in the edge of the hole and draw the stress distribution $\sigma_{\vartheta\vartheta}$ and σ_{rr}.

Problem 4 (3 points)

A steel beam is loaded by a force F (see figure). The beam has flanges of thickness t and width a at the upper and lower edge. The stiffeners at the middle and at the ends of the beam have dimensions a x $2a$ x t. The width of the beam is a, the height is $2a$ and the span is $12a$.



We want to calculate the deflection of the beam using complementary energy. The approximated stress distribution in the beam is drawn below. The panels have a homogeneous shear stress. The forces in flanges and stiffeners vary linear over the length



a Show that the complimentary energy of a flange is

$$E_c = \frac{1}{6}\frac{\hat{N}^2 l}{E\,a\,t}$$

where $\hat{N}$ is the force in the end of a flange or a stiffener and l is the length.

b Calculate the total complementary energy of the beam. Choose a prescribed displacement u where the force F is attached. Neglect the Poison's effect so that the shear modulus is $G = E/2$.

c Express the deflection u in the force F.

Tentamen b16, 12 januari 1998
Answers to Problem 1

The box-girder of this problem is known in The Netherlands as nabla beam applied in the “Deltawerken” in the dam of the “Haringvliet” estuary.

a Weightless plates

We choose w_2 of the middle cell larger than w_1 of the corner cell.

Equilibrium of the weight less plates of the corner cell gives

$$p\,\tfrac{1}{2}a\,a\,\tfrac{1}{2}\sqrt{3} = a\,s\frac{w_1}{h} + a\,s\frac{w_1}{h} - a\,s\frac{w_2 - w_1}{h}$$

Equilibrium of the weight less plates of the middle cell gives

$$p\,\tfrac{1}{2}a\,a\,\tfrac{1}{2}\sqrt{3} = a\,s\frac{w_2 - w_1}{h} + a\,s\frac{w_2 - w_1}{h} + a\,s\frac{w_2 - w_1}{h}$$

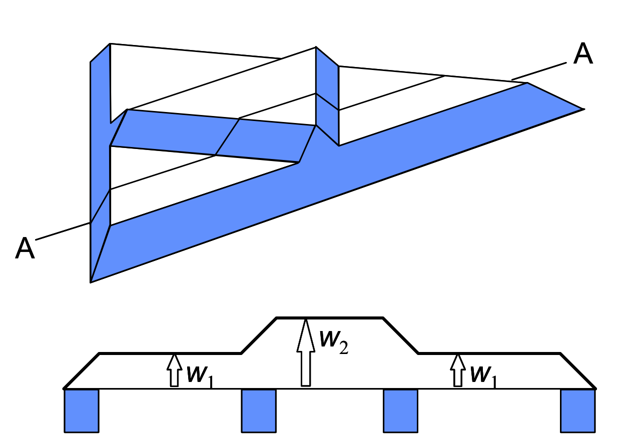
This we can simplify to

$$p\,\tfrac{1}{4}a\sqrt{3} = \frac{s}{h}(3\,w_1 - w_2)$$

$$p\,\tfrac{1}{4}a\sqrt{3} = \frac{s}{h}\,3\,(w_2 - w_1)$$

from which w_1 and w_2 can be solved.

$$\boxed{\begin{aligned} w_1 &= \tfrac{1}{6}\sqrt{3}\,\frac{p}{s}ah \\ w_2 &= \tfrac{1}{4}\sqrt{3}\,\frac{p}{s}ah \end{aligned}}$$



Section A-A

b Torsion Stiffness

From the membrane we go to the ϕ-bubble with the following substitutions.

$$w = \phi$$

$$p = 2\vartheta$$

$$s = \frac{1}{G}$$

So

$$\phi_1 = \tfrac{1}{3}\sqrt{3}\;\vartheta G\,ah$$

$$\phi_2 = \tfrac{1}{2}\sqrt{3}\;\vartheta G\,ah$$

The torsion moment equals two times the volume of the ϕ-bubble.

$$M_w = 2\left(\tfrac{1}{2}a\,a\tfrac{1}{2}\sqrt{3}\,\phi_1 + \tfrac{1}{2}a\,a\tfrac{1}{2}\sqrt{3}\,\phi_1 + \tfrac{1}{2}a\,a\tfrac{1}{2}\sqrt{3}\,\phi_1 + \tfrac{1}{2}a\,a\tfrac{1}{2}\sqrt{3}\,\phi_2\right)$$
$$= a^2\,\tfrac{1}{2}\sqrt{3}\,\left(3\,\phi_1 + \phi_2\right)$$

Substitution of the previous relations in the latter gives

$$\begin{aligned} M_w &= a^2\,\tfrac{1}{2}\sqrt{3}\left(3\,\tfrac{1}{3}\sqrt{3}\,\vartheta G\,ah + \tfrac{1}{2}\sqrt{3}\,\vartheta G\,ah\right) \\ &= a^2\,\tfrac{1}{2}3(1+\tfrac{1}{2})\,\vartheta G\,ah \\ &= G\,\tfrac{9}{4}\,a^3 h\,\,\vartheta \end{aligned}$$

For a wire frame model of the beam the torsion moment is

$$M_w = GI_w\,\vartheta\,.$$

Therefore, the torsion stiffness is

$$\boxed{GI_w = G\,\tfrac{9}{4}\,a^3 h}$$

c Shear stress

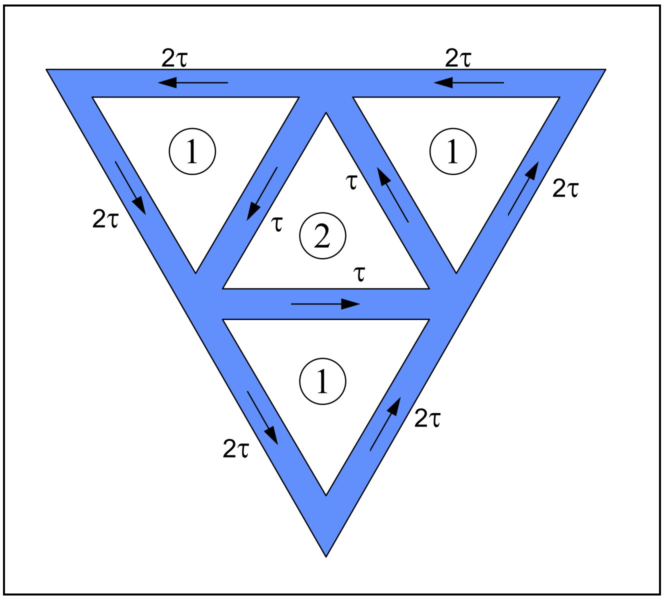
The shear stress is the slope of the ϕ-bubble. We first rewrite the relation of the torsion moment

$$\vartheta G\,ah = \frac{4}{9}\frac{M_w}{a^2}$$

and express ϕ_1 and ϕ_2 in the torsion moment

$$\phi_1 = \tfrac{1}{3}\sqrt{3}\,\tfrac{4}{9}\frac{M_w}{a^2}$$

$$\phi_2 = \tfrac{1}{2}\sqrt{3}\,\tfrac{4}{9}\frac{M_w}{a^2}$$



In the outside walls of the box-girder is the shear stress

$$\frac{\phi_1}{h} = \frac{4}{27}\sqrt{3}\,\frac{M_w}{a^2 h} = 2\,\tau$$

In the interior walls is the shear stress

$$\frac{\phi_2 - \phi_1}{h} = \frac{\tfrac{2}{9}\sqrt{3}\,\dfrac{M_w}{a^2} - \tfrac{4}{27}\sqrt{3}\,\dfrac{M_w}{a^2}}{h} = \frac{2}{27}\sqrt{3}\,\frac{M_w}{a^2 h} = \tau$$

d Warping (Dutch: welving)

When warping is locally restrained the box-girder will locally be stiffer than calculated in this problem (see lecture book Direct Methods, page 197).

Answer to Problem 2

Underneath a concentrated load the bending moment goes to infinity (see lecture book Direct Methods, Figure 4.20). Probably the element mesh that was selected by the structural engineer was very fine because otherwise the local peak would not have shown up. It is not useful to select an even finer mesh (answer A) because this will result in even larger moments. The explanation for the very large moments is that around the concentrated load the plate theory is not accurate over a distance of approximately the plate thickness.
Redistributing the concentrated load over an area (answer B) will indeed reduce the moments, however this takes much effort. The resulting moment over a distance of two times the plate thickness (answer D) is very suitable to dimension the reinforcement because this does not compromise equilibrium. This can be seen as spreading the peak moment. However, this also takes much effort to calculate. The moment is accurate at a distance of approximately the plate thickness (answer C).

So, only answer A is really wrong. Answer B and D are impractical but possible. Answer C is the best.

Answer to Problem 3

We can solve the constants using the boundary conditions. The boundary conditions for this case are.
- On the edge of the hole $r = a$ are no normal stresses σ_{rr}.
- Far from the hole the stresses σ_{rr} and $\sigma_{\vartheta\vartheta}$ equal $-p$.

When r becomes very large ln(r) does not approach a specific value. Instead it continues to grow. The stress should become equal to $-p$ for large r. This can only be if C_4 equals zero.

$$C_4 = 0$$

When r becomes very large than $\frac{1}{r^2}$ becomes very small. So

$$\sigma_{rr}\big|_{r \to \infty} = 2\,C_2 = -p \qquad\qquad C_2 = -\frac{p}{2}$$

The same result would have been found if we had considered $\sigma_{\vartheta\vartheta}$.

If $r = a$ than

$$\sigma_{rr} = 2\,C_2 + C_3\,\frac{1}{a^2} = -p + C_3\,\frac{1}{a^2} = 0 \qquad C_3 = p\,a^2$$

So

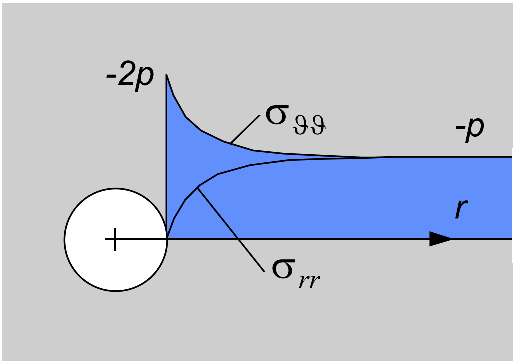
$$\sigma_{rr} = 2C_2 + C_3 \frac{1}{r^2} = -p + p\,a^2 \frac{1}{r^2} = -p\,(1 - \frac{a^2}{r^2})$$

$$\sigma_{\vartheta\vartheta} = 2C_2 - C_3 \frac{1}{r^2} = -p - p\,a^2 \frac{1}{r^2} = -p\,(1 + \frac{a^2}{r^2})$$

On the edge of the hole

$$\sigma_{\vartheta\vartheta} = -p\,(1 + \frac{a^2}{a^2}) \qquad \boxed{\sigma_{\vartheta\vartheta} = -2\,p}$$

The stress distribution becomes



Answers to Problem 4

a Complementary Energy

The force in a bar is linear from zero to some value $\hat{N}$

$$N = \frac{x}{l}\hat{N}$$

The complimentary energy in a bar is

$$E_c = \frac{1}{2}\int_0^l \frac{N^2}{EA} dx = \frac{1}{2}\int_0^l \frac{\left(\frac{x}{l}\hat{N}\right)^2}{E\,at} dx = \frac{\hat{N}^2}{2l^2 E\,at}\int_0^l x^2 dx = \frac{\hat{N}^2}{2l^2 E\,at}\frac{1}{3}x^3\Big|_0^l = \frac{1}{6}\frac{\hat{N}^2 l}{E\,at}$$

The complementary energy in a panel is

$$E_c = V\,\tfrac{1}{2}\tau\gamma = V\,\tfrac{1}{2}\frac{\tau^2}{G} = 6a\;2a\,t\;\;\frac{1}{2}\;\frac{(\frac{F}{4at})^2}{G} = \frac{3}{8}\frac{F^2}{Gt}$$

b The total complementary energy is the energy in the bars plus the energy in the panels minus the energy of position.

$$E_c = 4\,\frac{1}{6}\frac{(\frac{3}{2}F)^2\,6a}{E\,a\,t} + 2\,\frac{1}{6}\frac{(\frac{1}{2}F)^2\,2a}{E\,a\,t} + \frac{1}{6}\frac{F^2\,2a}{E\,a\,t} + 2\,\frac{3}{8}\frac{F^2}{Gt} - Fu$$

$$E_c = 9\frac{F^2}{E\,t} + \frac{1}{6}\frac{F^2}{E\,t} + \frac{1}{3}\frac{F^2}{E\,t} + \frac{3}{4}\frac{F^2}{Gt} - Fu$$

$$E_c = \frac{19}{2}\frac{F^2}{E\,t} + \frac{3}{4}\frac{F^2}{Gt} - Fu$$

Using $G = \frac{1}{2}E$ this becomes

$$E_c = 11\frac{F^2}{E\,t} - Fu$$

c In our minds u is a prescribed displacement and F is the resulting support reaction. The complete stress distribution in the beam is expressed in F. The complementary energy must be minimal as to the parameter F

$$0 = \frac{dE_c}{dF} = 22\frac{F}{E\,t} - u$$

Therefore, the relation between force and displacement is

$$\boxed{u = 22\,\frac{F}{E\,t}}$$

Technische Universiteit Delft
Faculteit der Civiele Techniek
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

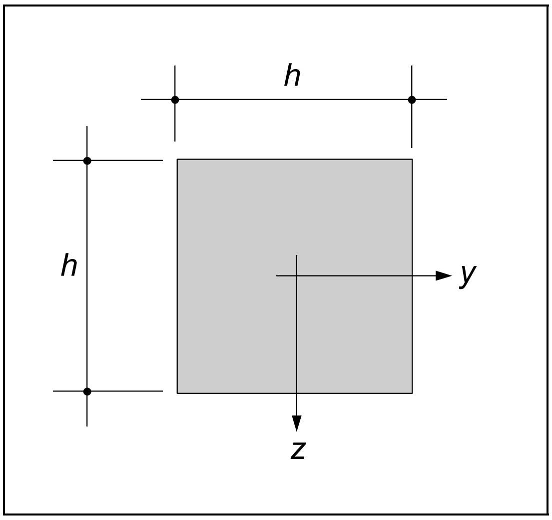
Tentamen Elasticiteitstheorie b16
18 juni 1998 van 9.00 -12.00 uur

Problem 1 (4 points)

A prismatic beam is loaded by torsion. The cross-section of the beam is square with dimension h (see figure). We want to calculate the torsion stiffness of the beam. Therefore, we consider a slice of length Δx.

The torsion moment causes shear stresses in the cross-section, which can be derived from a function ϕ. We choose the following function as an approximation.

$$\phi = A\,(1-4\frac{y^2}{h^2})(1-4\frac{z^2}{h^2})$$



where A is a yet unknown constant. The shear stresses in the cross-section are calculated by.

$$\sigma_{xy} = \frac{\partial \phi}{\partial z}$$

$$\sigma_{xz} = -\frac{\partial \phi}{\partial y}$$

a Show that ϕ fulfils the boundary conditions.

b Give a formula with which the resulting torsion moment can be calculated (You do not need to evaluate the formula).

The formula is evaluated for you with the following result

$$M_w = \tfrac{8}{9} A h^2$$

c Calculate the largest stress in the cross-section and express this in the torsion moment.

d Give the formula for calculating the complimentary energy of the slice (You do not need to evaluate the formula).

The formula has been evaluated for you with the following result

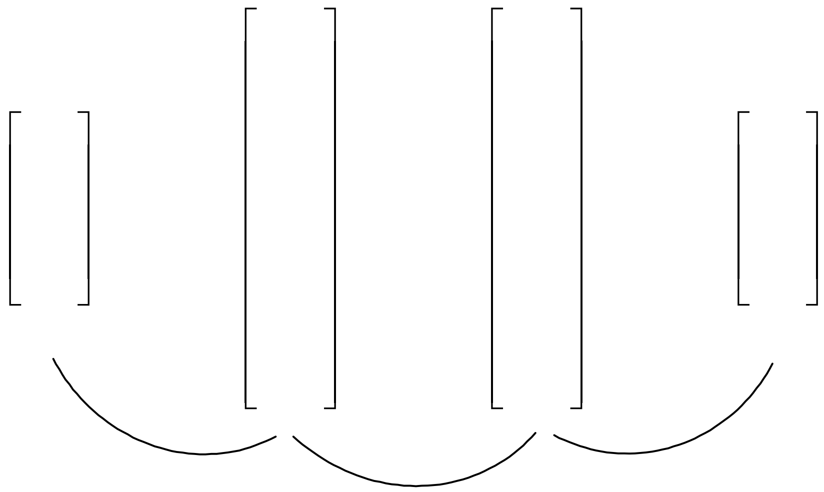
$$E_c = \frac{128}{45}\frac{A^2}{G}\Delta x$$

where G is the shear modulus of the material.

e Calculate the torsion stiffness GI_w of the beam.
Suggestion: Make the complementary energy of the slice equal to the complimentary energy of a part of the wire frame model.

Problem 2 (2 points)

Elasticity theory in three dimensions has a number of variables for describing displacements, strains and so forth. Write these quantities in the framework below and and write the names of the relations. (You do not need to give formula).



Problem 3 (4 points)

A storage tank for liquid needs to be jacked up for maintenance (see the figure at the next page). The steel bottom plate is elevated from the foundation at which it is normally resting. We want to check the stresses in the bottom plate.

The bottom plate is axial symmetrical and the edges can be assumed clamped. The liquid is removed from the tank so that only the self-weight *p* of the plate is relevant. The radius of the tank is *a*, the plate thickness is *h,* the elasticity modulus is *E* and the Poison’s ratio is ν.

The differential equation of the deflection *w* of the plate is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right)=\frac{p}{K}$$

where *K* is the plate stiffness. The general solution of this differential equation is

$$w = C_1 + C_2\, r^2 + C_3 \ln r + C_4\, r^2 \ln r + \frac{p\, r^4}{64\, K}$$

a Give the boundary conditions of the bottom plate (You do not need to solve the constants.).

The boundary conditions have been processed for you and we find the deflection

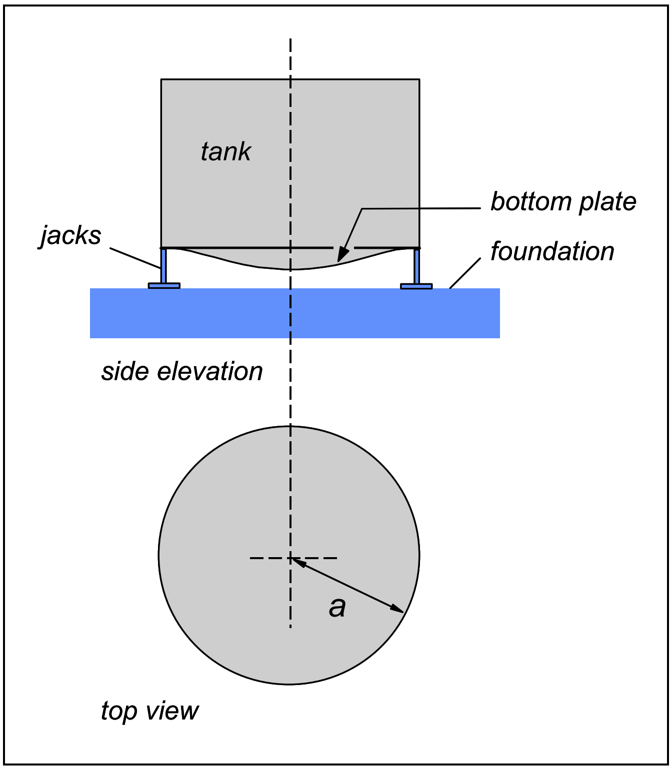
$$w = \frac{p}{64\,K}(a^2 - r^2)^2$$

b Calculate the extreme moment in the plate.

c Calculate the steel stress based on the extreme moment in question **b**. Use the following quantities:

a = 15 m
p = 900 N/m^2
h = 0.012 m
E = 210 10^9 N/m^2
ν = 0.1

Calculate also the largest deflection of the plate.



Probably you will notice that the calculated stress in question **c** is much larger than can be carried by normal structural steel. An obvious conclusion is that jacking up cannot take place in this way. However, recently this project has been executed in the described way to the complete satisfaction of the owner (Dutch: opdrachtgever).

d Explain why the bottom plate does not fail in the jack up process.

Tentamen b16, 18 juni 1998
Answers to Problem 1

a Boundary Conditions

The stresses are found by differentiating ϕ.

$$\sigma_{xy} = A\,(1-4\frac{y^2}{h^2})(-8\frac{z}{h^2})$$

$$\sigma_{xz} = -A\,(-8\frac{y}{h^2})(1-4\frac{z^2}{h^2})$$

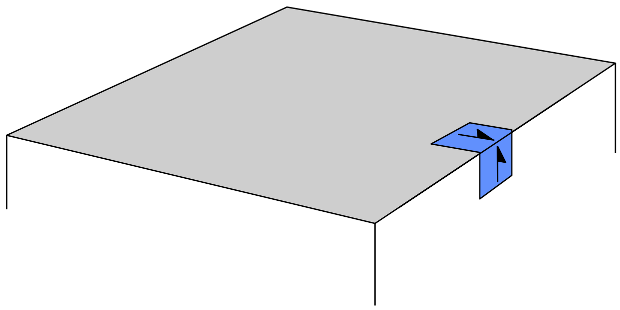
The stress σ_{xy} at the edges $y = h/2$ and $y = -h/2$ is

$$\sigma_{xy} = A\,(1-4\frac{\left(\frac{h}{2}\right)^2}{h^2})(-8\frac{z}{h^2}) = 0$$

The stress σ_{xz} at the edges $z = h/2$ and $z = -h/2$ is

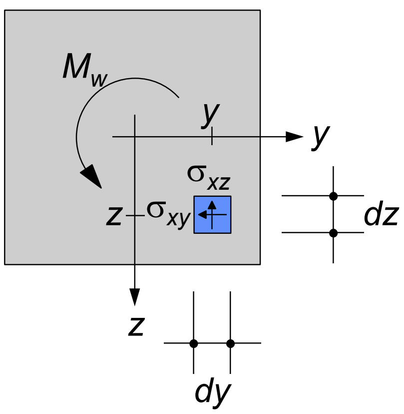
$$\sigma_{xz} = -A\,(-8\frac{y}{h^2})(1-4\frac{\left(\frac{h}{2}\right)^2}{h^2}) = 0$$

Therefore, the shear stress perpendicular to the edge of the beam is zero. This is indeed necessary because at the surface of the beam, are no shear stresses and shear stresses at perpendicular planes are equal.



b Torsion Moment

The torsion moment is the resultant of the shear stresses over the section area



$$M_w = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} y\,\sigma_{xz}\,dy\,dz - z\,\sigma_{xy}\,dy\,dz$$

As proved in the lecture book this is equal to two times the volume of the ϕ-bubble (Direct Methods, page 169).

c Largest Stress

The stress σ_{xy} is largest if y = 0 and z = -h/2.

$$\sigma_{xy} = A\,(1-4\frac{0^2}{h^2})(-8\frac{-\frac{h}{2}}{h^2}) = 4\frac{A}{h}$$

The stress σ_{xz} has the same extreme value.
We can rewrite the expression for the moment as

$$A = \frac{9}{8}\frac{M_w}{h^2}$$

Using this the largest stress becomes

$$\boxed{\sigma = \frac{M_w}{\frac{2}{9}h^3}}$$

This result is 6% smaller than an accurate computation by the finite element method.

d Complementary Energy
The complimentary energy is the integral of the energy in all particles in the slice. The force at a particle is $\sigma\, dy\, dz$ and the displacement of the force is $\gamma\ \Delta x$.

$$E_c = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \tfrac{1}{2}\sigma_{xy}\, dy\, dz\ \gamma_{xy}\Delta x + \tfrac{1}{2}\sigma_{xz}\, dy\, dz\ \gamma_{xz}\Delta x$$

e Torsion Stiffness
The moment in the wire frame model is

$$M_w = GI_w\ \vartheta$$

The energy in a length Δx of this model is

$$E_c = \tfrac{1}{2}M_w\ \vartheta\,\Delta x = \frac{1}{2}\frac{M_w^2}{GI_w}\Delta x = \frac{1}{2}\frac{\left(\frac{8}{9}Ah^2\right)^2}{GI_w}\Delta x = \frac{32}{81}\frac{A^2h^4}{GI_w}\Delta x$$

The energy in the slice must be equal to the energy in the wire frame part

$$\frac{128}{45}\frac{A^2}{G}\Delta x = \frac{32}{81}\frac{A^2h^4}{GI_w}\Delta x$$

Or

$$\boxed{GI_w = G\frac{5}{36}h^4}$$

A computation with the finite element method is just 1% stiffer (Lecture book, Direct Methods, Fig. 6.38).

Answer to Problem 2

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \qquad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} \qquad \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

kinematic relation | *constitutive relation* | *equilibrium relation*

Answers to Problem 3

a Boundary Conditions

On the edge ($r = a$) the plate is clamped, therefore, the local displacement and slope both equal zero.

$$w = 0$$

$$\frac{dw}{dx} = 0$$

In the origin of the reference frame ($r = 0$) the slope is zero because of symmetry. In the origin the shear force q is also zero because a small cylinder in the origin must be in equilibrium.

$$\frac{dw}{dx} = 0$$

$$q = 0$$

b Extreme Moment

The deflection line is

$$w = \frac{p}{64K}(a^2 - r^2)^2$$

Differentiation gives

$$\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{p}{64K} 2(a^2 - r^2)(-2r) = \frac{-p}{16K}(a^2 r - r^3)$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} = \frac{-p}{16K}(a^2 - 3r^2)$$

Substitution in the kinematic relations gives

$$\kappa_{rr} = -\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} = \frac{p}{16K}(a^2 - 3r^2)$$

$$\kappa_{\vartheta\vartheta} = -\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{\mathrm{p}}{16\mathrm{K}}(a^2 - r^2)$$

Substitution in the constitutive relations gives

$$m_{rr} = K(\kappa_{rr} + \nu\,\kappa_{\vartheta\vartheta}) = K\,\frac{p}{16K}(a^2 - 3r^2 + \nu(a^2 - r^2)) = \frac{p}{16}(a^2(1+\nu) - r^2(3+\nu))$$

$$m_{\vartheta\vartheta} = K(\nu\,\kappa_{\mathrm{rr}} + \kappa_{\vartheta\vartheta}) = K\,\frac{p}{16K}(\nu\,(a^2 - 3r^2) + (a^2 - r^2)) = \frac{p}{16}(a^2(1+\nu) - r^2(1+3\,\nu))$$

The moments in the middle of the plate (r = 0) are

$$m_{rr} = \frac{p\,a^2}{16}(1+\nu)$$

$$m_{\vartheta\vartheta} = \frac{p\,a^2}{16}(1+\nu)$$

The moments at the edges of the plate (r = a) are

$$m_{rr} = \frac{p}{16}(a^2(1+\nu) - a^2(3+\nu)) = \frac{p\,a^2}{16}(-2)$$

$$m_{\vartheta\vartheta} = \frac{p}{16}(a^2(1+\nu) - a^2(1+3\,\nu)) = \frac{p\,a^2}{16}(-2\,\nu)$$

Therefore, the extreme moment is

$$m_{rr} = \frac{p\,a^2}{16}(-2)$$

$$\boxed{m_{rr} = -\tfrac{1}{8}pa^2}$$

c Steel Stresses

Stress is moment over section modulus

$$\sigma = \frac{M}{W}$$

We consider a plate part with a width of 1 m.

$$W = \tfrac{1}{6}bh^2 = \tfrac{1}{6}\ 1\ 0.012^2 = 24\,10^{-6}\mathrm{m}^3$$

Therefore

$$\sigma = \frac{1 m_{rr}}{W} = \frac{\frac{1}{8}pa^2}{W} = \frac{\frac{1}{8}\,900\,15^2}{24\,10^{-6}}$$

$$\boxed{\sigma = 1055\ 10^6\ \mathrm{N/m^2}}$$

The larges deflection of the plate is

$$w = \frac{p\,a^4}{64\,K}$$

where the plate stiffness is

$$K = \frac{E\,h^3}{12\,(1-\nu^2)} = \frac{210\ 10^9\ 0.012^3}{12\,(1-0.01)} = 30545 \quad \mathrm{Nm}$$

The deflection becomes

$$w = \frac{900\ 15^4}{64\ 30545}$$

$$\boxed{w = 23\ \mathrm{m}}\ \ !$$

d Explanation

Membrane stresses (Dutch: zeilwerking) in the bottom plate cause the deflection and stresses to be much smaller than follows from the plate theory. However, even if these geometrical nonlinear effects are taken into account the calculated edge stresses will be larger than the yield stress. Therefore, at the edges plastic deformation occurs during the jack up process of the tank.

Consequently we need to conclude that the applied linear elastic model is not suitable to analyse this problem due to the large deflections.

Technische Universiteit Delft
Faculteit der Civiele Techniek
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
15 oktober 1998 van 12.00 -15.00 uur

Problem 1 (1 point)

In the force method one or more compatibility equations can be derived by

1 adding the kinematic equations
2 eliminating the strains from the kinematic relations
3 eliminating the displacements from the kinematic relations

Problem 2 (5 points)

We consider two axial symmetrical elements, a ring and a disk loaded in their plane. These elements will be analysed separately and subsequently connected.

Ring
The radius of the ring is 2*a* (Figure 1). The cross-section area is *A*. It is loaded by a pressure *p* per unit of circumference. The ring material has an elasticity modulus E_r .

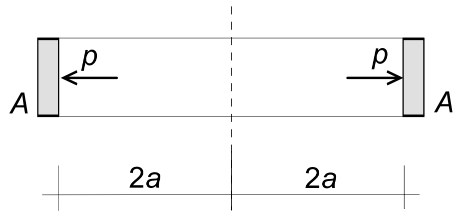


Figure 1. Cross-section of the ring

Disk
The radius of the disk is 2*a* and its thickness is *t* (Figure 2). The disk has a hole in the middle of radius *a*. The disk is loaded by a pressure *p* per unit of circumference. The material of the disk has an elasticity modulus *E* and a Poison's ratio $\nu = 0$.

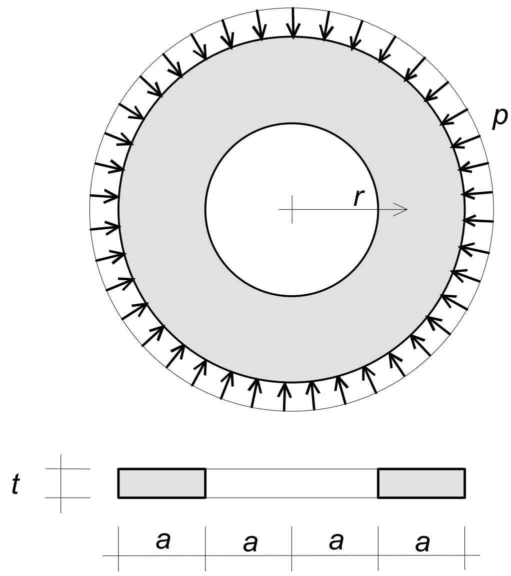


Figure 2. Disk

Kinematic equations of the disk

$$\varepsilon_{rr} = \frac{du}{dr}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r}$$

Equilibrium equation of the disk

$$\frac{d}{dr}(r\sigma_{rr}) - \sigma_{\theta\theta} = 0$$

Using the force method the solution is

$$\sigma_{rr} = \frac{\phi}{r}$$

$$\sigma_{\theta\theta} = \frac{d\phi}{dr}$$

$$\phi = C_1 \frac{a^2}{r} + C_2 r$$

a Derive the kinematic equation of $\varepsilon_{\theta\theta}$ for the disk. (Suggestion, consider how the shape changes due to displacement *u*).

b Derive the equilibrium equation for the disk. (Suggestion, draw an elementary part of the disk).

c Calculate the stresses in the disk due to the load *p*.

d Derive the following equation for the disk. u_1 is the displacement of the outer edge.

$$u_1 = -\frac{10}{3}\frac{pa}{Et}$$

e Derive the following equation for the ring.

$$u_2 = 4\frac{pa^2}{E_r A}$$

The ring is heated *T* degrees. It now fits exactly around the disk. Subsequently the ring is cooled down to its normal temperature. The linear expansion coefficient of both materials is α. $E_r = 30E$ and $A = \frac{1}{5}at$.

f Calculate the stresses in the connected disk and the ring.

Problem 3 (4 points)

Consider the structural system of Figure 3. It consists of two beams, which are connected by a hinge. The left beam has an infinite bending stiffness. The right-hand beam has a bending stiffness *EI*. The left support is a free hinge and the right-hand support is a clamp. The left beam is supported by a spring. The spring stiffness is *K*. The system is loaded by a force *F* at the middle hinge and a moment *T* at the left beam.

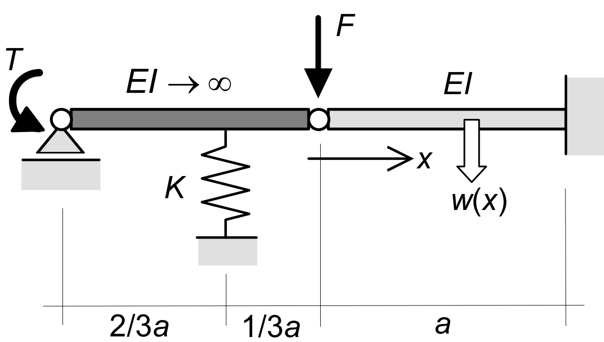


Figure 3. Structural system

The system will be calculated using the principle of minimum potential energy. The following displacement is assumed for the deflection of the right-hand beam. (Note that the x-axis starts at the middle hinge.)

$$w = \tfrac{1}{2}\left(\frac{x^3}{a^3} - 3\frac{x}{a} + 2 \right) C$$

The following expression can be derived for the potential energy of the system.

$$E_{pot} = \left(\tfrac{2}{9}K + \tfrac{3}{2}\frac{EI}{a^3} \right) C^2 - \left(F - \frac{T}{a} \right) C$$

a What is the unit of the constant C? How can it be interpreted?

b Derive the expression of the potential energy of the system.

c In the following we assume the values.

$$K = \tfrac{27}{4}\frac{EI}{a^3} \qquad T = \tfrac{1}{4}Fa$$

Calculate the constant C.

d Make a drawing of the moment line and give its extreme values.

e Are the calculated results approximations or exact solutions of the system? Explain your answer.

Tentamen b16, 15 oktober 1998
Answer to Problem 1

The correct answer is answer 3. The compatibility equations are derived by eliminating the displacements from the kinematic equations.

Answers to Problem 2

a Kinematic Relation
Consider a circle of radius r. The circle increases due to the loading to a radius $r + u$. The circle length before loading is $2\pi r$. The circle length after loading is $2\pi(r + u)$. Therefore, the strain is

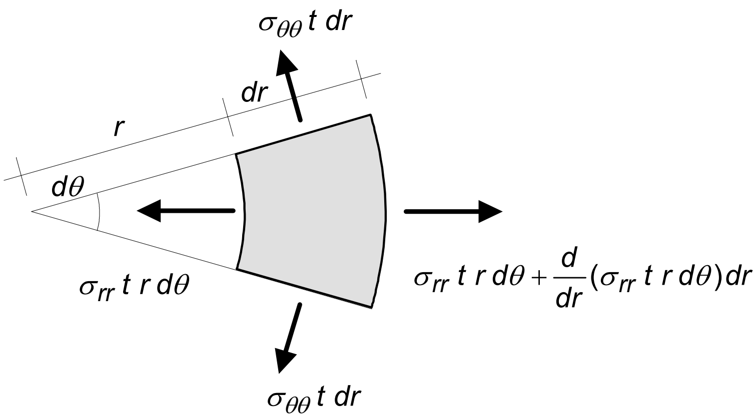
$$\varepsilon_{\theta\theta} = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

b Equilibrium Equations
The resulting force in the left section is $-\sigma_{rr}\, t\, r\, d\theta$. The resulting force in the right-hand section is $\sigma_{rr}\, t\, r\, d\theta + \frac{d}{dr}(\sigma_{rr}\, t\, r\, d\theta)dr$. The resulting forces in the top and bottom section are $\sigma_{\theta\theta}\, t\, dr$. The latter produce a force $-\sigma_{\theta\theta}\, t\, dr\, d\theta$ due to the angle $d\theta$.
Equilibrium in the r direction gives

$$\frac{d}{dr}(\sigma_{rr}\, t\, r\, d\theta)dr - \sigma_{\theta\theta}\, t\, dr\, d\theta = 0\,.$$

This can be simplified by division by $t\, dr\, d\theta$

$$\frac{d}{dr}(\sigma_{rr}\, r) - \sigma_{\theta\theta} = 0\,.$$



c Stresses
We know that

$$\phi = C_1 \frac{a^2}{r} + C_2 r$$

Substitution gives

$$\sigma_{rr} = \frac{\phi}{r} = C_1 \frac{a^2}{r^2} + C_2$$

$$\sigma_{\theta\theta} = \frac{d\phi}{dr} = -C_1 \frac{a^2}{r^2} + C_2$$

Applying the boundary conditions we obtain two equations with two unknown.

$$r = a \rightarrow \sigma_{rr} = 0 = C_1 \frac{a^2}{a^2} + C_2 \rightarrow 0 = C_1 + C_2$$

$$r = 2a \rightarrow \sigma_{rr} = -\frac{p}{t} = C_1 \frac{a^2}{(2a)^2} + C_2 \rightarrow -\frac{p}{t} = C_1 \tfrac{1}{4} + C_2$$

The solution is

$$C_1 = \tfrac{4}{3}\frac{p}{t}$$

$$C_2 = -\tfrac{4}{3}\frac{p}{t}$$

Therefore, the stresses are

$$\sigma_{rr} = \tfrac{4}{3}\frac{p}{t}\frac{a^2}{r^2} - \tfrac{4}{3}\frac{p}{t} \rightarrow \boxed{\sigma_{rr} = \tfrac{4}{3}\frac{p}{t}\left(\frac{a^2}{r^2} - 1\right)}$$

$$\sigma_{\theta\theta} = -\tfrac{4}{3}\frac{p}{t}\frac{a^2}{r^2} - \tfrac{4}{3}\frac{p}{t} \rightarrow \boxed{\sigma_{\theta\theta} = -\tfrac{4}{3}\frac{p}{t}\left(\frac{a^2}{r^2} + 1\right)}$$

d Displacement of the Disk Edge

We know that $\varepsilon_{\theta\theta} = \frac{u}{r}$. Therefore,

$$u(r) = r\varepsilon_{\theta\theta} = r\frac{\sigma_{\theta\theta}}{E} = -\tfrac{4}{3}\frac{p}{t}\left(\frac{a^2}{r^2} + 1\right)\frac{r}{E}$$

$$r = 2a \rightarrow u_1 = -\tfrac{4}{3}\frac{p}{t}\left(\frac{a^2}{(2a)^2} + 1\right)\frac{2a}{E} = -\tfrac{4}{3}\frac{p}{t}\left(\tfrac{1}{4} + 1\right)\frac{2a}{E}$$

$$\boxed{u_1 = -\tfrac{10}{3}\frac{pa}{tE}}$$

Note that u_1 is directed inwards.

e Displacement of the Ring

We assume that the ring is thin in the radial direction. Therefore, moments can be neglected.

The normal force N in a cross-section of the ring can be found from equilibrium of a segment $d\theta$. The resultant of the load p is $p2a\,d\theta$. The resultant of the normal forces N at both sections is $-N\,d\theta$. Equilibrium gives

$$N = 2ap\,.$$

The ring has the same kinematic equation as the disk.

$$\varepsilon_{\theta\theta} = \frac{u}{r} = \frac{u_2}{2a}$$

The constitutive equation is

$$N = E_r A\, \varepsilon_{\theta\theta} \,.$$

From the previous three equations we can derive

$$u_2 = 2a\, \varepsilon_{\theta\theta} = 2a \frac{N}{E_r A} = 2a \frac{2a\, p}{E_r A} = 4 \frac{a^2\, p}{E_r A} \,.$$

f System of Ring and Disk

The temperature rise of the ring is included in the constitutive equation.

$$N = E_r A(\varepsilon_{\theta\theta} - \alpha T_r)$$

From the equilibrium equation, kinematic equation and constitutive equation we derive

$$u_2 = 2a\, \varepsilon_{\theta\theta} = 2a \left(\frac{N}{E_r A} + \alpha T_r \right) = 2a \left(\frac{2a\, p}{E_r A} + \alpha T_r \right)$$

After the ring is fitted its temperature drops T degrees. Therefore,

$$u_2 = 2a \left(\frac{2a\, p}{E_r A} - \alpha T \right)$$

The displacements of disk and ring will be equal.

$$u_1 = u_2$$

$$-\tfrac{10}{3} \frac{p\, a}{t\, E} = 2a \left(\frac{2a\, p}{E_r A} - \alpha T \right)$$

Using, $E_r = 30E$ and $A = \tfrac{1}{5} at$ we derive

$$-\tfrac{10}{3} \frac{p\, a}{t\, E} = 2a \left(\frac{2a\, p}{30E \tfrac{1}{5} at} - \alpha T \right) = 2a \left(\frac{p}{3Et} - \alpha T \right) = \frac{2a\, p}{3Et} - 2a\alpha T$$

$$2a\alpha T = \frac{2a\, p}{3Et} + \tfrac{10}{3} \frac{p\, a}{t\, E} = \tfrac{12}{3} \frac{a\, p}{Et}$$

$$\alpha T = 2 \frac{p}{Et}$$

$$p = \tfrac{1}{2} \alpha T E t \,.$$

Subsequently, the displacement and stresses can be calculated.

$$u_1 = -\tfrac{10}{3} \frac{p\, a}{t\, E} = -\tfrac{10}{6} a T$$

Stresses in the disk

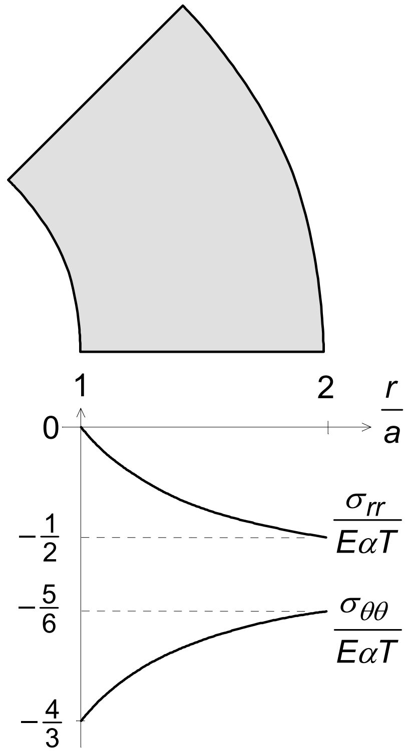
$$\sigma_{rr} = \tfrac{4}{3}\frac{p}{t}\left(\frac{a^2}{r^2}-1\right) = \tfrac{4}{3}\frac{\tfrac{1}{2}\alpha TEt}{t}\left(\frac{a^2}{r^2}-1\right) \rightarrow \boxed{\sigma_{rr} = \tfrac{2}{3}\alpha TE\left(\frac{a^2}{r^2}-1\right)}$$

$$\sigma_{\theta\theta} = -\tfrac{4}{3}\frac{p}{t}\left(\frac{a^2}{r^2}+1\right) = -\tfrac{4}{3}\frac{\tfrac{1}{2}\alpha TEt}{t}\left(\frac{a^2}{r^2}+1\right) \rightarrow \boxed{\sigma_{\theta\theta} = -\tfrac{2}{3}\alpha TE\left(\frac{a^2}{r^2}+1\right)}$$

Stresses in the ring

$$N = 2a\,p = 2a\,\tfrac{1}{2}\alpha TEt = a\alpha TEt$$

$$\sigma_{\theta\theta} = \frac{N}{A} = \frac{a\alpha TEt}{\tfrac{1}{5}at} \rightarrow \boxed{\sigma_{\theta\theta} = 5\alpha TE}$$



<u>Remarks</u>
The internal radius of the ring has been used in all calculations. We could also have used the radius to the centreline of the ring cross-section. However, this has very little effect on the results when the ring is thin in the radial direction.

Damaged train wheels are often turned off on a lathe (Dutch: afdraaien op een draaibank) and fitted with steel tyres. The method of this problem can be used to model this process.

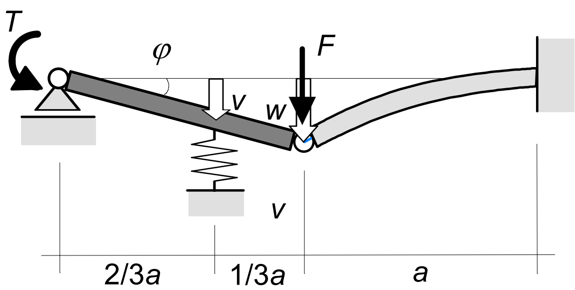
Answers to Problem 3

a <u>Constant *C*</u>
C has the unit of length. For example meter [m].

When x = 0 then $w(0) = C$. Therefore, C is the displacement of the middle hinge.

b <u>Potential Energy</u>

$$E_{pot} = E_s + E_p$$



E_s consists of two parts, due to the right-hand beam and due to the spring.
E_p consists of two parts, due to the force F and due to the moment T.
Together this gives

$$E_{pot} = \tfrac{1}{2}Kv^2 + \tfrac{1}{2}\int_{x=0}^{a} EI\kappa^2 dx - Fw - T\varphi$$

where,

$v = \frac{2}{3}C$ is the shortening of the spring,

$w = C$ is the displacement of the force F,

$\varphi = -\dfrac{C}{a}$ is the rotation of the left beam,

$\kappa = -\dfrac{d^2w}{dx^2} = -\dfrac{3x}{a^3}C$ is the curvature of the right-hand beam.

Substitution in the equation of the potential energy gives.

$$E_{pot} = \tfrac{1}{2}K\tfrac{4}{9}C^2 + \tfrac{1}{2}\int_{x=0}^{a} EI\frac{9}{a^6}x^2C^2dx - FC + T\frac{C}{a}$$

$$E_{pot} = K\tfrac{2}{9}C^2 + \tfrac{9}{2}\frac{EIC^2}{a^6}\int_{x=0}^{a} x^2dx - FC + T\frac{C}{a}$$

$$E_{pot} = K\tfrac{2}{9}C^2 + \tfrac{9}{2}\frac{EIC^2}{a^6}\tfrac{1}{3}a^3 - FC + T\frac{C}{a}$$

$$E_{pot} = K\tfrac{2}{9}C^2 + \tfrac{3}{2}\frac{EIC^2}{a^3} - FC + T\frac{C}{a}$$

$$E_{pot} = \left(\tfrac{2}{9}K + \tfrac{3}{2}\frac{EI}{a^3}\right)C^2 - \left(F - \frac{T}{a}\right)C$$

c Constant C

Using $K = \frac{27}{4}\dfrac{EI}{a^3}$ and $T = \frac{1}{4}Fa$ the potential energy becomes

$$E_{pot} = \left(\tfrac{2}{9}\tfrac{27}{4}\frac{EI}{a^3} + \tfrac{3}{2}\frac{EI}{a^3}\right)C^2 - \left(F - \frac{\tfrac{1}{4}Fa}{a}\right)C$$

$$E_{pot} = 3\frac{EI}{a^3}C^2 - \tfrac{3}{4}FC$$

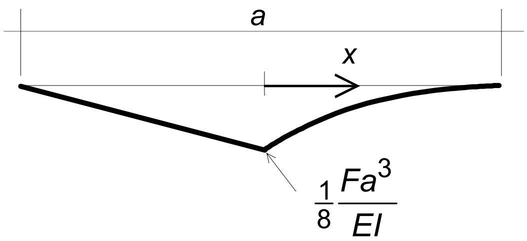
$$\frac{d}{dC}E_{pot} = 6\frac{EI}{a^3}C - \tfrac{3}{4}F = 0$$

$$\boxed{C = \tfrac{1}{8}\frac{Fa^3}{EI}}$$

d Deflections and Moments

Largest deflection

$$w(0) = C = \tfrac{1}{8}\frac{Fa^3}{EI}$$



Moment in the right-hand beam

$$M = EI\kappa = -EI\frac{3x}{a^3}C = -EI\frac{3x}{a^3}\frac{1}{8}\frac{Fa^3}{EI} = -\frac{3}{8}Fx$$

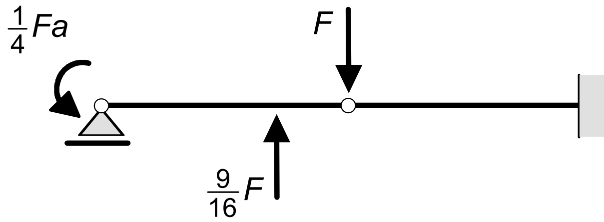
The force in the spring is

$$F = -Kv = -K\frac{2}{3}C = -\frac{27}{4}\frac{EI}{a^3}\frac{2}{3}\frac{1}{8}\frac{Fa^3}{EI} = -\frac{9}{16}F$$

Moment equilibrium of the left beam gives the shear force D left of the middle hinge

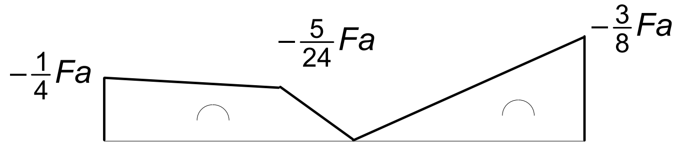
$$\frac{1}{4}Fa = Da - \frac{9}{16}F\frac{2}{3}a$$

$$D = \frac{5}{8}F$$



from which the moment at the spring can be calculated.

$$M = -\frac{5}{8}F\frac{1}{3}a = -\frac{5}{24}Fa$$



e Approximation or Exact

The results are exact if the assumed displacement function $w(x)$ gives a moment line which is in equilibrium with the load and fulfills the dynamic boundary conditions.

Equilibrium $q = -\dfrac{d^2M}{dx^2} = \dfrac{d^2}{dx^2}\dfrac{3}{8}Fx = 0$, which is correct.

Dynamic boundary condition $M = \frac{3}{8}Fx \quad x = 0 \rightarrow M = 0$, which is correct.

Therefore, the results are exact.

Technische Universiteit Delft
Faculteit der Civiele Techniek
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
29 oktober 1999 van 14.00 -17.00 uur

Problem 1 (5 points)

A non-prismatic beam has a length L and is simply supported at both ends (Figure 1). The bending stiffness at the left support is EI_o and at the right-hand support $\frac{3}{4}EI_o$. Between the supports the bending stiffness varies linear. The beam is loaded by a moment M_1 at the left support

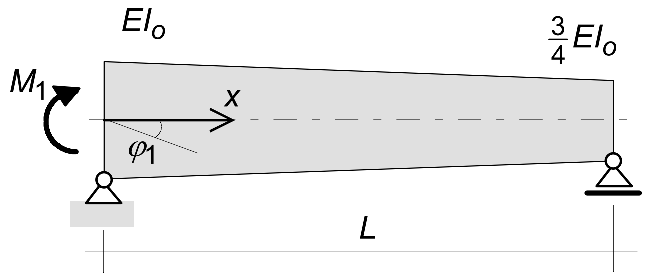


Figure 1. Non-prismatic beam

A structural designer calculates the rotation φ_1 of the beam at the left support. He or she uses the standard formula (Dutch: vergeet-mij-nietje) for prismatic beams and the average bending stiffness of the beam.

a What is the percentage error that the structural designer makes?
Apply one of Castigliano's theorems to calculate the rotation φ_1. In this problem you can use either the displacement method or the force method. Explain your selection of the interpolation function.

b If the displacement method were used: Would the force method give a larger or smaller value for the rotation?
If the force method were used: Would the displacement method give a larger or smaller value for the rotation?

<ins>Suggestion</ins>
If you need to calculate an integral for which you do not know the solution, use Simpson's rule for numerical integration (Figure 2.).

$$\int_0^L f(x)dx = \tfrac{1}{6}L(f_1 + 4f_2 + f_3)$$

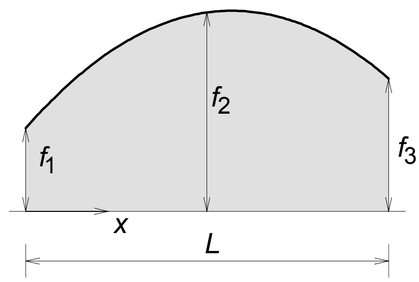


Figure 2. Simpson's rule

Problem 2 (5 points)

Consider the composite bridge shown in Figure 3. The continuous lines are reinforced concrete plates and the dotted lines are steel trusses. Figure 4 shows part of one of the trusses. The thickness of all concrete walls is t and the shear modulus of the concrete is G.

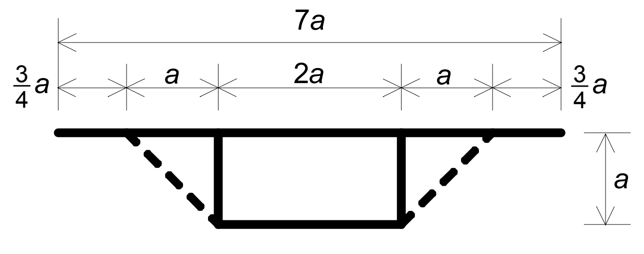


Figure 3. Cross-section of a composite bridge

We need to calculate the torsion stiffness and the torsion stresses in the cross-section. Therefore, we replace the trusses by homogenous isotropic plates with the same shear modulus G as the concrete plates and a fictitious thickness t_f. The extension stiffness of the truss bars is EA_s.

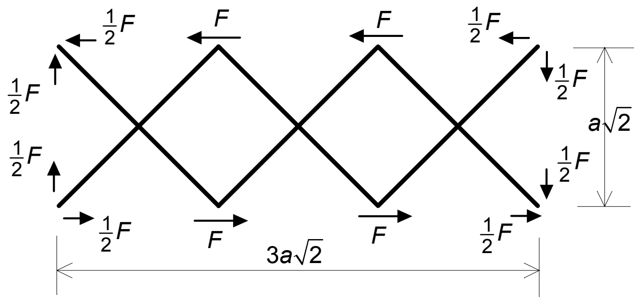


Figure 4. Elevation of one of the trusses

a The fictitious thickness of the wall is $t_f = \dfrac{EA_s}{2aG}$. Use energy to derive this fictitious thickness (Figure 5).

b Assume that $t_f = \frac{1}{2}\sqrt{2}t$. Calculate the torsion stiffness of the cross-section of the bridge.

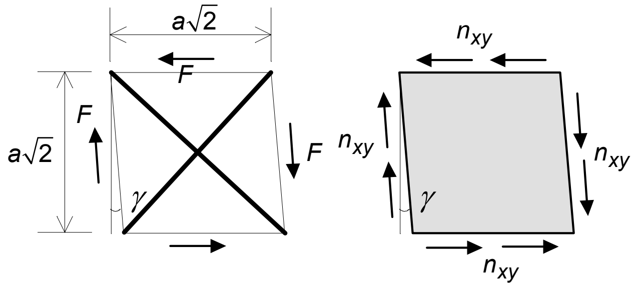


Figure 5. Truss and isotropic plate

c Calculate the shear stresses due to a torsion moment M_t.

Tentamen Ctme5141, 29 oktober 1999
Answers to Problem 1

a Method of the Structural Designer

Formula of the prismatic beam $\varphi_1 = \dfrac{M_1 L}{3EI}$

Average bending stiffness $EI = \frac{1}{2}\left(EI_o + 0.75 EI_o\right) = 0.875 EI_o$

Substitution $\varphi_1 = \dfrac{M_1 L}{3EI} = \dfrac{M_1 L}{3 * 0.875 EI_o}$ $\quad \boxed{\varphi_1 = 0.381 \dfrac{M_1 L}{EI_o}}$

Force Method (Minimum complementary energy or Castigliano 2)

Moment line $M = M_1(1 - \dfrac{x}{L})$

The linear moment line is in equilibrium with the load. Therefore, it can be used in complementary energy. The beam is statically determined. Therefore, we can easily see that the linear moment line can describe the exact solution too. Consequently, approximations are not introduced here.

Bending stiffness $EI = EI_o(1 - \dfrac{x}{4L})$

Complementary energy $E_c = \frac{1}{2}\displaystyle\int_0^L \frac{M^2}{EI} dx$

Substitution $E_c = \frac{1}{2}\displaystyle\int_0^L \frac{M_1^2 (1 - \frac{x}{L})^2}{EI_o (1 - \frac{x}{4L})} dx = \frac{M_1^2}{2EI_o}\int_0^L f(x)\, dx$

$$f(x) = \frac{(1 - \frac{x}{L})^2}{(1 - \frac{x}{4L})}$$

Approximation of the integral

using Simpson's rule $f(0) = \dfrac{(1-0)^2}{(1-0)} = 1$

$$f(\tfrac{1}{2}L) = \frac{(1 - \frac{1}{2})^2}{(1 - \frac{1}{8})} = \frac{2}{7}$$

$$f(L) = \frac{(1-1)^2}{(1 - \frac{1}{4})} = 0$$

$$E_c = \frac{M_1^2}{2EI_o}\frac{L}{6}(1 + 4 * \tfrac{2}{7} + 0) = \tfrac{5}{28}\frac{M_1^2 L}{EI_o} = 0.179 \frac{M_1^2 L}{EI_o}$$

Intermezzo The integral can also be calculated exactly

$$E_c = \frac{M_1^2}{2EI_o}(72 \ln 2 - 36 \ln 3 - 10)L = 0.178 \frac{M_1^2 L}{EI_o}$$

Complementary energy $E_{compl} = E_c - M_1 \varphi_1 \quad 0 = \dfrac{\partial E_{compl}}{\partial M_1} = \dfrac{\partial E_c}{\partial M_1} - \varphi_1$

$$\varphi_1 = \frac{\partial E_c}{\partial M_1} = 0.179 * 2 \frac{M_1 L}{EI_o} \quad \boxed{\varphi_1 = 0.358 \frac{M_1 L}{EI_o}}$$

Error of the structural designer $\dfrac{0.381-0.358}{0.358}*100\% = 6\%$ too large

<u>Displacement Method</u> (Minimum potential energy or Castigliano 1) Alternative answer

Deflection line $w(x) = C(1-\dfrac{x}{L})\dfrac{x}{L}$

This is the simplest function that still fulfils the kinematic boundary conditions

$$w(0) = 0$$

$$w(L) = 0$$

The rotation of the beam $\varphi = \dfrac{dw}{dx} = C\dfrac{1}{L}\dfrac{x}{L} + C(1-\dfrac{x}{L})\dfrac{1}{L}$

$$\varphi_1 = \varphi(0) = C\frac{1}{L}$$

Curvature $\kappa = -\dfrac{d^2w}{dx^2} = C\dfrac{1}{L}\dfrac{1}{L} + C\dfrac{1}{L}\dfrac{1}{L} = 2\dfrac{C}{L^2} = 2\dfrac{\varphi_1}{L}$

Strain energy $E_s = \frac{1}{2}\int_0^L EI\kappa^2 dx$

Substitution $E_s = \frac{1}{2}\int_0^L EI_o(1-\dfrac{x}{4L})(2\dfrac{\varphi_1}{L})^2 dx =$

$$= 2EI_o(\frac{\varphi_1}{L})^2 \int_0^L (1-\frac{x}{4L})dx = 2EI_o(\frac{\varphi_1}{L})^2 \frac{7}{8}L = \tfrac{7}{4}EI_o\frac{\varphi_1^2}{L}$$

Potential energy $E_{pot} = E_s - M_1\varphi_1 \quad 0 = \dfrac{\partial E_{pot}}{\partial \varphi_1} = \dfrac{\partial E_s}{\partial \varphi_1} - M_1$

$$M_1 = \frac{\partial E_s}{\partial \varphi_1} = 2\tfrac{7}{4}EI_o\frac{\varphi_1}{L} \qquad \boxed{\varphi_1 = 0.286\frac{M_1L}{EI_o}}$$

Error of the structural designer $\dfrac{0.381-0.286}{0.286}*100\% = 33\%$ too large

b <u>Displacement versus Force Method</u>

The displacement method gives a too stiff solution, so a too small φ_1. The force method gives the exact solution except for the approximation of the in the integration rule.

Answers to Problem 2

a <u>Fictitious Thickness</u>

The energy in the plate parts should be equal to the energy in the truss walls. From this we calculate the fictitious thickness t_f .

Energy in a plate part

$$E_c = \frac{1}{2}\frac{\left(\frac{n_{xy}}{t_f}\right)^2}{G}(a\sqrt{2})^2 t_f = \frac{n_{xy}^2}{G\,t_f}a^2$$

$$n_{xy}\,a\sqrt{2} = F$$

$$E_c = \frac{\left(\frac{F}{a\sqrt{2}}\right)^2}{G\,t_f}a^2 = \frac{F^2}{2G\,t_f}$$

Energy in two truss bars

$$E_c = \frac{1}{2}\frac{N_1^2}{EA_s}2a + \frac{1}{2}\frac{N_2^2}{EA_s}2a$$

$$N_1 = -\tfrac{1}{2}F\sqrt{2} \quad N_2 = \tfrac{1}{2}F\sqrt{2}$$

$$E_c = \frac{1}{2}\frac{(-\frac{1}{2}F\sqrt{2})^2}{EA_s}2a + \frac{1}{2}\frac{(\frac{1}{2}F\sqrt{2})^2}{EA_s}2a = \frac{F^2 a}{EA_s}$$

To equate

$$\frac{F^2}{2G\,t_f} = \frac{F^2 a}{EA_s}$$

$$t_f = \frac{EA_s}{2aG}$$

b <u>Torsion Stiffness</u>
We use the membrane analogy.

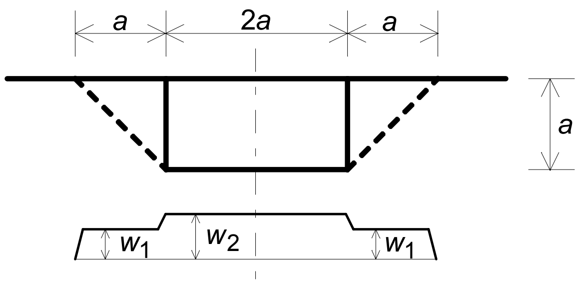
$$S\left(\frac{w_1}{t_f}a\sqrt{2}\right) + S\left(\frac{w_1}{t}a\right) - S\left(\frac{w_2 - w_1}{t}a\right) = \tfrac{1}{2}a^2 p$$

$$S\left(\frac{w_2}{t}a\right) + S\left(\frac{w_2}{t}a\right) + S\left(\frac{w_2 - w_1}{t}a\right) = a^2 p$$

Simplified

$$4w_1 - w_2 = \frac{1}{2}\frac{atp}{S}$$

$$-w_1 + 3w_2 = \frac{atp}{S}$$



From which we can solve

$$w_1 = \frac{5}{22}\frac{atp}{S}$$

$$w_2 = \frac{9}{22}\frac{atp}{S}$$

Substituting $S = \dfrac{1}{G}$ and $p = 2\theta$ we obtain the ϕ-bubble.

$$\phi_1 = \tfrac{5}{11} at\theta G$$
$$\phi_2 = \tfrac{9}{11} at\theta G$$

The torsion moment M_t is two times the volume of the ϕ-bubble.

$$M_t = 2(a^2\phi_1 + 2a^2\phi_2) = 2a^2(\phi_1 + 2\phi_2) = 2a^2(\tfrac{5}{11} at\theta G + 2\tfrac{9}{11} at\theta G)$$
$$M_t = \tfrac{46}{11} a^3 t\theta G$$

Therefore,

$$I_t = \frac{M_t}{G\theta}$$
$$\boxed{I_t = \tfrac{46}{11} a^3 t}$$

c Shear Stresses

The shear stress is the slope of the ϕ-bubble. At the previous page we found

$$\begin{aligned} \phi_1 &= \tfrac{5}{11} at\theta G \\ \phi_2 &= \tfrac{9}{11} at\theta G \end{aligned}.$$

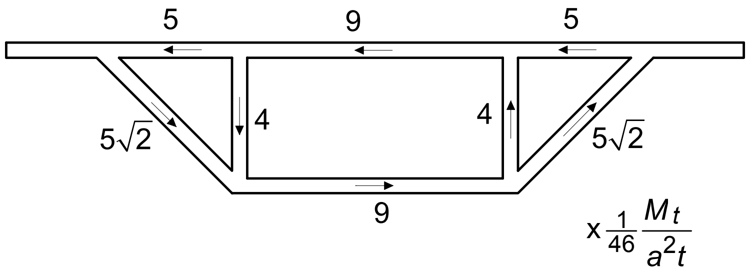
To eliminate θ we substitute $\tfrac{11}{46}\dfrac{M_t}{a^2} = at\theta G$, which gives

$$\begin{aligned} \phi_1 &= \tfrac{5}{46}\frac{M_t}{a^2} \\ \phi_2 &= \tfrac{9}{46}\frac{M_t}{a^2} \end{aligned}.$$

The shear stresses in the top and bottom plates are $\tau = \dfrac{\phi_2}{t} = \tfrac{9}{46}\dfrac{M_t}{a^2 t}$

The shear stresses in the vertical plates are $\tau = \dfrac{\phi_2 - \phi_1}{t} = \dfrac{\tfrac{9}{46}\dfrac{M_t}{a^2} - \tfrac{5}{46}\dfrac{M_t}{a^2}}{t} = \tfrac{4}{46}\dfrac{M_t}{a^2 t}$

The shear stresses in the fictitious trusses are $\tau = \dfrac{\phi_1}{\tfrac{1}{2}\sqrt{2}t} = \dfrac{\tfrac{5}{46}\dfrac{M_t}{a^2}}{\tfrac{1}{2}\sqrt{2}t} = \tfrac{10}{46}\dfrac{M_t}{\sqrt{2}a^2 t} = \tfrac{5}{46}\sqrt{2}\dfrac{M_t}{a^2 t}$



Technische Universiteit Delft
Faculteit der Civiele Techniek en Geowetenschappen
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
Dinsdag 11 januari 2000, 9:00 –12:00 uur

Problem 1

A high-rise building has a tube frame structure. The cross-section is modelled as a tube loaded in torsion (Figure 1). The walls are plates of a homogenous isotropic material of thickness h and shear modulus G.

We calculate the tube using the membrane analogy. You can assume that the wall thickness is small compared to the width of the tube.

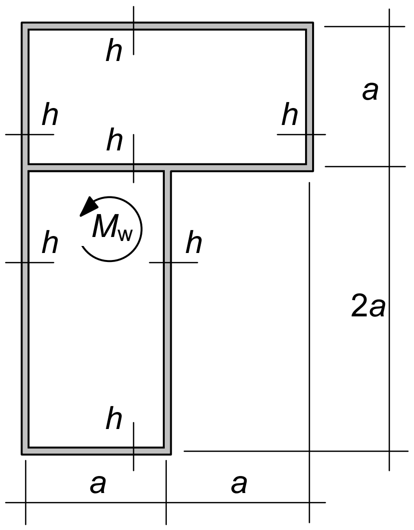


Figure 1. Cross-section of the tube

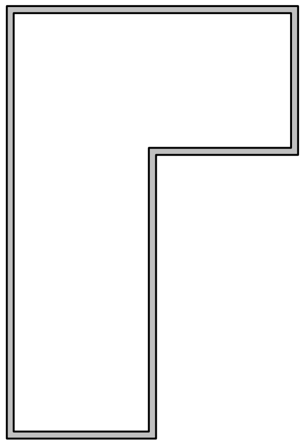


Figure 2. Alternative cross-section

a Calculate the position of the weightless plates.

b Calculate the torsion stiffness GI_t of the cross-section.

c Calculate the shear stresses in the cross-section due to a torsion moment M_t and draw the stresses in the correct direction.

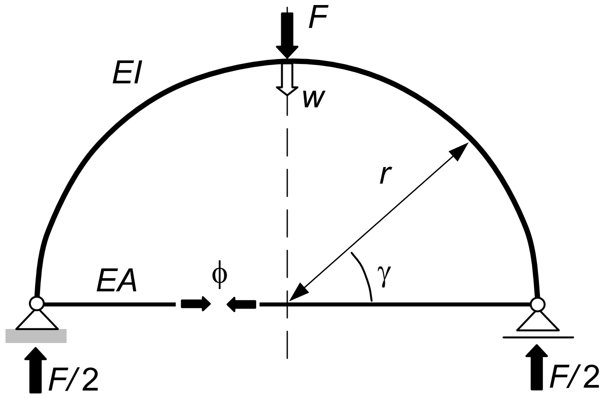
As an alternative it is suggested to leave out the interior wall. This cross-section has been drawn in Figure 2.

d Does the interior wall contribute much to the torsion stiffness? Explain your answer.

Problem 2

An arch and a tension bar are idealised according to the figure below. The materials are linear elastic. The bending stiffness of the arch is *EI* and the tension stiffness of the arch is infinitely large. The tension stiffness of the tension bar is *EA*.

The structure is calculated by complementary energy. We choose the force in the tension bars as redundant ϕ (Dutch: statisch onbepaalde).



a Express the arch moment as a function of ϕ, r, F and γ. (Due to symmetry only halve the arch needs to be considered.)

b Give the formula for the complimentary energy of the structure.

c Evaluation of the complimentary energy gives the following result. Derive this result.

$$E_{\text{compl}} = \frac{r^3}{EI}[\phi^2 \tfrac{1}{4}\pi - \tfrac{1}{2}\phi F + F^2(\tfrac{1}{16}\pi - \tfrac{1}{4})] + \frac{\phi^2 r}{EA}$$

d Calculate ϕ.

e Calculate the deflection w of the arch top.

f Assume that the calculation would have been made by the direct method instead of complementary energy. Would we have found different answers to question **d** and **e**? Explain your answer.

$$\int_0^{\pi/2} \sin\gamma \, d\gamma = 1$$

$$\int_0^{\pi/2} \cos\gamma \, d\gamma = 1$$

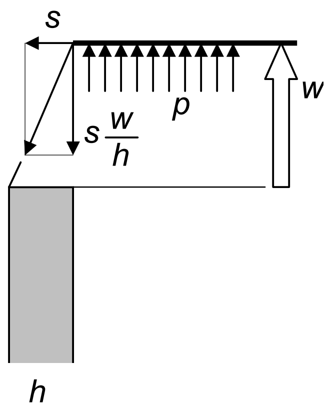
$$\int_0^{\pi/2} \sin^2\gamma \, d\gamma = \frac{\pi}{4}$$

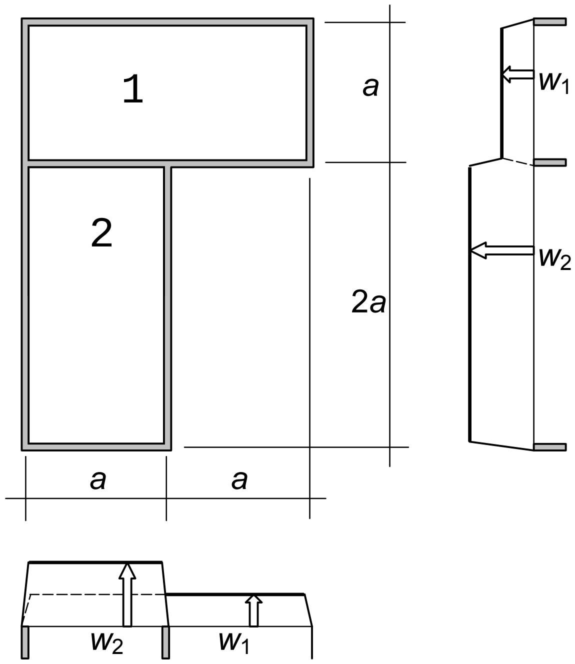
$$\int_0^{\pi/2} \cos^2\gamma \, d\gamma = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin\gamma\cos\gamma \, d\gamma = \frac{1}{2}$$

Tentamen Ctme5141, 11 januari 2000
Answer to Problem 1

a Weightless plates





Equilibrium plate 1 $\quad p\,a\,2a = 2as\dfrac{w_1}{h} + as\dfrac{w_1}{h} + as\dfrac{w_1}{h} + as\dfrac{w_1}{h} - as\dfrac{w_2 - w_1}{h}$

Equilibrium plate 2 $\quad p\,a\,2a = 2as\dfrac{w_2}{h} + 2as\dfrac{w_2}{h} + as\dfrac{w_2}{h} + as\dfrac{w_2 - w_1}{h}$

This can be simplified to

$$2ah\frac{p}{s} = 6w_1 - w_2$$

$$2ah\frac{p}{s} = 6w_2 - w_1$$

From which w_1 and w_2 can be solved.

$$w_1 = \tfrac{1}{5}2ah\frac{p}{s}$$

$$w_2 = \tfrac{1}{5}2ah\frac{p}{s}$$

b Torsion Stiffness

From the bubble we go the ϕ-bubble using the following substitutions.

$$w = \phi$$

$$p = 2\vartheta$$

$$s = \frac{1}{G}$$

Therefore

$$\phi_1 = \phi_2 = \tfrac{1}{5}2ah2\vartheta G = \tfrac{4}{5}ah\vartheta G$$

The torsion moment is two times the volume of the ϕ-bubble.

$$M_w = 2(a\,2a\,\phi_1 + a\,2a\,\phi_2) = 4a^2(\phi_1 + \phi_2) = 4a^2(\tfrac{4}{5} + \tfrac{4}{5})ah\vartheta G = \tfrac{32}{5}a^3h\vartheta G$$

For a wire frame model of the beam we have

$$M_w = GI_w \vartheta$$

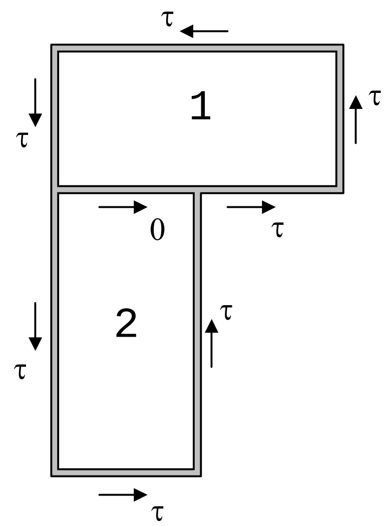
Therefore the torsion stiffness is

$$GI_w = G \tfrac{32}{5} a^3 h$$

c Shear Stresses

The shear stress is the slope of the ϕ-bubble. First we rewrite the equation for the torsion moment

$$ah\vartheta G = \tfrac{5}{32} \frac{M_w}{a^2}$$



and express ϕ_1 and ϕ_2 in the torsion moment.

$$\phi_1 = \phi_2 = \tfrac{4}{5} ah\vartheta G = \tfrac{1}{8} \frac{M_w}{a^2}$$

This gives for the shear stress

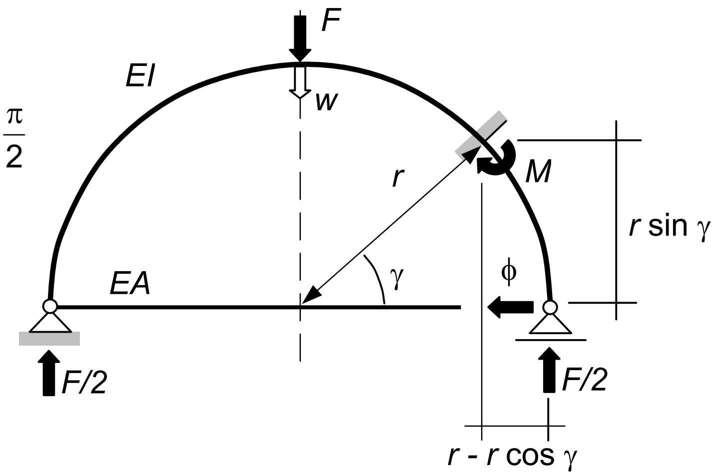
$$\tau = \frac{\phi_1}{h} = \tfrac{1}{8} \frac{M_w}{a^2 h} .$$

d Alternative

The interior wall does not contribute to the torsion stiffness. After all ϕ_1 and ϕ_2 are equal and the shear stress in the wall is zero.

Answers to Problem 2

a Moment line



$$M = \phi r \sin\gamma - F/2(r - r\cos\gamma) \qquad 0 < \gamma < \frac{\pi}{2}$$

b Complementary energy

$$E_{compl} = 2 \int_{\gamma=0}^{\pi/2} \frac{M^2}{2EI} ds + \int_{x=0}^{2r} \frac{\phi^2}{2EA} dx$$

c Evaluation

$$\begin{aligned} E_{compl} &= \frac{1}{EI} \int_{\gamma=0}^{\pi/2} M^2 ds + \frac{\phi^2}{2EA} \int_{x=0}^{2r} dx \\ &= \frac{1}{EI} \int_{\gamma=0}^{\pi/2} M^2 \, r \, d\gamma + \frac{\phi^2}{2EA} 2r \\ &= \frac{r}{EI} \int_{\gamma=0}^{\pi/2} [\phi r \sin\gamma - \tfrac{1}{2} F (r - r\cos\gamma)]^2 d\gamma + \frac{\phi^2 r}{EA} \end{aligned}$$

$$= \frac{r^3}{EI} \int_{\gamma=0}^{\pi/2} [\phi \sin\gamma - \tfrac{1}{2} F(1-\cos\gamma)]^2 d\gamma + \frac{\phi^2 r}{EA}$$

$$= \frac{r^3}{EI} \int_{\gamma=0}^{\pi/2} [\phi^2 \sin^2\gamma - 2\phi \sin\gamma \tfrac{1}{2} F(1-\cos\gamma) + \tfrac{1}{4} F^2 (1-\cos\gamma)^2] d\gamma + \frac{\phi^2 r}{EA}$$

$$= \frac{r^3}{EI} \int_{\gamma=0}^{\pi/2} [\phi^2 \sin^2\gamma - \phi F \sin\gamma + \phi F \sin\gamma \cos\gamma + \tfrac{1}{4} F^2 - \tfrac{1}{2} F^2 \cos\gamma + \tfrac{1}{4} F^2 \cos^2\gamma] d\gamma + \frac{\phi^2 r}{EA}$$

$$= \frac{r^3}{EI} [\phi^2 \frac{\pi}{4} - \phi F + \phi F \frac{1}{2} + \tfrac{1}{4} F^2 \frac{\pi}{2} - \tfrac{1}{2} F^2 + \tfrac{1}{4} F^2 \frac{\pi}{4}] + \frac{\phi^2 r}{EA}$$

$$= \frac{r^3}{EI} [\phi^2 \tfrac{1}{4}\pi - \tfrac{1}{2}\phi F + F^2 (\tfrac{3}{16}\pi - \tfrac{1}{2})] + \frac{\phi^2 r}{EA}$$

Therefore, the answer provided in question **2c** is wrong. We continue with the correct answer.

d Redundant

$$0 = \frac{\partial E_{\text{compl}}}{\partial \phi}$$

$$0 = \frac{r^3}{EI} [\tfrac{1}{2}\phi\pi - \tfrac{1}{2} F] + 2\frac{\phi r}{EA}$$

$$\phi = \frac{F}{\dfrac{4EI}{r^2 EA} + \pi}$$

e Deflection

For this we assume that the deflection w of the arch top is imposed. The complementary energy becomes

$$E_{\text{compl}} = \frac{r^3}{EI} [\phi^2 \frac{1}{4}\pi - \frac{1}{2}\phi F + F^2 (\frac{3}{16}\pi - \frac{1}{2})] + \frac{\phi^2 r}{EA} - Fw$$

We can calculate the support reaction F by minimising the complimentary energy.

$$0 = \frac{\partial E_{\text{compl}}}{\partial F} = \frac{r^3}{EI} [-\frac{1}{2}\phi + 2F(\frac{3}{16}\pi - \frac{1}{2})] - w$$

We now know the relation between w and F. It does not matter any longer which has been imposed.

$$w = \frac{r^3 F}{EI} \left[\frac{3}{4}\pi - 1 - \frac{1}{2} \frac{r^2 EA}{4EI + \pi r^2 EA} \right]$$

f We would have found the same answers because the moment line of question **a** is not an approximation.

Technische Universiteit Delft
Faculteit der Civiele Techniek en Geowetenschappen
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
Vrijdag 27 oktober 2000, 14:00 –17:00 uur

Problem 1

A four-cell box-girder is loaded in torsion (Figure 1). All wall parts have the same thickness t. The centre to centre (Dutch: hart-op-hart) distances are all a. The thickness t can be considered very small compared to a.

a Which parts of the four cell box-girder can be neglected in calculating the torsion stiffness?

b How many weightless plates do we need to consider in calculating the torsion stiffness by the membrane analogy?

c Calculate the torsion stiffness.

d Determine the shear stresses in all wall parts due to a torsion moment M_t. Express the stresses in $M_t/(a^2t)$.

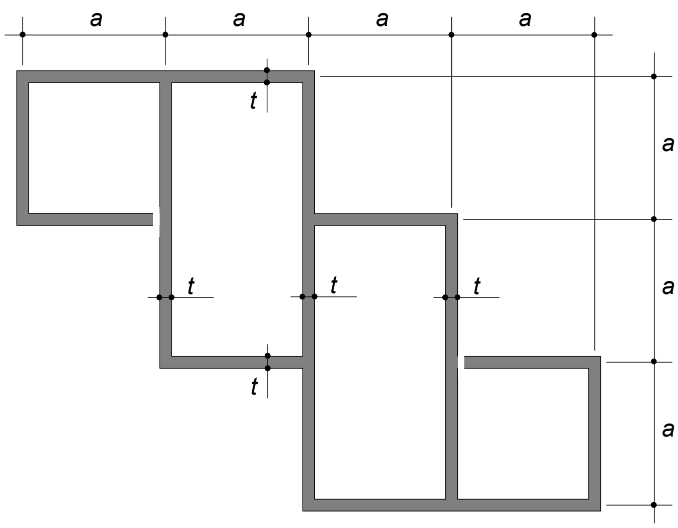


Figure 1. Cross-section of a box-girder

Problem 2

Consider an axial symmetrical plate which is loaded perpendicular to its plane (Figure 2). The curvature in the radial direction is $\kappa_{rr} = -\dfrac{d^2w}{dr^2}$. What is the curvature $\kappa_{\theta\theta}$ in the tangential direction?

A $\kappa_{\theta\theta} = 0$

B $\kappa_{\theta\theta} = -\dfrac{1}{r}\dfrac{dw}{dr}$

C $\kappa_{\theta\theta} = -\dfrac{w}{r^2}$

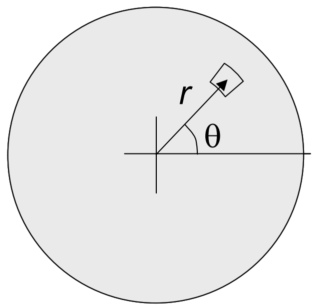


Figure 2. Axial symmetric plate

Problem 3

A non-prismatic beam is loaded by a moment M_1 (Figure 3). At the left end the beam is simply supported and at the right-hand side it is clamped. The bending stiffness EI varies according to

$$EI(x) = EI_1(1-\frac{x}{L}) + \tfrac{1}{2}EI_1\frac{x}{L}.$$

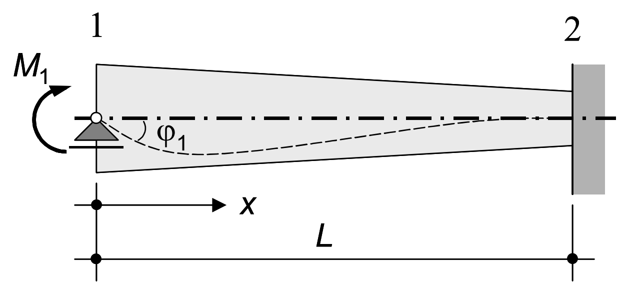


Figure 3. Non-prismatic beam

The following function is proposed for the moment line.

$$M(x) = M_1(1-\frac{x}{L}) + A\frac{x}{L}$$

where A is a constant that will be determined later.

a Draw the moment line. To do so make an estimate of constant A.

b Is the proposed moment line suitable or application in the principle of minimum complimentary energy? Is this moment line an approximation? Explain your answer.

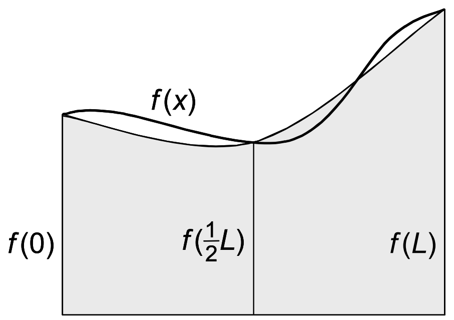
c Give the formula of the complimentary energy of the beam.

d Show that the complimentary energy can be evaluated to the following result. Use Simpson's rule (Figure 4).

$$E_{compl} = \frac{L}{EI_1}(0.194\ \ M_1^2 + 0.222\ \ M_1A + 0.278\ \ A^2)$$

e Calculate constant A.

f Calculate the rotation φ_1 of the left end of the beam.



$$\int_{x=0}^{L} f(x)dx = \tfrac{1}{6}L\left(f(0)+4f(\tfrac{1}{2}L)+f(L)\right)$$

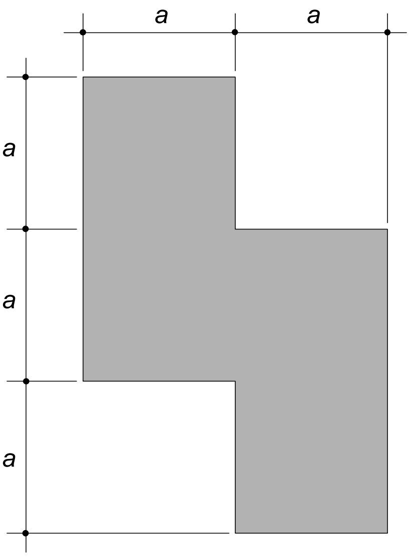
Figure 4. Simpson's rule for approximation of the integral of $f(x)$

Tentamen CTme5141, 27 oktober 2000
Answers to Problem 1 (4 points)

a The cells at the left top and right bottom are not closed. The contribution of these wall parts is calculated as that of a strip ($\sum l\,t^3 = 2x3\,a\,t^3$). This is much smaller than the contribution of the closed cells, therefore the open cells can be neglected.

b For the calculation is just one weightless plate required. After all, when we rotate the cross-section over π rad we obtain the same shape. Would the weightless plates of the closed cells have different displacements than the drawn cross-section and the rotated section would have different solutions. This is not possible, so the weightless plates will have the same displacements.

c We consider the weightless plate as draw below. The total circumference O of the plate is
$O = 2(2a) + 6(a) = 10a$

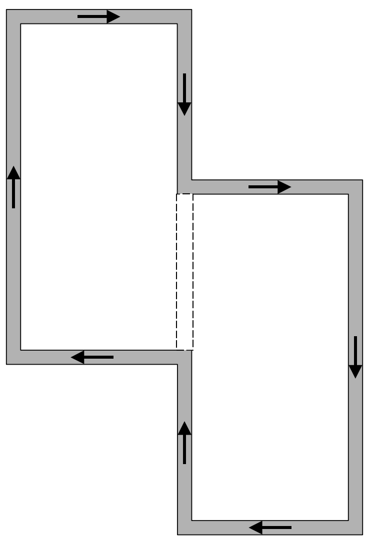


The surface A of the plate is
$A = 2(2a\;a) = 4a^2$

Equilibrium of the plate gives
$$\frac{w}{t} S\,10a = p\,4a^2$$
Therefore
$$w = \tfrac{2}{5}\frac{pat}{S}$$

Transition from w to ϕ.
$$p = 2\theta$$
$$S = \frac{1}{G}$$
$$\phi = \tfrac{4}{5} G\,at\,\theta$$

The torsion moment is two times the volume of the volume of the ϕ-bubble.
$$M_t = 2A\phi = 2(4a^2)\tfrac{4}{5}G\,at\,\theta = G\tfrac{32}{5}a^3 t\,\theta$$



We also know that $M_t = G I_t\,\theta$
Therefore $I_t = \tfrac{32}{5}a^3 t$

d The shear stress in the wall parts is equal to the slope of the ϕ-bubble. In all wall parts is the size of the shear stress the same.

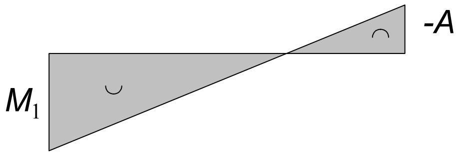
$$\sigma = \frac{\phi}{t} = \frac{\tfrac{4}{5}G\,at\,\theta}{t} = \tfrac{4}{5}G\,a\,\theta = \tfrac{4}{5}G\,a\frac{M_t}{G\tfrac{32}{5}a^3 t}$$
$$\sigma = \tfrac{1}{8}\frac{M_t}{a^2 t}$$

Answer to Problem 2 (1 point)

The correct answer is **B**. (Lecture book, Direct Methods, page 122)

Answers to Problem 3 (5 points)

a The constant A is the moment at the right-hand side of the beam. From the expected curvature of the beam we can conclude that A will be negative.



b Yes, the proposed moment line is suitable for complementary energy because it is in equilibrium with the loading M_1=M(0) and $\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -q = 0$. The moment line is not an approximation because it can describe the real moment line.

c In general the complimentary energy of beam consists of an internal part and an external part.

$$E_{compl} = \int_{x=o}^{L} \tfrac{1}{2}\frac{M^2}{EI}dx - F_1 u_{o2} - F_2 u_{o2} - M_2 \varphi_{o2}$$

where u_{o1}, u_{o2} and θ_{o2} are imposed displacements (free support and clamp) and F_1, F_2 and M_2 are the corresponding support reactions. The complimentary energy of the beam in this problem is

$$E_{compl} = \int_{x=o}^{L} \tfrac{1}{2}\frac{M^2}{EI}dx$$

After all, the imposed displacements u_{o1}, u_{o2} and θ_{o2} equal zero.

d Evaluation of the complimentary energy

$$E_{compl} = \int_{x=o}^{L} \tfrac{1}{2}\frac{\left(M_1(1-\frac{x}{L}) + A\frac{x}{L}\right)^2}{EI_1(1-\frac{x}{L}) + \frac{1}{2}EI_1\frac{x}{L}}dx$$

$$E_{compl} = \frac{1}{2EI_1}\int_{x=o}^{L} \frac{\left(M_1(1-\frac{x}{L}) + A\frac{x}{L}\right)^2}{1-\frac{1}{2}\frac{x}{L}}dx$$

$$E_{compl} = \frac{1}{2EI_1}\int_{x=o}^{L} f(x)dx \qquad f(x) = \frac{\left(M_1(1-\frac{x}{L}) + A\frac{x}{L}\right)^2}{1-\frac{1}{2}\frac{x}{L}}$$

$$f(0) = M_1^2$$

$$f(\tfrac{1}{2}L) = \tfrac{1}{3}(M_1 + A)^2$$

$$f(L) = 2A^2$$

$$E_{compl} \approx \frac{1}{2EI_1}\tfrac{1}{6}L\Big(f(0)+4f(\tfrac{1}{2}L)+f(L)\Big)$$

$$E_{compl} = \frac{1}{2EI_1}\tfrac{1}{6}L\Big(M_1^2+\tfrac{4}{3}(M_1+A)^2+2A^2)\Big)$$

$$E_{compl} = \frac{1}{2EI_1}\tfrac{1}{6}L(\tfrac{7}{3}M_1^2+\tfrac{8}{3}M_1A+\tfrac{10}{3}A^2)$$

$$E_{compl} = \frac{L}{EI_1}(0.194\, M_1^2+0.222\, M_1A+0.278\, A^2)$$

> **Intermezzo**
> The integral can also be calculated exactly. The result is
> $$E_{compl} = \frac{L}{2EI_1}[(2M_1-A)A+(M_1-2A)^2(2\ln(2)-1)]$$
> $$E_{compl} = \frac{L}{EI_1}(0.193\, M_1^2+0.227\, M_1A+0.273\, A^2)$$

e Complementary energy needs to be minimal with respect to the parameters of the moment line.

$$\frac{d\,E_{compl}}{d\,A}=0$$

From this A is solved.

$$\frac{d\,E_{compl}}{d\,A}=\frac{L}{EI_1}(0.222\, M_1+0.556\, A)=0$$

$$A=\frac{-0.222\, M_1}{0.556}=-0.399M_1$$

f The rotation φ_1 is not part of the expression of the complementary energy. Therefore we use a trick. We assume that φ_1 is imposed, so $\varphi_1=\varphi_{o1}$. The complementary energy becomes

$$E_{compl} = \frac{L}{EI_1}(0.194\, M_1^2+0.222\, M_1A+0.278\, A^2)-M_1\varphi_{o1}.$$

Two parameters determine the moment line. These are A and the support moment M_1. The complimentary energy again needs to be minimal with respect to these parameters.

$$\frac{d\,E_{compl}}{d\,A}=\frac{L}{EI_1}(0.222\, M_1+0.556\, A)=0$$

$$\frac{d\,E_{compl}}{d\,M_1}=\frac{L}{EI_1}(0.388\, M_1+0.222\, A)-\varphi_{o1}=0$$

In principle, form these two equations we can solve the unknown A and M_1. However, since we know the relation between φ_{o1} and M_1 it does not matter which was imposed and which was calculated. Therefore we can also write

$$A=\frac{-0.222\, M_1}{0.556}=-0.399M_1$$

$$\varphi_1=\frac{L}{EI_1}(0.388\, M_1+0.222\, A)=0.299\frac{M_1 L}{EI_1}$$

Technische Universiteit Delft
Faculteit der Civiele Techniek en Geowetenschappen
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
Dinsdag 9 januari 2001, 9:00 – 12:00 uur

Problem 1 (3 points)

A new software tool has become available for calculation of circular plates*. We want to check the program with a manual calculation. Therefore, the computation of Figure 1 has been performed.

Formulae

$$p = D\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}(r\frac{dw}{dr})))$$

$$m_{rr} = -D(w'' + \frac{\nu}{r}w')$$

$$m_{\vartheta\vartheta} = -D(\frac{1}{r}w' + \nu w'')$$

$$q_r = -D(w''' + \frac{1}{r}w'' - \frac{1}{r^2}w')$$

$$w' = \frac{pr^3}{16D} + 2rC_2 + \frac{C_3}{r} + (r + 2r\ln(r))C_4$$

$$w'' = \frac{3pr^2}{16D} + 2C_2 - \frac{C_3}{r^2} + (3 + 2\ln(r))C_4$$

$$w''' = \frac{3pr}{8D} + \frac{2C_3}{r^3} + \frac{2C_4}{r}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

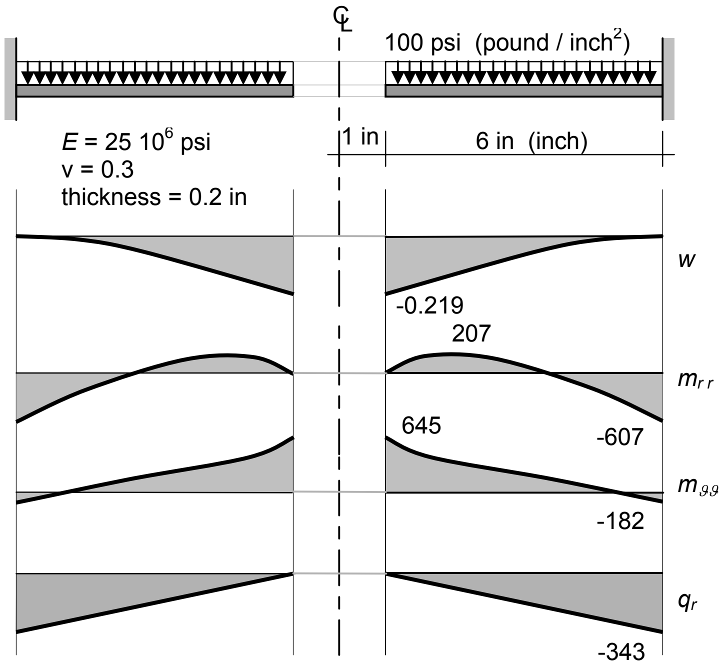


Figure 1. Computation results of the software tool

a What are the units of w, m_{rr}, $m_{\vartheta\vartheta}$ and q_r in Figure 1.

b Clearly an axial symmetrical plate can be modelled by a differential equation. The total solution of this differential equation is

$$w(r) = \frac{p\,r^4}{64D} + C_1 + C_2 r^2 + C_3 \ln(r) + C_4 r^2 \ln(r)$$

Give the boundary conditions that can be used to calculate the constants C_1 , C_2 , C_3 and C_4. Are the graphs of Figure 1 in agreement with these boundary conditions? Explain your answer.

c Derive the four equations from which the constants can be solved. (You can leave D, ν and p in the equations. You do not need to solve the constants.)

* WinPlate, Archon Engineering, Columbia, Missouri, U.S.A. Shareware, http://www.archoneng.com/

d The constants have been solved for you. The result is

$$C_1 = 0.2248 \quad C_2 = -0.006428 \quad C_3 = -0.02548 \quad C_4 = -0.0006825$$

Use this to check the largest deflection in Figure 1. Explain possible differences.

Problem 2 (4 points)

A prismatic tube with thick walls is loaded by torsion. The cross-section of the tube is square with dimension *h* and wall thickness *t* (Figure 2). We want to calculate the torsion stiffness and the largest shear stress. Therefore, we considerer a slice of length Δx.

The torsion moment causes shear stresses in the cross-section, which can be derived from a function ϕ . We select the following function as an approximation.

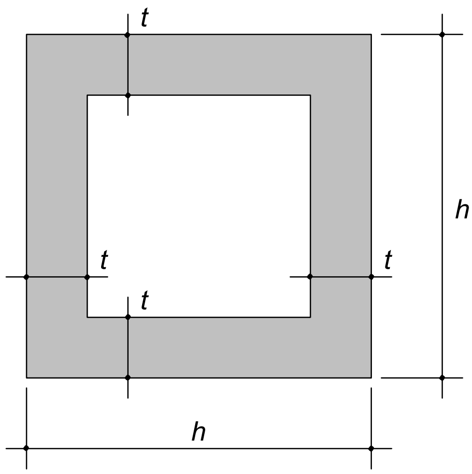


Figure 2. Cross-section of the tube

$$\phi = \begin{cases} A(1-4\dfrac{y^2}{h^2}) & |y| > |z| \quad \text{en} \quad \tfrac{1}{2}h - t \le |y| \le \tfrac{1}{2}h \\ A(1-4\dfrac{z^2}{h^2}) & |z| \ge |y| \quad \text{en} \quad \tfrac{1}{2}h - t \le |z| \le \tfrac{1}{2}h \\ A(1-(1-2\dfrac{t}{h})^2) & |y|,|z| < \tfrac{1}{2}h - t \end{cases}$$

where *A* is a jet unknown constant. Function ϕ has been drawn in Figure 3. The shear stresses in the cross-section are calculated by

$$\sigma_{xy} = \frac{\partial \phi}{\partial z}$$

$$\sigma_{xz} = -\frac{\partial \phi}{\partial y}$$

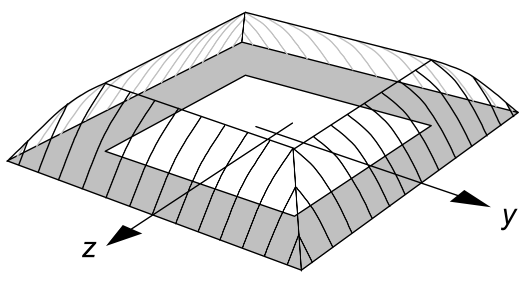


Figure 3. The ϕ bubble

a Use Figure 3 to show that ϕ fulfils the boundary conditions.

b Give the formula for calculating the resulting torsion moment (You do not need to evaluate the formula).

The formula has been evaluated for you with the following result.

$$M_w = Ah^2\left(1-(1-2\frac{t}{h})^4\right)$$

c Calculate the largest stress in the cross-section and express this in the torsion moment.

d Give the formula for calculating the complementary energy of the slice of length Δx (You do not need to evaluate the formula).

The formula has been evaluated for you with the following result

$$E_c = 4\frac{A^2}{G}\left(1-(1-2\frac{t}{h})^4\right)\Delta x$$

where G is the shear modulus of the material

e Calculate the torsion stiffness GI_w of the beam.
Suggestion: Make the complementary energy of the slice equal to the complementary energy of a part of wire frame model.

Problem 3 (3 points)

A statically indetermined truss is loaded by a concentrated load F (Figure 4). All bars have a cross-section area A and an elasticity modulus E. The bars are connected with hinges to the nodes. The diagonal bars are not connected in the middle. The complementary energy of the truss is

$$E_{compl} = \frac{l}{EA}\left((\tfrac{3}{4}+\tfrac{1}{2}\sqrt{2})F^2 + (\tfrac{5}{2}+2\sqrt{2})\phi^2 - (2+\tfrac{3}{2}\sqrt{2})F\phi\right)$$

a Explain the parameter ϕ in the equation of the complementary energy. How can ϕ be calculated?

b Calculate the deflection of the concentrated load.

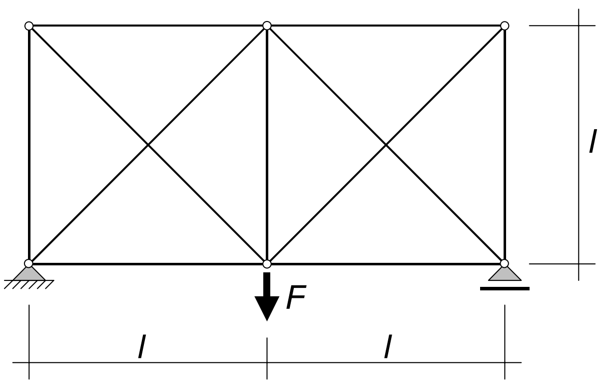


Figure 4. Statically indetermined truss

Tentamen Ctme5141, 9 januari 2001
Answers to Problem 1

a Units

w	inch; in
m_{rr}, $m_{\theta\theta}$	inch-pound / inch; pound; lb
q_r	pound / inch; lb / in

b Boundary Conditions

1. $w(7) = 0$
2. $\dfrac{\mathrm{d}w}{\mathrm{d}r}(7) = 0$
3. $m_{rr}(1) = 0$
4. $q_r(1) = 0$

In Figure 3 we see that the deflection line w at the clamped support (r = 7) does not have a slope. This agrees with boundary conditions 1 and 2. We also see that m_{rr} and q_r equal zero at the free edge (r = 1). This agrees with boundary conditions 3 and 4. Therefore the graphs fulfil the boundary conditions.

c System of Equations

$$m_{rr} = -D[w'' + \frac{\nu}{r}w']$$

$$= -D[\frac{3pr^2}{16D} + \frac{\nu}{r}\frac{pr^3}{16D} + 2C_2 + \frac{\nu}{r}2C_2 r - \frac{C_3}{r^2} + \frac{\nu}{r}\frac{C_3}{r} + 2C_4 \ln r + 2C_4 + C_4 + \frac{\nu}{r}2C_4 r \ln r + \frac{\nu}{r}C_4 r]$$

$$= -\frac{pr^2}{16}(\nu + 3) - 2D(\nu + 1)C_2 - \frac{D(\nu - 1)}{r^2}C_3 - D[2\ln r(\nu + 1) + \nu + 3]C_4$$

$$q_r = -D[w''' + \frac{1}{r}w'' - \frac{1}{r^2}w']$$

$$= -D[\frac{3pr}{8D} + 2\frac{C_3}{r^3} + \frac{2C_4}{r} + \frac{1}{r}\frac{3pr^2}{16D} + \frac{1}{r}2C_2 - \frac{1}{r}\frac{C_3}{r^2} + \frac{1}{r}2C_4 \ln r + \frac{1}{r}3C_4$$

$$- \frac{1}{r^2}\frac{pr^3}{16D} - \frac{1}{r^2}2C_2 r - \frac{1}{r^2}\frac{C_3}{r} - \frac{1}{r^2}2C_4 r \ln r - \frac{1}{r^2}C_4 r]$$

$$= -\frac{pr}{2} - \frac{4D}{r}C_4$$

$$\begin{cases} w(7) = \dfrac{p7^4}{64D} + C_1 + C_2 7^2 + C_3 \ln 7 + C_4 7^2 \ln 7 = 0 \\ w'(7) = \dfrac{p7^3}{16D} + 2C_2 7 + \dfrac{C_3}{7} + 2C_4 7 \ln 7 + C_4 7 = 0 \\ m_{rr}(1) = -\dfrac{p}{16}(\nu + 3) - 2D(\nu + 1)C_2 - D(\nu - 1)C_3 - D[2\ln 1(\nu + 1) + \nu + 3]C_4 = 0 \\ q_r(1) = -\dfrac{p}{2} - 4DC_4 = 0 \end{cases}$$

d Numbers

$$D=\frac{25*10^6*0.2^3}{12(1-0.3^2)}=18315 \text{ in-lb (Pronounce: inch-pound)}$$

$$w(1)=\frac{100*1^4}{64D}+C_1+C_2 1^2+C_3\ln 1+C_4 1^2\ln 1=0.218 \text{ in}$$

The graph shows a deflection of –0.219 in. Apparently the program uses a different positive sign convention than the lecture book. The small difference in the number is definitely caused by round off errors.

Encore (not an exam question)
The other values in the figure are checked below. From the figure we estimate the maximum moment at r = 2.2 in.

$$m_{rr}(2{,}2)=-\frac{100*2.2^2}{16}3.3-2*18315*1.3\ \ C_2-\frac{18315*-0.7}{2.2^2}\ \ C_3-18315\ \ [2\ln 2.2*1.3+3.3]\ \ C_4$$
$$=-99.83-47619\ \ C_2+2649\ \ C_3-97985\ \ C_4=205 \text{ Correct}$$

$$m_{rr}(7)=-\frac{100*\ \ 7^2}{16}3.3-2*1\ \ 8315\ \ *1.3\ \ C_2-\frac{18315*\ \ -0{,}7}{7^2}\ \ C_3-18315\ \ [2\ln 7*1.3+3.3]\ \ C_4$$
$$=-1010.6-47619\ \ C_2+261.6\ \ C_3-153102\ \ C_4=-607 \text{ Correct}$$

$$m_{\theta\theta}=-D[\frac{1}{r}w'+\nu w'']$$
$$=-D[\frac{1}{r}(\frac{pr^3}{16D}+2rC_2+\frac{C_3}{r}+(r+2r\ln(r))C_4)+\nu(\frac{3pr^2}{16D}+2C_2-\frac{C_3}{r^2}+(3+2\ln(r))C_4)]$$
$$=-D[\frac{pr^2}{16D}+2C_2+\frac{C_3}{r^2}+(1+2\ln(r))C_4+\frac{3\nu pr^2}{16D}+2\nu C_2-\frac{\nu C_3}{r^2}+\nu(3+2\ln(r))C_4]$$
$$=-D[(1+3\nu)\frac{pr^2}{16D}+2(1+\nu)C_2+\frac{1-\nu}{r^2}C_3+[1+2\ln(r)+\nu(3+2\ln(r))]C_4]$$
$$m_{\theta\theta}(1)=-18315\ \ [1.9\frac{100}{16*18315}+2.6\ \ C_2+0.7C_3+1.9\ \ C_4]=645 \text{ Correct}$$
$$m_{\theta\theta}(7)=-18315\ \ [\ 1{,}9\frac{100*7^2}{16*18315}+2.6\ \ C_2+\frac{0.7}{7^2}C_3+[1+2\ln(7)+0.3*(3+2\ln(7))]\ \ C_4]$$
$$=-581.9-47619\ \ C_2-261.6\ \ C_3-127461\ \ C_4$$
$$=-182 \text{ Klopt}$$

$$q_r(7)=-\frac{100*7}{2}-\frac{4*18315}{7}C_4=-343 \qquad \text{Correct too}$$

Answers to Problem 2

a Boundary Conditions

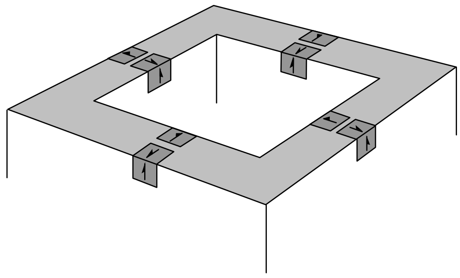
At the internal and external edges the shear stresses that are perpendicular to the edge should be zero.

$$\sigma_{xy} = 0 \text{ if } |y| = \tfrac{1}{2}h \text{ and } |z| \le |y|,\ |y| = \tfrac{1}{2}h - t \text{ and } |z| \le |y|$$

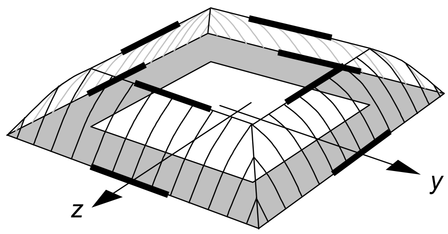
$$\sigma_{xz} = 0 \text{ if } |z| = \tfrac{1}{2}h \text{ and } |y| \le |z|,\ |z| = \tfrac{1}{2}h - t \text{ and } |y| \le |z|$$

This follows from moment equilibrium of infinitesimal cubes in the edges.

The formulae of the stress shows that the shear stress is zero when the ϕ bubble does not have a slope in the direction perpendicular to the stress. The thick lines in the right-hand figure show that the ϕ bubble correctly does not have a slope at the edges. Therefore, the boundary conditions are fulfilled.



Equilibrium of infinitesimal cubes in the edges of the tube



The ϕ bubble does indeed not have a slope at the edges of the tube.

b Torsion Moment
The torsion moment is the resultant of the shear stress over the cross-section area.

$$M_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} y\sigma_{xz}dydz - z\sigma_{xy}dydz$$

As shown in the lecture book, this equals two times the volume of the ϕ bubble (Direct Methods, page 169).

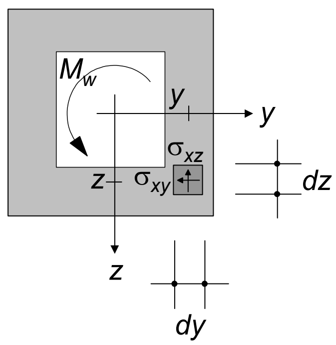
c Largest Stress
The stress is largest where the slope of the ϕ bubble is largest. For example, σ_{xz} is largest when y = h/2.

$$\sigma_{xz} = -\frac{\partial \phi}{\partial y} = \begin{cases} A8\dfrac{y}{h^2} & |y| > |z| \text{ en } \tfrac{1}{2}h - t \le |y| \le \tfrac{1}{2}h \\ 0 & |z| \ge |y| \text{ en } \tfrac{1}{2}h - t \le |z| \le \tfrac{1}{2}h \\ 0 & |y|,|z| < \tfrac{1}{2}h - t \end{cases}$$

$$\sigma_{xz}(y = \tfrac{1}{2}h, z = z) = 4\frac{A}{h}$$

The expression for the torsion moment can be reworked as

$$A = \frac{M_t}{h^2\left(1-(1-2\frac{t}{h})^4\right)}$$



This makes the largest stress

$$\sigma = \frac{4M_t}{h^3}\frac{1}{1-(1-2\frac{t}{h})^4}$$

d Complimentary Energy

The complementary energy is the integral over a slice Δx of the energy in all particles. The force at one particle is $\sigma\, dy\, dz$ and the displacement of this force is $\gamma\,\Delta x$.

$$E_c = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tfrac{1}{2}\sigma_{xy} dy\, dz\, \gamma_{xy}\Delta x + \tfrac{1}{2}\sigma_{xz} dy\, dz\, \gamma_{xz}\Delta x$$

e Torsion Stiffness

The torsion moment is

$$M_t = GI_t\theta$$

The energy in a length Δx of this model is

$$E_c = \tfrac{1}{2}M_t\theta\Delta x = \tfrac{1}{2}\frac{M_t^2}{GI_t}\Delta x$$

The energy of the slice can also be expressed in the torsion moment

$$E_c = 4\frac{A^2}{G}\left(1-(1-2\frac{t}{h})^4\right)\Delta x = 4\frac{\dfrac{M_t^2}{h^4\left(1-(1-2\frac{t}{h})^4\right)^2}}{G}\left(1-(1-2\frac{t}{h})^4\right)\Delta x =$$

$$= \frac{4M_t^2}{Gh^4}\frac{1}{1-(1-2\frac{t}{h})^4}\Delta x$$

The energy of a part of the wire frame model should be equal to the energy of the slice.

$$\tfrac{1}{2}\frac{M_t^2}{GI_t}\Delta x = \frac{4M_t^2}{Gh^4}\frac{1}{1-(1-2\frac{t}{h})^4}\Delta x$$

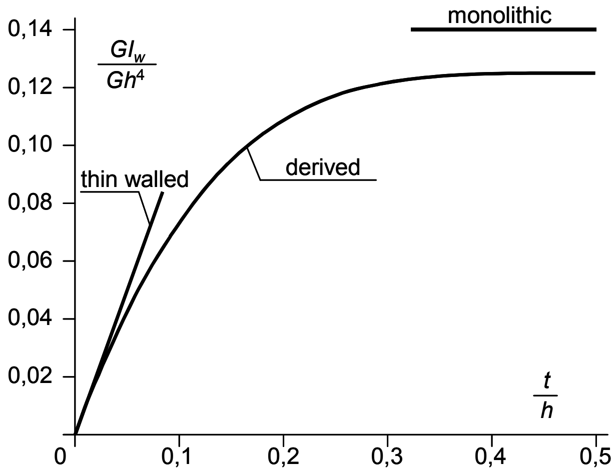
Or

$$GI_t = G\tfrac{1}{8}(h^4-(h-2t)^4)$$

Encore (not an exam question)

The calculated tube is situated between two extreme tubes. If the walls are thin ($t << h$) we can calculate the torsion stiffness simply using a weightless plate $GI_t = t\,h^3G$. If the cross-section is monolithic ($t = h/2$) we can find the torsion stiffness in a table $GI_t = 0.141\ \ h^4G$

(See lecture book Direct Methods at page 183) (See also this book page 25). With this we can plot the following graph.



The derived formula appears to be accurate for thin and thick wall tubes. However, the formula is not accurate for very thick wall tubes and monolith cross-sections.

Answers to Problem 3

a Parameter ϕ

Parameter ϕ is the redundant (Dutch: statisch onbepaalde) in the force flow of the truss. ϕ can be solved from the equation

$$\frac{dE_{compl}}{d\phi} = 0 .$$

b Deflection

We impose the deflection u^o. The complementary energy becomes

$$E_{compl} = \frac{l}{EA}\left((\tfrac{3}{4} + \tfrac{1}{2}\sqrt{2})F^2 + (\tfrac{5}{2} + 2\sqrt{2})\phi^2 - (2 + \tfrac{3}{2}\sqrt{2})F\phi\right) - Fu^o .$$

The force F is an unknown support reaction, which also determines the force flow. The complementary energy needs to be minimal with respect to the parameters that determine the force flow.

$$\begin{cases} \dfrac{dE_{compl}}{d\phi} = 0 \\ \dfrac{dE_{compl}}{dF} = 0 \end{cases}$$

From this the deflection can be solved.

$$\begin{cases} \dfrac{l}{EA}\left(2(\tfrac{5}{2} + 2\sqrt{2})\phi - (2 + \tfrac{3}{2}\sqrt{2})F\right) = 0 \\ \dfrac{l}{EA}\left(2(\tfrac{3}{4} + \tfrac{1}{2}\sqrt{2})F - (2 + \tfrac{3}{2}\sqrt{2})\phi\right) - u^o = 0 \end{cases}$$

$$\begin{cases} (5+4\sqrt{2})\phi = (2+\frac{3}{2}\sqrt{2})F \\ \frac{l}{EA}\left((\frac{3}{2}+\sqrt{2})F-(2+\frac{3}{2}\sqrt{2})\phi\right) = u^o \end{cases}$$

$$\frac{l}{EA}\left((\frac{3}{2}+\sqrt{2})F-(2+\frac{3}{2}\sqrt{2})\frac{2+\frac{3}{2}\sqrt{2}}{5+4\sqrt{2}}F\right) = u^o$$

$$u^o = \frac{7+5\sqrt{2}}{5+4\sqrt{2}}\frac{F\ \ l}{EA} = 1.32\frac{F\ \ l}{EA}$$

Technische Universiteit Delft
Faculteit der Civiele Techniek en Geowetenschappen
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
Vrijdag 26 oktober 2001, 14.00 – 17.00 uur

Problem 1 (4 points)

Consider a symmetric structural system consisting of three high walls in one plane (Figure 1). The extension stiffness' are EA_1, $2EA_2$ and EA_1 respectively. The walls are supported by springs at the bottom. (pile foundation). The spring stiffness' are K_1, $2K_2$ and K_1 respectively. The two rows of horizontal connection beams between the walls are each modelled as a system of distributed springs of stiffness k. The height of the structural system is l. The x-as start at the bottom of the wall (x=0) and end at the top of the wall (x=l).

The two outside walls have a temperature of T degrees higher that the middle wall. The linear extension coefficient of the walls is α.

A structural designer wants to calculate the shear force s in the connection beams. The structural designer uses symmetry and uses only the left part of the structure.

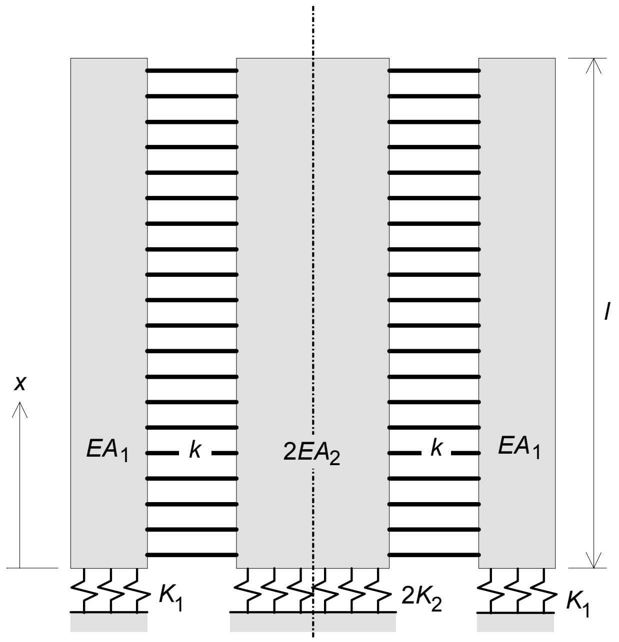


Figure 1. Three connected high walls

a Give the framework that shows the degrees of freedom, deformations, stress quantities and loading in the structural system. (Make drawings that explain the quantities and their positive directions).

b Which is most suitable to solve this problem, the force method of the displacement method? Explain your answer.

c Formulate for the structural system
- the kinematic equations
- the constitutive equations
- the equilibrium equations

(if need be supplement the drawings of question **a**).

d The structural designer uses the <u>displacement method</u>.
Using the kinematic, constitutive and equilibrium equations he derives two differential equations. The solution of the differential equation has four coefficients that need to be determined by four boundary conditions. Two boundary conditions occur at the top ($x = l$) of the wall and two boundary conditions occur at the bottom ($x = 0$) of the wall.

Formulate the boundary conditions at the top of the wall? Evaluate these to an equation in the wall displacements u_1 and u_2.

e Formulate the boundary conditions at the bottom of the wall. (Draw a small finite slice at the bottom of each wall with a height d. Thus, the slice goes from x=0 to x=d. d is small compared to l. Draw the forces on the slices. Formulate the equilibrium conditions of the slices and subsequently let d approach to zero.) Evaluate the equilibrium equations to equations in the wall displacements u_1 and u_2.

f Bonus Question (one point if completely correct)
Two descending exponential functions appear in the solution of the displacements and stress quantities. One of them is descending from the bottom of the system and the other from the top of the system. We select the stiffness such that exponential functions can be neglected at a distance of $l/3$ form the bottom or the top respectively.

Questions
- Consider the situation that K_1 is infinitely large and K_2 has a finite value. Draw the expected distribution of the normal forces in the walls and shear forces in the connection beam. (You do not need to calculate something. It is sufficient to just reason). Write down whether the normal forces are tension or compression.
- Draw the direction of the shear force s in the distributed springs on each of the walls.
- Write down on which position(s) the normal forces N_1 and N_2 (in absolute sense) are equal.

Problem 2 (3 points)

A hollow core slab (Dutch: kanaalplaat) with 11 cells is modelled as a thin wall cross-section (Figure 3). Just six webs are taken into account in the model. All walls have a thickness t.

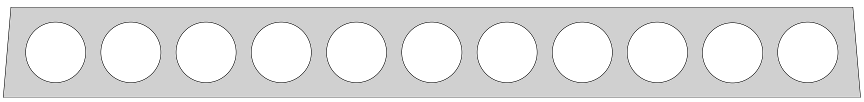


Figure 2. Cross-section of a hollow core slab

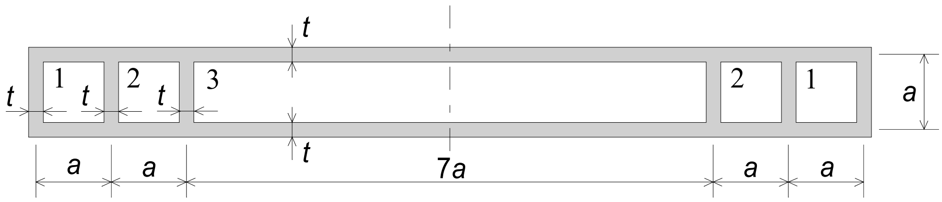


Figure 3. Model of the hollow core slab for calculation of the torsion properties

a Why are some of the webs left out of the model?

b Formulate the equations for calculation of the torsion stiffness according to the membrane analogy. Use symmetry. (You do not need to evaluate the equations.)

c The equations are evaluated for you with the following result.

$$w_1 = \frac{85}{232}\frac{pat}{s} \qquad w_2 = \frac{108}{232}\frac{pat}{s} \qquad w_3 = \frac{115}{232}\frac{pat}{s}$$

where w_1 is the displacement of the plate above cell 1, w_2 that of cell 2 and w_3 that of cell 3. s is the membrane stress and p is the pressure underneath the weightless plates. Calculate the torsion stiffness GI_t of the cross-section.

d Calculate the shear stresses in the cross-section as function of the torsion moment. Draw the shear stresses in the correct direction.

Problem 3 (3 points)

A square plate of thickness *t* carries an evenly distributed load *q* (Figure 4.). At each edge the plate is simply supported. According to elasticity theory we assume that the kinematic, constitutive and equilibrium equations are linear. As you know an exact solution of the deflection of this plate does not exist. However, it can be approximated. A calculation using potential energy is very suitable because we can accurately estimate the deformation of the plate.

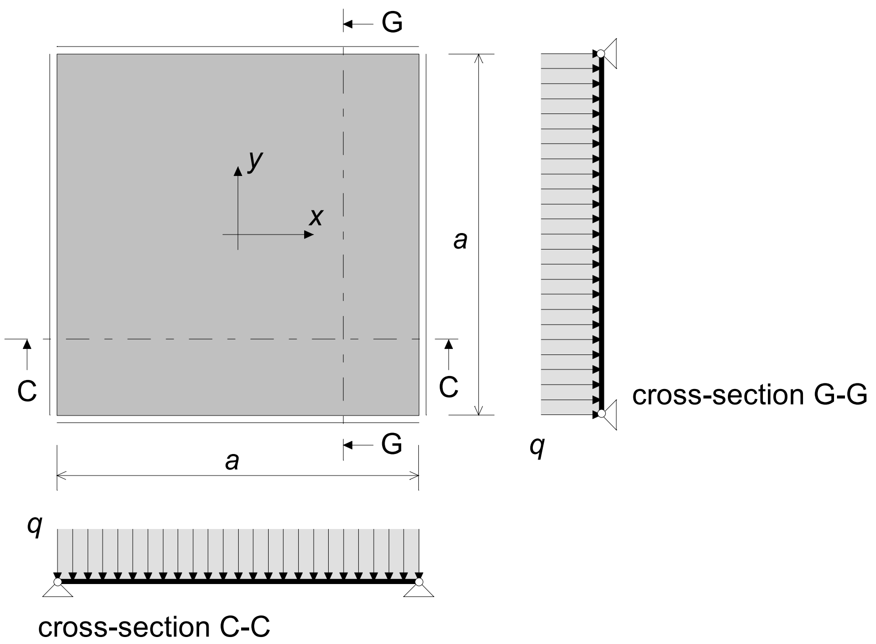


Figure 4. Square plate with distributed load

The following deflection function is assumed

$$w(x,y) = A(1-\left(\frac{2x}{a}\right)^2)(1-\left(\frac{2y}{a}\right)^2) + B(1-\left(\frac{2x}{a}\right)^4)(1-\left(\frac{2y}{a}\right)^4)$$

where *A* and *B* are coefficients that need to be determined later.

a Show that the deflection function fulfils the kinematic boundary conditions of the plate.

b Give the formula for calculation of the potential energy of the plate. (You do not need to evaluate the formula.)

c Using the deflection function the potential energy E_{pot} of the plate can be calculated. This has been done for you. The following equations have been used.

$$m_{xx} = D(\kappa_{xx} + \nu\kappa_{yy}) \qquad \kappa_{xx} = -\frac{\partial^2 w}{\partial x^2} \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

$$m_{yy} = D(\kappa_{yy} + \nu\kappa_{xx}) \qquad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$m_{xy} = \tfrac{1}{2}D(1-\nu)\rho_{xy} \qquad \rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

The result is

$$E_{\text{pot}} = \tfrac{256}{11025}\frac{D}{a^2}(17712B^2 + 10248BA + 2695A^2) - \tfrac{4}{225}qa^2(36B + 25A)$$

Solve the coefficients A and B from the equation of the potential energy.

d Calculate the deflection in the middle of the plate as a function of a, D and q.

e Since now A and B are known the potential energy of the plate can be calculated. This has been done for you with the following result.

$$E_{\text{pot}} = -\tfrac{103}{121760}\frac{q^2a^6}{D}$$

This potential energy is an approximation of the exact potential energy because the assumed deflection function in the beginning of this problem is an approximation.

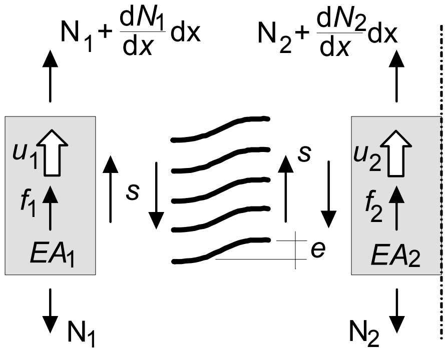
Will the exact potential energy of the plate be larger of smaller than the approximation? Explain your answer. (You do not need to make calculations.)

Tentamen Ctme5141, 26 oktober 2001
Answers to Problem 1

a <u>Framework</u>

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ e \end{bmatrix} \quad \begin{bmatrix} N_1 \\ N_2 \\ s \end{bmatrix} \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

kinematic equations — constitutive equations — equilibrium equations



(The positive directions of *s* and *e* can also be chosen differently.)

b <u>Method</u>
The force method is most suitable. The displacement method results in two differential equations (because two degrees of freedom, so two equilibrium equations). The force method results in one differential equation. (because one redundant, because two equilibrium equations and three stress quantities).

c <u>Equations</u>
Kinematic equations

$$\varepsilon_1 = \frac{du_1}{dx}$$

$$\varepsilon_2 = \frac{du_2}{dx}$$

$$e = u_2 - u_1$$

Constitutive equations

$$N_1 = EA_1(\varepsilon_1 - \alpha T)$$

$$N_2 = EA_2\,\varepsilon_2$$

$$s = k\,e$$

Equilibrium equations

$$\frac{dN_1}{dx} + s + f_1 = 0$$

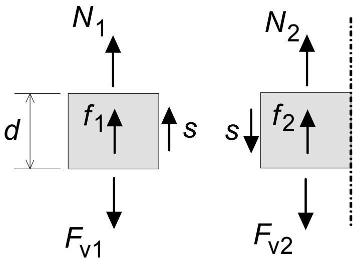
$$\frac{dN_2}{dx} - s + f_2 = 0 \qquad f_1 = f_2 = 0$$

d <u>Boundary Conditions $x = l$</u>

$$N_1(l) = 0 \rightarrow \varepsilon_1(l) = \alpha T \rightarrow \quad \boxed{\begin{array}{l} \left.\dfrac{du_1}{dx}\right|_{x=l} = \alpha T \\ \left.\dfrac{du_2}{dx}\right|_{x=l} = 0 \end{array}}$$

$$N_2(l) = 0 \rightarrow \varepsilon_2(l) = 0 \rightarrow$$

e Boundary Conditions $x = 0$



$$N_1 + d(f_1 + s) - F_{v1} = 0$$
$$N_2 + d(f_2 - s) - F_{v2} = 0$$

$\lim d \rightarrow 0$:

$$N_1(0) - F_{v1} = 0 \quad \text{met} \quad N_1(0) = EA_1\left(\left.\frac{du_1}{dx}\right|_{x=0} - \alpha T\right) \quad \text{en} \quad F_{v1} = K_1 u_1(0)$$

$$N_2(0) - F_{v2} = 0 \quad \text{met} \quad N_2(0) = EA_2 \left.\frac{du_2}{dx}\right|_{x=0} \quad \text{en} \quad F_{v2} = K_2 u_2(0)$$

Result:

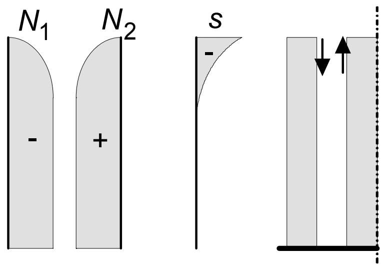
$$\boxed{\begin{array}{l} EA_1\left(\left.\dfrac{du_1}{dx}\right|_{x=0} - \alpha T\right) - K_1 u_1(0) = 0 \\ EA_2 \left.\dfrac{du_2}{dx}\right|_{x=0} - K_2 u_2(0) = 0 \end{array}}$$

f Bonus Question

The described situation is in between two extremes.

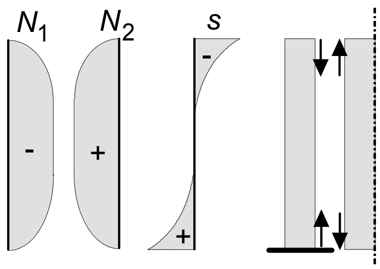
Extreme 1

$K_1 = \infty$ and $K_2 = \infty$ This situation is described in the lecture book (Direct Methods, page 36).

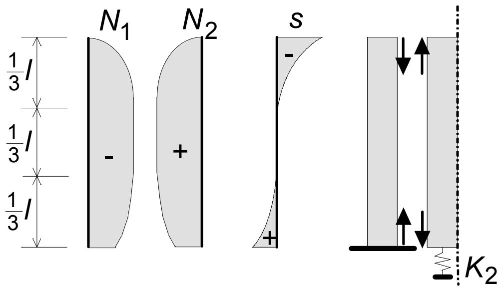


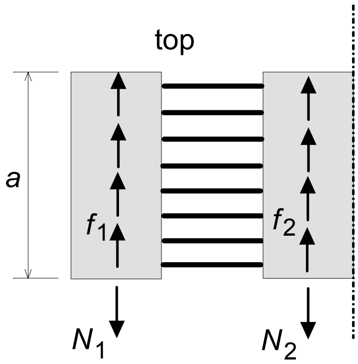
Extreme 2

$K_1 = \infty$ and $K_2 = 0$ The behaviour at the top and bottom is the same.



In between
$K_1 = \infty$ and K_2 has a finite value The result is in between the extremes.





Equilibrium of the top of the walls gives

$$\int_{x=l-a}^{l} (f_1 + f_2)\,dx - N_1 - N_2 = 0$$

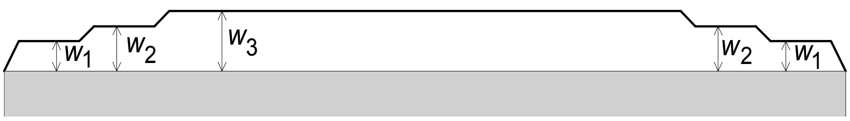
Since $f_1 = f_2 = 0$, it holds $|N_1(x)| = |N_2(x)|$. So, the normal forces are equal in absolute sense over the total height.

Answer to Problem 2

a Flanges

Some of the middle webs have been left out for two reasons. 1) They probably contribute little to the torsion properties. 2) The number of equations that need to be solved is now far less.

b Equations



Equilibrium of the plate above cell 1 $\quad pa^2 = 3as\frac{w_1}{t} - as\frac{w_2 - w_1}{t}$

Plate 2 $\quad pa^2 = as\frac{w_2 - w_1}{t} + 2as\frac{w2}{t} - as\frac{w_3 - w_2}{t}$

Plate 3 $\quad pa7a = 2as\frac{w_3 - w_2}{t} + 2(7a)s\frac{w_3}{t}$

c Torsion Stiffness

The membrane is transformed into the ϕ bubble by the following substitutions.

$$w = \phi \qquad p = 2\theta \qquad s = \frac{1}{G}$$

Therefore

$$\phi_1 = \tfrac{85}{232} 2\theta Gat \qquad \phi_2 = \tfrac{108}{232} 2\theta Gat \qquad \phi_3 = \tfrac{115}{232} 2\theta Gat$$

The torsion moment is two times the volume of the ϕ bubble.

$$M_t = 2\left(2\phi_1 a^2 + 2\phi_2 a^2 + \phi_3 a(7a)\right)$$
$$M_t = 2\left(2\tfrac{85}{232} 2\theta Gat\, a^2 + 2\tfrac{108}{232} 2\theta Gat\, a^2 + \tfrac{115}{232} 2\theta Gat\, a(7a)\right)$$
$$M_t = \tfrac{1191}{58} \theta G a^3 t$$

The moment in a wire frame model of the beam is

$$M_t = G I_t \theta$$

Therefore the torsion stiffness is

$$G I_t = \tfrac{1191}{58} G a^3 t$$

d <u>Shear Stresses</u>

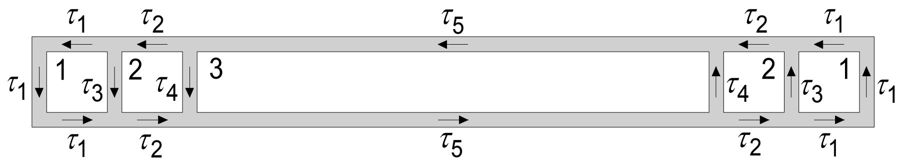
The shear stress is the slope of the ϕ bubble. First we rewrite the equation of the torsion moment

$$\theta Gat = \frac{M_t}{a^2} \frac{58}{1191}$$

and express ϕ_1, ϕ_2 and ϕ_3 in the torsion moment.

$$\phi_1 = \tfrac{85}{232} 2 \frac{M_t}{a^2} \frac{58}{1191} \qquad \phi_2 = \tfrac{108}{232} 2 \frac{M_t}{a^2} \frac{58}{1191} \qquad \phi_3 = \tfrac{115}{232} 2 \frac{M_t}{a^2} \frac{58}{1191}$$
$$\phi_1 = \tfrac{85}{2382} \frac{M_t}{a^2} \qquad \phi_2 = \tfrac{108}{2382} \frac{M_t}{a^2} \qquad \phi_3 = \tfrac{115}{2382} \frac{M_t}{a^2}$$

The shear stresses become

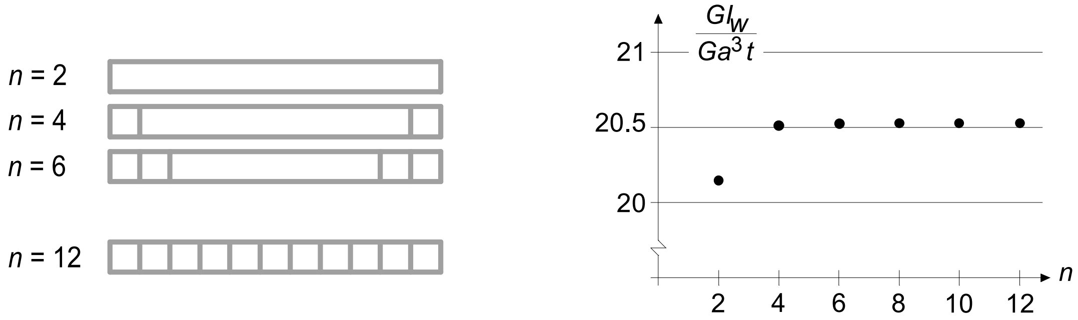


$$\tau_1 = \frac{\phi_1}{t} = \frac{85}{2382} \frac{M_t}{a^2 t}$$
$$\tau_2 = \frac{\phi_2}{t} = \frac{108}{2382} \frac{M_t}{a^2 t}$$
$$\tau_3 = \frac{\phi_2 - \phi_1}{t} = \frac{23}{2382} \frac{M_t}{a^2 t}$$

$$\tau_4 = \frac{\phi_3 - \phi_2}{t} = \frac{7}{2382}\frac{M_t}{a^2 t}$$

$$\tau_5 = \frac{\phi_3}{t} = \frac{115}{2382}\frac{M_t}{a^2 t}$$

Encore (not an exam question)
In the figure below the torsion stiffness is plotted as a function of the number of webs n in the model. It shows that a model with 4 flanges is sufficiently accurate for calculating the torsion stiffness.



The largest shear stress converges less quickly with increasing n (not plotted). The largest shear stress in the model with 12 webs is $\tau_{max} = \frac{225}{4624}\frac{M_t}{a^2 t}$ which is only 0.8% larger than that of the model with 6 webs [1]. Therefore, the model with 6 webs is more than sufficiently accurate.

Answers to Problem 3

a Boundary Conditions
The plate is simply supported on all edges. Therefore, the deflection w must be zero at the edges.

$$w(\tfrac{1}{2}a, y) = A(\tfrac{1}{4} - \frac{(\frac{1}{2}a)^2}{a^2})(\tfrac{1}{4} - \frac{y^2}{a^2}) + B(\tfrac{1}{16} - \frac{(\frac{1}{2}a)^4}{a^4})(\tfrac{1}{16} - \frac{y^4}{a^4}) = 0$$

$$w(-\tfrac{1}{2}a, y) = A(\tfrac{1}{4} - \frac{(-\frac{1}{2}a)^2}{a^2})(\tfrac{1}{4} - \frac{y^2}{a^2}) + B(\tfrac{1}{16} - \frac{(-\frac{1}{2}a)^4}{a^4})(\tfrac{1}{16} - \frac{y^4}{a^4}) = 0$$

$$w(x, \tfrac{1}{2}a) = A(\tfrac{1}{4} - \frac{x^2}{a^2})(\tfrac{1}{4} - \frac{(\frac{1}{2}a)^2}{a^2}) + B(\tfrac{1}{16} - \frac{x^4}{a^4})(\tfrac{1}{16} - \frac{(\frac{1}{2}a)^4}{a^4}) = 0$$

$$w(x, -\tfrac{1}{2}a) = A(\tfrac{1}{4} - \frac{x^2}{a^2})(\tfrac{1}{4} - \frac{(-\frac{1}{2}a)^2}{a^2}) + B(\tfrac{1}{16} - \frac{x^4}{a^4})(\tfrac{1}{16} - \frac{(-\frac{1}{2}a)^4}{a^4}) = 0$$

The kinematic boundary conditions are correctly satisfied.

It is noted that the dynamic boundary conditions do not need to be fulfilled for application of the principle of minimum potential energy.

b Potential Energy

$$E_{\text{pot}} = \tfrac{1}{2}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}(m_{xx}\kappa_{xx} + m_{yy}\kappa_{yy} + m_{xy}\rho_{xy})\,dx\,dy - q\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} w\,dx\,dy$$

c <u>Coefficients</u>

For the correct displacement field the potential energy is minimal.

$$\begin{cases} \dfrac{\partial E_{\text{pot}}}{\partial A} = \tfrac{256}{11025}\dfrac{D}{a^2}(10248B + 5390A) - \tfrac{4}{225}qa^2 25 = 0 \\ \dfrac{\partial E_{\text{pot}}}{\partial B} = \tfrac{256}{11025}\dfrac{D}{a^2}(35424B + 10248A) - \tfrac{4}{225}qa^2 36 = 0 \end{cases}$$

From these two equations the two coefficients A and B are solved.

$$\begin{cases} 535{,}4B + 281.6A = \dfrac{qa^4}{D} \\ 1285{,}2B + 371.8A = \dfrac{qa^4}{D} \end{cases}$$

$$\begin{cases} 535.4B + 281.6A = \dfrac{qa^4}{D} \\ 535.4B + 154.9A = 0.4166\dfrac{qa^4}{D} \end{cases}$$

$$\begin{cases} 535.4B + 281.6A = \dfrac{qa^4}{D} \\ 126.7A = 0.5834\dfrac{qa^4}{D} \end{cases}$$

$$\begin{cases} 535.4B = \dfrac{qa^4}{D} - 281.6A \\ A = 0.004605\dfrac{qa^4}{D} \end{cases}$$

$$\begin{cases} 535.4B = (1 - 281.6 \times 0.004605)\dfrac{qa^4}{D} = -0.2968\dfrac{qa^4}{D} \\ A = 0.004605\dfrac{qa^4}{D} \end{cases}$$

$$\begin{cases} B = -0.000554\dfrac{qa^4}{D} \\ A = 0.00460\dfrac{qa^4}{D} \end{cases}$$

d <u>Deflection</u>

$$w(0,0) = A(1-\left(\frac{0}{a}\right)^2)(1-\left(\frac{0}{a}\right)^2) + B(1-\left(\frac{0}{a}\right)^4)(1-\left(\frac{0}{a}\right)^4) = A + B$$

$$w(0,0) = 0.00460\frac{qa^4}{D} - 0.000554\frac{qa^4}{D}$$

$$w(0,0) = 0.00405 \frac{qa^4}{D}$$

e Larger or Smaller
We are looking for a solution with the smallest potential energy. The potential energy of a good approximation is small. The potential energy of the exact solution is smaller.

Encore (not an exam question)
Using the coefficients *A* and *B* we can calculate the moments in the middle of the plate.

$$m_{xx} = \tfrac{897}{24352} qa^2 (1+\nu)$$
$$m_{yy} = \tfrac{897}{24352} qa^2 (1+\nu)$$
$$m_{xy} = 0$$

Moments in plates have been published for many plate shapes and dimensions. Often these calculations have been performed using the finite difference method. (To date the finite element method would have been used.) For example in [2] we find the following formula for the plate of problem 2.

$$m_{xx} = 0.03676\, qa^2$$

If we assume that ν = 0 this moment is only 0.2% smaller than the calculated result.

Literature

[1] L. Span, “Grote openingen in kanaalplaten” (“Large Openings in Hollow Core Slab Floors”), Preliminary Graduation Report, Delft University of Technology, May 2001 (In Dutch).

[2] K. Stiglat, H. Wippel, “Platten”, Zweite Auflage, Verlag von Wilhelm Ernst & Sohn, Berlin 1973, p. 158 (In German).

Technische Universiteit Delft
Faculteit der Civiele Techniek en Geowetenschappen
Mechanica & Constructies

Vermeld rechtsboven op uw werk:
Naam en Studienummer

Tentamen Elasticiteitstheorie Ctme5141
Dinsdag 8 januari 2002, 9.00 – 12.00 uur

Problem 1 (3 points)

Consider a symmetric system consisting of three high walls that are in one plane (Figure 1). The extension stiffness' are respectively EA_1, $2EA_2$ and EA_1. The walls are supported by springs at the bottom (pile foundation). The spring stiffnesses' are respectively K_1, $2K_2$ and K_1. Two rows of horizontal connection beams between the walls are each modelled as distributed springs of stiffness k. The height of the structural system is l. The x-axis starts at the bottom of the walls (x=0) and ends at the top (x=l).

The two outer walls have a temperature that is T degrees higher than the middle wall. The linear extension coefficient of the walls is α.

A structural designer wants to calculate the shear force s in the connection beams. The structural designer uses symmetry and considers only the left part of the system.

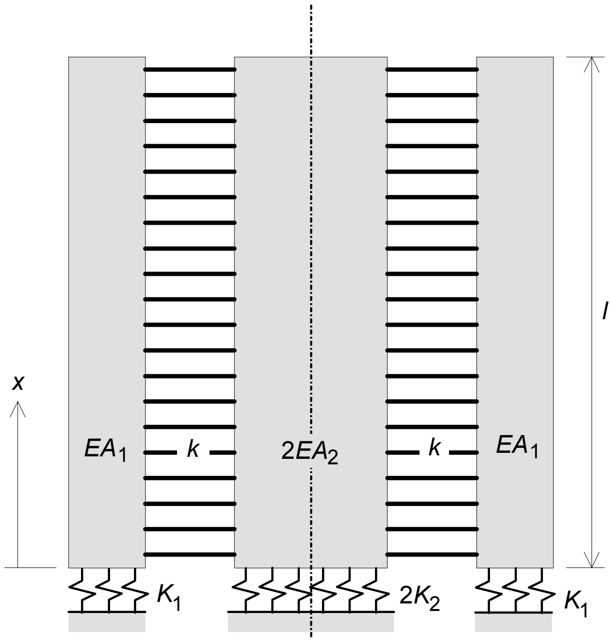


Figure 1. Three connected high walls

a Give the framework which shows the degrees of freedom, deformations, stress quantities and load quantities of the structural system. (Make drawings that explain the quantities and show their positive directions).

b Is the force method or the displacement method most suitable to solve this problem? Explain your answer.

c Write down the
- the kinematic equations
- the constitutive equations
- the equilibrium equations

(If need be supplement the drawings that were made for question **a**).

d The structural designer continues with the <u>force method</u>
- Propose the redundant(s) and show how the stress quantities can be expressed in the redundant.
- Derive the compatibility condition(s) from the kinematic equations.

e The structural designer derives one differential equation based on the redundant(s) and the compatibility conditions(s). (You do not need to derive this differential equation.) Two coefficients occur in the solution to the differential equation. These need to be determined by two boundary conditions. One boundary condition can be found in the top of the system (x = l) and on boundary condition can be found at the bottom (x = 0).

Formulate the boundary condition in the top of the system. Evaluate this to an equation in the redundant(s).

f Formulate the boundary condition at the bottom of the system. Evaluate this to an equation in the redundant(s).

g Bonus Question (one point if completely correct)
Two descending exponential functions appear in the solution of the displacements and stress quantities. One of them is descending from the bottom of the system and the other from the top of the system. We select the stiffness such that exponential functions can be neglected at a distance of $l/3$ form the bottom or the top respectively.

Questions

- Draw the expected distribution of the normal forces in the walls and the shear forces in the connection beams for the situation that K_1 and K_2 both have a finite value. Write down whether the normal forces are tension or compression. (calculation is not needed, just understanding).
- Draw in which direction the shear force s in the distributed springs acts on each of the adjacent walls.
- Write down shortly (only words) how the results change if the horizontal connection beams are not present over the middle third part of the height l.

Problem 2 (4 points)

A semi circular arch and tension bar are idealised according to Figure 2. The materials are linear-elastic. The bending stiffness of the arch is EI and the extension stiffness is infinitely large. The extension stiffness of the tension bar is EA. The distributed load q is constant per unit of arch length (for example self weight).

The structure will be calculated using complimentary energy. We choose the force in the tension bar as redundant ϕ. The moment in the arch can be expressed in ϕ.

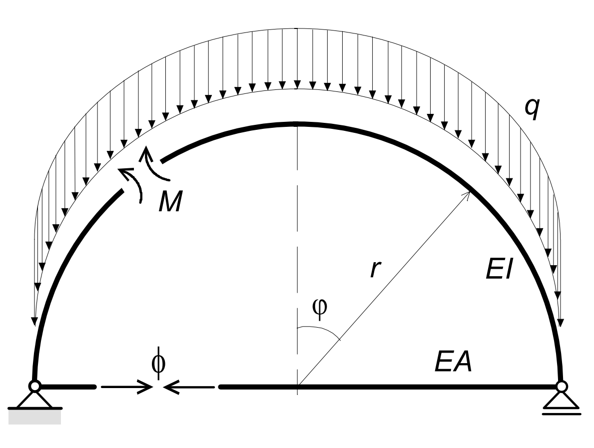


Figure 2. Arch and tension bar

$$M(\varphi) = q r^2 (\tfrac{1}{2}\pi - \varphi \sin\varphi) - r(\phi + q r)\cos\varphi$$

a Show that the moment $M(0)$ in the top of the arch is in equilibrium with the load.

b Give the formula for the complimentary energy of the structure. (You do not need to evaluate the formula.)

c The complementary energy has been evaluated for you with the following result.

$$E_{\text{compl}} = \frac{\pi r^3}{48EI}[12\phi^2 - 12 r\phi q + r^2 q^2 (7\pi^2 - 66)] + \frac{\phi^2 r}{EA}$$

Calculate ϕ. Use the parameter $\beta = \dfrac{\pi r^2 EA}{EI}$

d Calculate the moment $M(0)$ in the top of the arch. Can this moment have a negative value? Explain your answer.

Problem 3 (3 points)

A tunnel is loaded by a train that does not move. The tunnel is modelled as a ring with a bending stiffness EI. The surrounding soil is modelled with distributed springs of stiffness k per unit length. The wheel load by the train is modelled as a concentrated load F.

We assume the following function for the displacement w of the ring.

$$w(s) = A\cos\frac{s}{r} + B\cos(2\frac{s}{r}) + C\cos(3\frac{s}{r})$$

where A, B and C are coefficients that are to be determined.

The kinematic equation for the curvature κ in the ring is

$$\kappa(s) = -\frac{d^2w}{ds^2} - \frac{w}{r^2}$$

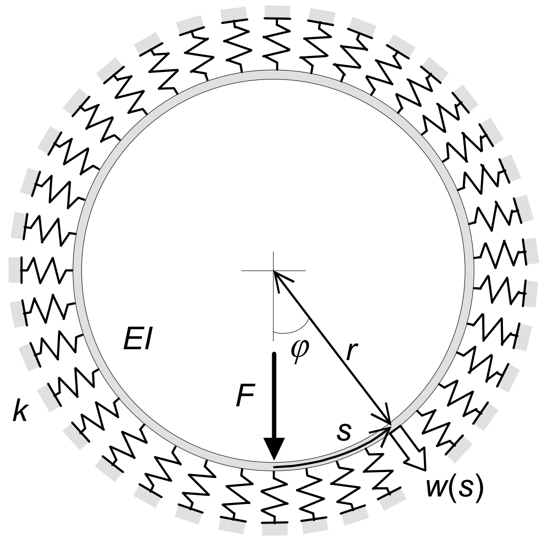


Figure 3. Spring supported ring

a The displacement function $w(s)$ consists of three terms. Which of the three terms give a rigid displacement of the tube? Show that the rigid displacement does not produce a curvature in the ring.

b Give the equation of the potential energy of the model. (You do not need to evaluate the equation.)

c The potential energy has been evaluated for you with the following result.

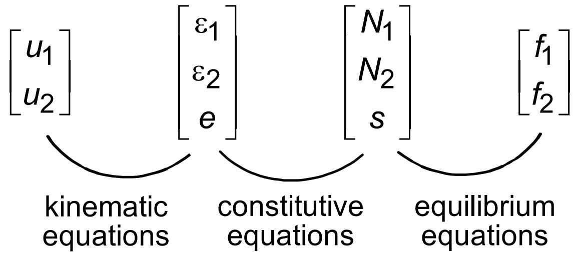
$$E_{\text{pot}} = \frac{\pi EI}{2r^3}(9B^2 + 64C^2) + \tfrac{1}{2}k\pi r(A^2 + B^2 + C^2) - F(A + B + C)$$

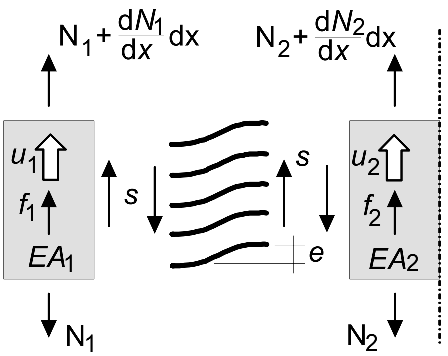
Solve the coefficients A, B and C op. Use the parameter $\beta = \dfrac{kr^4}{EI}$ and express A, B, and C in $\dfrac{Fr^3}{\pi EI}$

d Calculate the moment M in the tube at $\varphi = 0$.

Tentamen Ctme5141, 8 januari 2002
Answers to Problem 1

a <u>Framework</u>





(The positive direction of *s* and *e* can also be selected differently.)

b <u>Method</u>
The force method is most suitable. The displacement method results in two differential equations (because two degrees of freedom, so two equilibrium equations). The force method results in one differential equation. (because one redundant, which follows from two equilibrium equation and three stress quantities).

c <u>Equations</u>
Kinematic equations

$$\varepsilon_1 = \frac{du_1}{dx}$$

$$\varepsilon_2 = \frac{du_2}{dx}$$

$$e = u_2 - u_1$$

Constitutive equations

$$\varepsilon_1 = \frac{N_1}{EA_1} + \alpha T$$

$$\varepsilon_2 = \frac{N_2}{EA_2}$$

$$e = \frac{s}{k}$$

Equilibrium equations

$$\frac{dN_1}{dx} + s + f_1 = 0$$

$$\frac{dN_2}{dx} - s + f_2 = 0$$

$$f_1 = f_2 = 0$$

d <u>Redundant</u>

We choose $s = \phi$ and substitute this in the equilibrium equations.

$$\boxed{\begin{aligned}\frac{dN_1}{dx} &= -\phi \\ \frac{dN_2}{dx} &= \phi\end{aligned}}$$

<u>Compatibility Condition</u>

We use the kinematic equations and eliminate u_1 and u_2

$$\frac{de}{dx} = \frac{du_2}{dx} - \frac{du_1}{dx} \quad \to \quad \boxed{\frac{de}{dx} = \varepsilon_2 - \varepsilon_1}$$

e <u>Boundary Conditions $x = l$</u>

$$\left.\begin{aligned} N_1 = 0 \to \varepsilon_1 = \alpha T \\ N_2 = 0 \to \varepsilon_2 = 0 \end{aligned}\right\} \text{substitution in the compatibility condition}$$

$$\to \quad \frac{de}{dx} = -\alpha T \quad \to \quad \frac{d}{dx}\left(\frac{\phi}{k}\right) = -\alpha T \quad \to \quad \boxed{\frac{d\phi}{dx} = -k\,\alpha T}$$

f <u>Boundary Conditions $x = 0$</u>

$e = u_1 - u_2$

Substitution of $u_1 = \dfrac{F_{v1}}{K_1}$, $u_2 = \dfrac{F_{v2}}{K_2}$ and $e = \dfrac{\phi}{k}$ gives

$$\frac{\phi}{k} = \frac{F_{v1}}{K_1} - \frac{F_{v2}}{K_2}$$

From equilibrium of wall 1 follows $F_{v1} = \int_0^l \phi dx$.

From equilibrium of wall 2 follows $F_{v2} = -\int_0^l \phi dx$.

$$\frac{\phi}{k} = \frac{\int_0^l \phi dx}{K_1} - \frac{-\int_0^l \phi dx}{K_2}$$

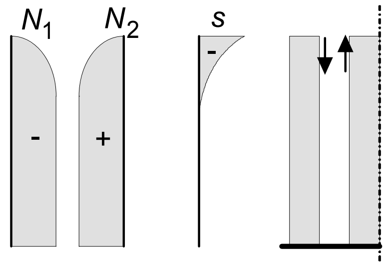
$$\frac{\phi}{k} = \left(\frac{1}{K_1} + \frac{1}{K_2}\right)\int_0^l \phi dx$$

$$\boxed{\frac{1}{k}\phi - \left(\frac{1}{K_1} + \frac{1}{K_2}\right)\int_0^l \phi dx = 0}$$

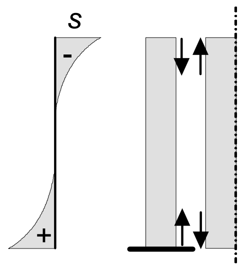
g <u>Bonus Question</u>

The described situation is in between two extremes.

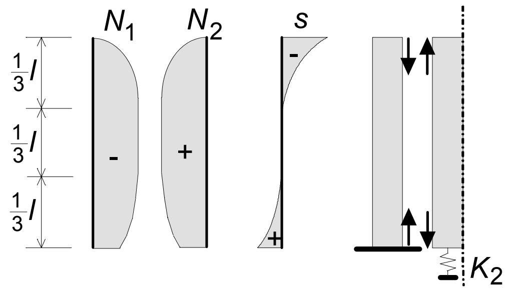
Extreme 1
$K_1 = \infty$ and $K_2 = \infty$. This situation is described in the lecture book (Direct Methods, page 36).



Extreme 2
$K_1 = \infty$ and $K_2 = 0$. The behaviour at the top and bottom is the same.



In between
$K_1 = \infty$ and K_2 is some finite value. The result is in between the extremes.

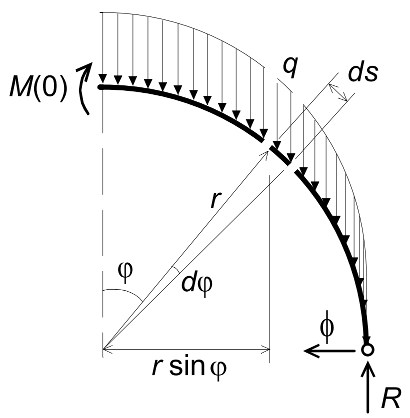


The shear force is zero over the middle 1/3 of the height. The results do not change if the connection beams are removed over the middle 1/3 of the height.

Answers to Problem 2

a Equilibrium

The load on a little part of the arch of length *ds* is *q ds*. The lever arm of this load is $r \sin\varphi$. Therefore we find for the equilibrium



$$M(0) = R\,r - \phi\,r - \int_{\varphi=0}^{\frac{1}{2}\pi} q\,ds\,r \sin\varphi \ .$$

The support reaction R is equal to the load q times the length of the semi arch.

$$R = q\,\tfrac{1}{2}\pi\,r$$

During evaluation we use $ds = r\,d\varphi$.

$$M(0) = q\tfrac{1}{2}\pi r^2 - \phi r - qr^2 \int_{\varphi=0}^{\frac{1}{2}\pi} \sin\varphi\, d\varphi$$

$$= q\tfrac{1}{2}\pi r^2 - \phi r - qr^2 \left[-\cos\varphi\right]_0^{\frac{1}{2}\pi}$$

$$= q\tfrac{1}{2}\pi r^2 - \phi r - qr^2 \left[-\cos\tfrac{1}{2}\pi + \cos 0\right]$$

$$= q\tfrac{1}{2}\pi r^2 - \phi r - qr^2 \left[-0 + 1\right]$$

$$= q\tfrac{1}{2}\pi r^2 - \phi r - qr^2$$

According to this exam problem the moment in the top equals

$$M(0) = q\,r^2(\tfrac{1}{2}\pi - 0\,sin\,0) \text{-} r(\phi + q\,r)cos\,0$$

$$= q\,r^2\,\tfrac{1}{2}\pi \text{-} r(\phi + q\,r)$$

which indeed is the same.

b Complementary Energy

$$E_{\text{compl}} = \int_{\varphi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2} M\,\kappa\, ds + \tfrac{1}{2}\phi\,\varepsilon\, 2r$$

When we substitute the constitutive equations and $ds = r\,d\varphi$ we obtain

$$E_{\text{compl}} = \int_{\varphi=-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{2}\frac{M^2}{EI} r\,d\varphi + \frac{1}{2}\frac{\phi^2}{EA}\,2r$$

c Force in the Bar

$$E_{\text{compl}} = \frac{\pi r^3}{48EI}[12\phi^2 - 12r\,\phi\,q + r^2q^2(7\pi^2 - 66)] + \frac{\phi^2 r}{EA}$$

$$\frac{\partial E_{\text{compl}}}{\partial\phi} = 0 \quad \rightarrow \quad \frac{\pi r^3}{48EI}[24\phi - 12r\,q] + \frac{2\phi r}{EA} = 0$$

$$\frac{\pi r^3 EA}{48EI}[24\phi - 12r\,q] + 2\phi r = 0$$

$$\frac{\beta r}{48}[24\phi - 12r\,q] + 2\phi r = 0$$

$$\tfrac{24}{48}\beta\phi - \tfrac{12}{48}\beta r\,q + 2\phi = 0$$

$$2\beta\phi + 8\phi = \beta r\,q$$

$$2\phi(\beta + 4) = \beta r\,q$$

$$\boxed{\phi = \tfrac{1}{2} q\,r\,\frac{\beta}{\beta + 4}}$$

d Moment in the Top

$$M(0) = q\tfrac{1}{2}\pi r^2 - \phi r - qr^2$$

$$= q\tfrac{1}{2}\pi r^2 - \tfrac{1}{2} q r \frac{\beta}{\beta+4}\, r - qr^2$$

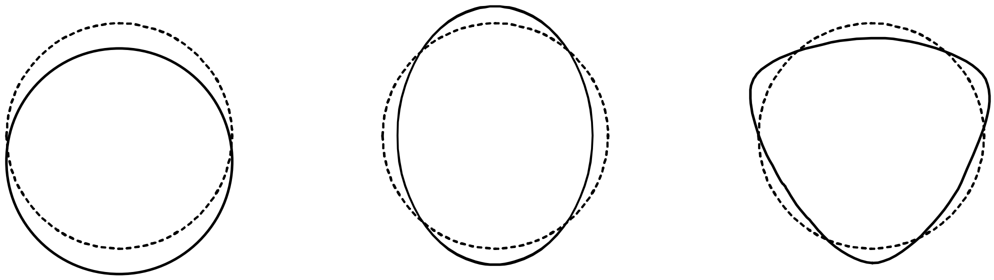
$$\boxed{M(0) = \tfrac{1}{2} q r^2 \left(\pi - 2 - \frac{\beta}{\beta+4} \right)}$$

The stiffness' EI and EA are always positive. Therefore $\beta \geq 0$. So $\frac{\beta}{\beta+4} < 1$.

So $M(0) > 0$ provided that $q > 0$.

Answers to Problem 3

a Rigid Displacement
We draw the terms of the displacement function.



$\cos\frac{s}{r}$ $\qquad$ $\cos 2\frac{s}{r}$ $\qquad$ $\cos 3\frac{s}{r}$

The rigid displacement is therefore $w = A\cos\frac{s}{r}$
Substitution in the kinematic equation gives

$$\kappa = -\frac{d^2 w}{ds^2} - \frac{w}{r^2} = -\frac{d^2}{ds^2}\left(A\cos\frac{s}{r} \right) - \frac{A\cos\frac{s}{r}}{r^2} = A\frac{1}{r^2}\cos\frac{s}{r} - \frac{A\cos\frac{s}{r}}{r^2} = 0$$

Therefore, the rigid displacement does not give a curvature of the ring.

b Potential Energy

$$E_{\text{pot}} = \int_{s=0}^{2\pi r} \tfrac{1}{2} M \kappa \, ds + \int_{s=0}^{2\pi r} \tfrac{1}{2} f\, w \, ds - F\, w(0)$$

where M is the moment in the ring and f is the spring force per unit of length. Using the constitutive equations we can write this as

$$E_{\text{pot}} = \int_{s=0}^{2\pi r} \tfrac{1}{2} EI \kappa^2 ds + \int_{s=0}^{2\pi r} \tfrac{1}{2} k\, w^2 ds - F\, w(0).$$

c Coefficients

$$E_{\text{pot}} = \frac{\pi EI}{2r^3}(9B^2 + 64C^2) + \tfrac{1}{2} k\pi r (A^2 + B^2 + C^2) - F(A + B + C)$$

$$\frac{\partial E_{\text{pot}}}{\partial A}=0 \quad \rightarrow \quad k\pi r\,A-F=0 \quad \rightarrow \quad A=\frac{F}{k\pi r}=\frac{Fr^3}{\pi EI}\frac{1}{\beta}$$

$$\frac{\partial E_{\text{pot}}}{\partial B}=0 \quad \rightarrow \quad \frac{\pi EI}{r^3}9B+k\pi r\,B-F=0 \quad \rightarrow \quad B=\frac{F}{9\dfrac{\pi EI}{r^3}+k\pi r}=\frac{Fr^3}{\pi EI}\frac{1}{\beta+9}$$

$$\frac{\partial E_{\text{pot}}}{\partial C}=0 \quad \rightarrow \quad \frac{\pi EI}{r^3}64C+k\pi r\,C-F=0 \quad \rightarrow \quad C=\frac{F}{64\dfrac{\pi EI}{r^3}+k\pi r}=\frac{Fr^3}{\pi EI}\frac{1}{\beta+64}$$

d Largest Moment

$$w=A\cos\frac{s}{r}+B\cos(2\frac{s}{r})+C\cos(3\frac{s}{r})$$

$$\frac{\partial w}{\partial s}=-\frac{1}{r}A\sin\frac{s}{r}-\frac{2}{r}B\sin(2\frac{s}{r})-\frac{3}{r}C\sin(3\frac{s}{r})$$

$$\frac{\partial^2 w}{\partial s^2}=-\frac{1}{r^2}A\cos\frac{s}{r}-\frac{4}{r^2}B\cos(2\frac{s}{r})-\frac{9}{r^2}C\cos(3\frac{s}{r})$$

$$\kappa=-\frac{d^2w}{ds^2}-\frac{w}{r^2}$$

$$=\frac{1}{r^2}A\cos\frac{s}{r}+\frac{4}{r^2}B\cos(2\frac{s}{r})+\frac{9}{r^2}C\cos(3\frac{s}{r})-\frac{1}{r^2}\left(A\cos\frac{s}{r}+B\cos(2\frac{s}{r})+C\cos(3\frac{s}{r})\right)$$

$$=\frac{3}{r^2}B\cos(2\frac{s}{r})+\frac{8}{r^2}C\cos(3\frac{s}{r})$$

$$\kappa\big|_{s=0}=\frac{3}{r^2}B+\frac{8}{r^2}C=\frac{3}{r^2}\frac{F\,r^3}{\pi EI}\frac{1}{\beta+9}+\frac{8}{r^2}\frac{F\,r^3}{\pi EI}\frac{1}{\beta+64}=\frac{F\,r}{\pi EI}\left(\frac{3}{\beta+9}+\frac{8}{\beta+64}\right)$$

$$M\big|_{s=0}=EI\,\kappa\big|_{s=0}=\frac{F\,r}{\pi}\left(\frac{3}{\beta+9}+\frac{8}{\beta+64}\right)$$

Encore (not an exam question)

We can also select more than three terms for the displacement function.

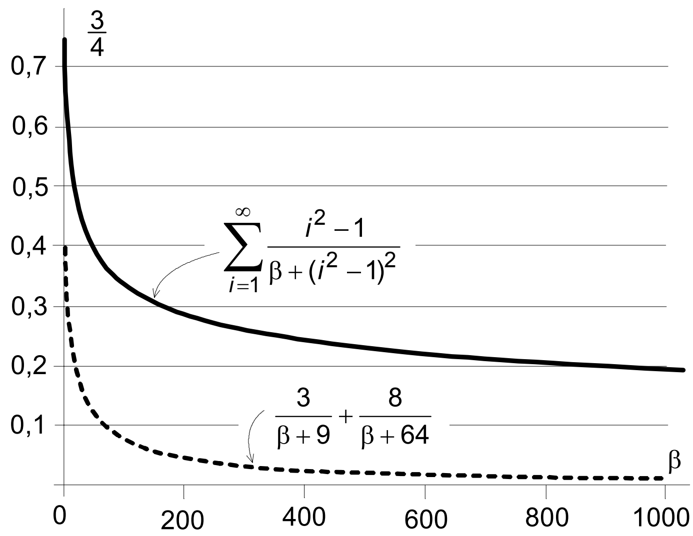
$$w=\sum_{i=1}^{\infty}A_i\cos\frac{i\,s}{r}$$

These coefficients A_i can be calculated in the same way.

$$A_i=\frac{F\,r^3}{\pi EI}\frac{1}{\beta+(i^2-1)^2} \qquad i=1,2,3\,...\,\infty$$

The largest moment becomes

$$M\big|_{s=0}=\frac{F\,r}{\pi}\sum_{i=1}^{\infty}\frac{i^2-1}{\beta+(i^2-1)^2}$$



The graph shows that the approximation using three terms substantially underestimates the largest moment.

Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5141 Theory of Elasticity
Monday 13 January 2003, 13:00 – 17:00 hours

Problem 1 (2 points)

Consider a very large plate with a circular hole made of a linear-elastic material. Figure 1 shows a rectangular part of this plate. Everywhere in the plate except close to the hole the principle stresses are σ_1 and σ_2. Around the hole this homogeneous stress field is disturbed and peak stresses σ_A and σ_B occur. If $\sigma_1 = \sigma_2 = \sigma$ the peak stresses are $\sigma_A = \sigma_B = 2\sigma$. If $\sigma_1 = -\sigma_2 = \sigma$ the peak stresses are $\sigma_A = 4\sigma$ and $\sigma_B = -4\sigma$.

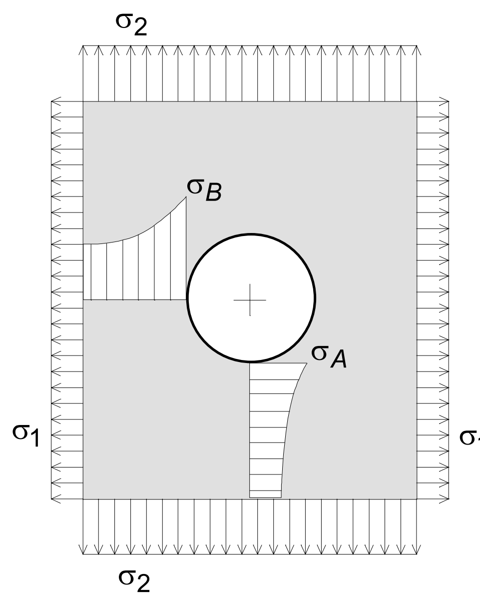


Figure 1. Stress field around a circular hole

a Use the provided information to determine σ_A and σ_B in case $\sigma_1 = \sigma$ and $\sigma_2 = 3\sigma$.

b We use polar coordinates and the force method to calculate the stress field. Which differential equation describes the problem for $\sigma_1 = -\sigma_2 = \sigma$?

c Again we use polar coordinates and the force method. For the case $\sigma_1 = \sigma_2 = \sigma$ a simpler differential equation may be used than in question **b**. Which is this differential equation?

Problem 2 (5 points)

A thin axisymmetric plate with an opening is simply supported at the outer edge (Figure 2.). The plate stiffness is *D* and the coefficient of lateral contraction is ν. We will consider three load cases. In the first load case the plate is loaded by an edge moment m_o along its outer edge $r = b$ and inner edge $r = a$. Questions **a**, **b** and **c** are on this load case. The remaining load cases are introduced after these questions.

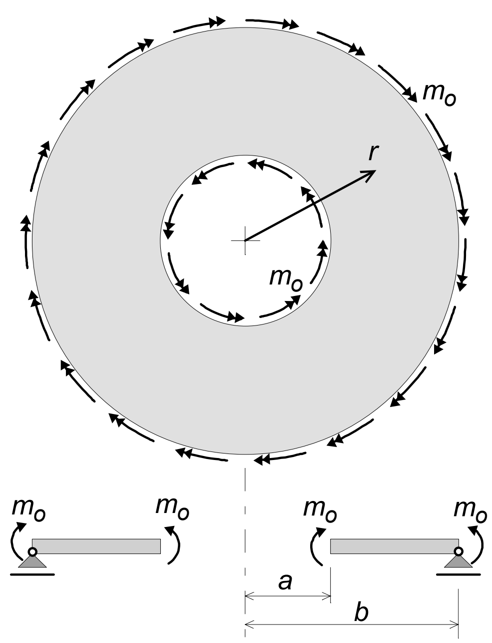


Figure 2. Axisymmetric plate

a Which differential equation describes this problem? Give the general solution having four integration constants. What do you notice about the particular solution?

b Specify four boundary conditions from which the constants in the general solution can be solved. (you do not need to elaborate the boundary conditions into expressions of the deflection *w*. You do not need not solve the constants.)

c Solving the four constants gives the following expression for the deflection *w.*

$$w(r) = \frac{m_o}{2D(1+\nu)}(b^2 - r^2)$$

Calculate the bending moments m_{rr} , $m_{\theta\theta}$ and the shear force v_r .

Subsequently the second load case is considered for the same plate. The edge moments m_o are removed and a temperature gradient is applied over the thickness of the plate. The temperature of the lower face is higher than the upper face. If the plate would be able to deform freely a curvature κ_T would occur in all directions. This curvature would not give any stress.

d Derive the following constitutive equations of the axisymmetric plate.

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta} - (1+\nu)\kappa_T\right)$$
$$m_{\theta\theta} = D\left(\nu\kappa_{rr} + \kappa_{\theta\theta} - (1+\nu)\kappa_T\right)$$

e In this case of temperature deformation the displacement field has the shape

$$w(r) = C_1 + C_2 r^2$$

Determine C_1 and C_2 and show that bending moments and shear force do not occur due to this temperature loading.

Subsequently the third load case is considered. To this end the temperature load is removed, the edge *r* = *b* is clamped and the temperature load is applied again.

f Will the plate be free of stresses again? Explain your answer. (Analysis is not needed.)

Problem 3 (3 points)

A triangular plate of thickness t carries an evenly distributed load q (Figure 4.). The plate is simply supported at each edge. According to elasticity theory we assume that the kinematic, constitutive and equilibrium equations are linear. An exact solution of the deflection of this plate does not exist. However, it can be approximated. A calculation using potential energy is very suitable because we can accurately estimate the deformation of the plate.

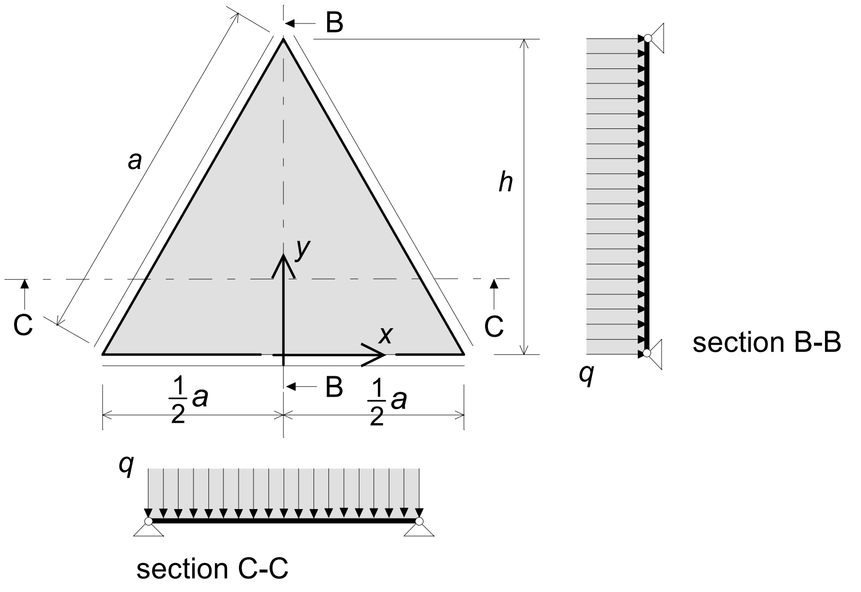


Figure 4. Triangular plate with distributed load

The following deflection function is assumed.

$$w(x,y) = A\frac{y}{h}(1-2\frac{x}{a}-\frac{y}{h})(1+2\frac{x}{a}-\frac{y}{h})$$

where A is a coefficient that needs to be determined later.

a Show that the deflection function fulfils the kinematic boundary conditions of the plate.

b Give the formula for calculation of the potential energy of the plate. (You do not need to evaluate the formula.)

c Using the deflection function the potential energy E_{pot} of the plate can be calculated. This has been done for you. The following equations have been used.

$$m_{xx} = D(\kappa_{xx} + \nu\kappa_{yy}) \qquad \kappa_{xx} = -\frac{\partial^2 w}{\partial x^2} \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

$$m_{yy} = D(\kappa_{yy} + \nu\kappa_{xx}) \qquad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$m_{xy} = \tfrac{1}{2}D(1-\nu)\rho_{xy} \qquad \rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

The result is

$$E_{\text{pot}} = \tfrac{16}{9}\sqrt{3}\,\frac{DA^2(2+\nu)}{a^2} - \tfrac{1}{60}\sqrt{3}\,qAa^2$$

Solve the coefficient A from the potential energy equation.

d Calculate the deflection in the middle of the plate as a function of a, D and q.

e Since now A is known the potential energy of the plate can be calculated. This has been done for you with the following result.

$$E_{\text{pot}} = -\frac{\sqrt{3}}{25600}\frac{q^2a^6}{D(2+\nu)}$$

The table below shows three alternative deflection functions for this problem and the calculated potential energy. Which deflection function provides the most accurate approximation of the plate behaviour? Explain your answer.

	Deflection Function	Potential Energy
1	$w(x,y) = A\frac{y}{h}(1-2\frac{x}{a}-\frac{y}{h})(1+2\frac{x}{a}-\frac{y}{h})$	$E_{\text{pot}} = -\frac{\sqrt{3}}{25600}\frac{q^2a^6}{D(2+\nu)}$
2	$w(x,y) = A\left(\frac{y}{h}(1-2\frac{x}{a}-\frac{y}{h})(1+2\frac{x}{a}-\frac{y}{h})\right)^2$	$E_{\text{pot}} = -\frac{\sqrt{3}}{86016}\frac{q^2a^6}{D(2+\nu)}$
3	$w(x,y) = A\left(\frac{y}{h}(1-2\frac{x}{a}-\frac{y}{h})(1+2\frac{x}{a}-\frac{y}{h})\right)^3$	$E_{\text{pot}} = -\frac{13\sqrt{3}}{1689600}\frac{q^2a^6}{D(2+\nu)}$

Exam CT5141, 13 January 2003
Answers to Problem 1

a Peak Stresses
For linear-elastic materials we can apply the principle of superposition.
We have two load cases
1. $\sigma_1 = \sigma$, $\sigma_2 = \sigma$, $\sigma_A = 2\sigma$, $\sigma_B = 2\sigma$
2. $\sigma_1 = \sigma$, $\sigma_2 = -\sigma$, $\sigma_A = 4\sigma$, $\sigma_B = -4\sigma$

From this we can make a new case that consists of two times case 1 minus case 2.
3. $\sigma_1 = \sigma$, $\sigma_2 = 3\sigma$, $\sigma_A = 0$, $\sigma_B = 8\sigma$

b Differential Equation
The formulation in polar coordinates of a plate loaded in extension is

$$\nabla^2\nabla^2\phi = 0 \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}$$

(See lecture book Direct Methods page 107.)

c Differential Equation
Since the plate is very large the far boundaries have very little influence on the problem. Therefore, the geometry can be considered symmetric. In load case 1, the stress is equal in all directions (Mohr's circle reduces to a point.). Since the geometry and loading are symmetric the differential equation can be simplified.

$$L\phi = 0 \qquad L = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r$$

(See lecture book Direct Methods page 99.)
Note that this ϕ is a different one than the ϕ in question **b**.

Answers to Problem 2

a Differential Equation and Solution

$$D\nabla^2\nabla^2 w = 0 \qquad \nabla^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}$$

$$w = C_1 + C_2 r^2 + C_3 \ln r + C_4 r^2 \ln r$$

Since there is not a distributed load at the plate surface there is not a particular solution.
(See lecture book Direct Methods pages 124, 127.)

b Boundary Conditions

$$r = a \rightarrow \begin{array}{l} v_r = 0 \\ m_{rr} = m_o \end{array} \qquad r = b \rightarrow \begin{array}{l} w = 0 \\ m_{rr} = m_o \end{array}$$

c Bending Moments and Shear Force

$$w = \frac{m_o}{2D(1+\nu)}(b^2 - r^2)$$

$$\kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = \frac{m_o}{D(1+\nu)}$$

$$\kappa_{rr} = -\frac{d^2w}{dr^2} = \frac{m_o}{D(1+\nu)}$$

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta}\right) = D\left(\frac{m_o}{D(1+\nu)} + \nu\frac{m_o}{D(1+\nu)}\right) = m_o$$

$$m_{\theta\theta} = D\left(\kappa_{\theta\theta} + \nu\kappa_{rr}\right) = D\left(\frac{m_o}{D(1+\nu)} + \nu\frac{m_o}{D(1+\nu)}\right) = m_o$$

$$v_r = \frac{1}{r}\frac{d(r\, m_{rr})}{dr} - \frac{m_{\theta\theta}}{r} = \frac{1}{r}\frac{d(r\, m_o)}{dr} - \frac{m_o}{r} = 0$$

d Constitutive Equations

Without temperature the constitutive equations read

$$m_{rr} = D(\kappa_{rr} + \nu\kappa_{\theta\theta})$$
$$m_{\theta\theta} = D(\kappa_{\theta\theta} + \nu\kappa_{rr})$$

Part of the curvature is due to temperature loading. This part does not cause moments.

$$m_{rr} = D\left((\kappa_{rr} - \kappa_T) + \nu(\kappa_{\theta\theta} - \kappa_T)\right) = D\left(\kappa_{rr} - \kappa_T + \nu\kappa_{\theta\theta} - \nu\kappa_T\right) = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta} - (1+\nu)\kappa_T\right)$$
$$m_{\theta\theta} = D\left((\kappa_{\theta\theta} - \kappa_T) + \nu(\kappa_{rr} - \kappa_T)\right) = D\left(\kappa_{\theta\theta} - \kappa_T + \nu\kappa_{rr} - \nu\kappa_T\right) = D\left(\kappa_{\theta\theta} + \nu\kappa_{rr} - (1+\nu)\kappa_T\right)$$

e Moments and Shear Forces

$$w = C_1 + C_2 r^2$$

$$\kappa_{\theta\theta} = -\frac{1}{r}\frac{dw}{dr} = -\frac{1}{r}2C_2 r = -2C_2$$

$$\kappa_{rr} = -\frac{d^2w}{dr^2} = -2C_2$$

$$m_{rr} = D\left(\kappa_{rr} + \nu\kappa_{\theta\theta} - (1+\nu)\kappa_T\right) = D\left(-2C_2 - \nu 2C_2 - (1+\nu)\kappa_T\right) = -D\left(2C_2 + \kappa_T\right)(1+\nu)$$

$$m_{\theta\theta} = D\left(\kappa_{\theta\theta} + \nu\kappa_{rr} - (1+\nu)\kappa_T\right) = D\left(-2C_2 - \nu 2C_2 - (1+\nu)\kappa_T\right) = -D\left(2C_2 + \kappa_T\right)(1+\nu)$$

$$v_r = \frac{1}{r}\frac{d(r\, m_{rr})}{dr} - \frac{m_{\theta\theta}}{r} = \frac{1}{r}\frac{d}{dr}\left(-rD\left(2C_2 + \kappa_T\right)(1+\nu)\right) + \frac{D\left(2C_2 + \kappa_T\right)(1+\nu)}{r} =$$
$$= \frac{1}{r}\left(-D\left(2C_2 + \kappa_T\right)(1+\nu)\right) + \frac{D\left(2C_2 + \kappa_T\right)(1+\nu)}{r} = 0$$

Boundary conditions

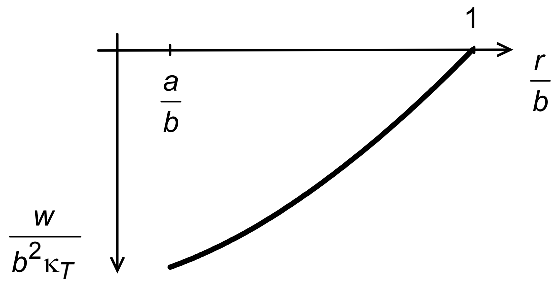
$$r = a \rightarrow \begin{matrix} v_r = 0 \\ m_{rr} = 0 \end{matrix} \qquad r = b \rightarrow \begin{matrix} w = 0 \\ m_{rr} = 0 \end{matrix}$$

These are fulfilled only if $C_2 = -\frac{1}{2}\kappa_T$ and $C_1 = \frac{1}{2}b^2\kappa_T$.

Therefore, $m_{rr} = m_{\theta\theta} = v_r = 0$

f Clamped Edge

The displacement function that is calculated in question **e** is drawn below. Clearly a rotation dw/dr occurs at the edge $r = b$. Preventing this rotation will lead to moments in the plate.



Answers to Problem 3

a Boundary Conditions

The plate is simply supported on all edges. Therefore, the deflection w must be zero at the edges.

$$w(x,0) = A\frac{0}{h}(1-2\frac{x}{a}-\frac{0}{h})(1+2\frac{x}{a}-\frac{0}{h}) = 0$$

$$w(-\frac{a(h-y)}{2h},y) = A\frac{y}{h}(1-2\frac{-\frac{a(h-y)}{2h}}{a}-\frac{y}{h})(1+2\frac{-\frac{a(h-y)}{2h}}{a}-\frac{y}{h}) = A\frac{y}{h}(2-2\frac{y}{h})(0) = 0$$

$$w(\frac{a(h-y)}{2h},y) = A\frac{y}{h}(1-2\frac{\frac{a(h-y)}{2h}}{a}-\frac{y}{h})(1+2\frac{\frac{a(h-y)}{2h}}{a}-\frac{y}{h}) = A\frac{y}{h}(0)(2-2\frac{y}{h}) = 0$$

The kinematic boundary conditions are correctly satisfied.

It is noted that the dynamic boundary conditions do not need to be fulfilled for application of the principle of minimum potential energy.

b Potential Energy

$$E_{\text{pot}} = \int_0^h \int_{-\frac{a(h-y)}{2h}}^{\frac{a(h-y)}{2h}} \tfrac{1}{2}(m_{xx}\kappa_{xx} + m_{yy}\kappa_{yy} + m_{xy}\rho_{xy})\,dx\,dy - \int_0^h \int_{-\frac{a(h-y)}{2h}}^{\frac{a(h-y)}{2h}} q\,w\,dx\,dy$$

c Coefficient

For the correct displacement field the potential energy is minimal.

$$\frac{\partial E_{\text{pot}}}{\partial A} = \tfrac{32}{9}\sqrt{3}\,\frac{DA(2+\nu)}{a^2} - \tfrac{1}{60}\sqrt{3}\,qa^2 = 0$$

From this equation coefficient A is solved.

$$A = \tfrac{3}{640}\frac{qa^4}{D(2+\nu)}$$

A mistake has been made in the potential energy of problem 3. The correct potential energy is

$$E_{\text{pot}} = \tfrac{32}{9}\sqrt{3}\,\frac{DA^2}{a^2} - \tfrac{1}{60}\sqrt{3}\,qAa^2$$

In this answer we continue with the incorrect potential energy as if it were correct.

d Deflection

$$w(0,\tfrac{1}{3}h)=A\tfrac{1}{3}(1-\tfrac{1}{3})(1-\tfrac{1}{3})=A\tfrac{4}{27}=\frac{3}{640}\frac{qa^4}{D(2+\nu)}\frac{4}{27}$$

$$\boxed{w(0,\tfrac{1}{3}h)=\frac{1}{1440}\frac{qa^4}{D(2+\nu)}}$$

e Most Accurate

The exact solution has the smallest potential energy. In the table below the fractions in the potential energy are evaluated. Clearly, deflection function 1 gives the smallest potential energy. Therefore it is the best of the three approximations.

1	$E_{\text{pot}}=-0.676\ 10^{-4}\dfrac{q^2a^6}{D(2+\nu)}$
2	$E_{\text{pot}}=-0.201\ 10^{-4}\dfrac{q^2a^6}{D(2+\nu)}$
3	$E_{\text{pot}}=-0.133\ 10^{-4}\dfrac{q^2a^6}{D(2+\nu)}$

Encore

The problem introduction mistakenly states that an exact solution does not exist. This particular problem does have an exact analytical solution. The displacement function is

$$w(x,y)=\frac{\sqrt{3}}{96}\frac{q}{aD}(y^3-y^2a\sqrt{3}+\tfrac{3}{4}ya^2-3x^2y)(\tfrac{1}{4}a^2-y^2+\tfrac{1}{3}y\sqrt{3}a-x^2)$$

The deflection in the middle is

$$w(0,\tfrac{1}{3}h)=\frac{1}{1728}\frac{qa^4}{D}$$

The moments in the middle are

$$m_{xx}=\tfrac{1}{72}qa^2(1+\nu)$$
$$m_{yy}=\tfrac{1}{72}qa^2(1+\nu)$$
$$m_{xy}=0$$

which are the largest principle moments. The minimum potential energy of the plate is

$$E_{pot}=\frac{-\sqrt{3}}{35840}\frac{q^2a^6}{D}$$