CT5141 Theory of Elasticity
Delft University of Technology

Problem

A structural system consists of two bars connected by a hinge (Fig.1). The system is hinged to the foundation. The hinges are stiffened by rotation springs. The springs are linear-elastic and have a stiffness *c*. The bars cannot deform (infinite stiffness). The system is loaded by a vertical force *F*. The direction of the force does not change when the system deforms.

We want to understand the behaviour of this system for large deformations.

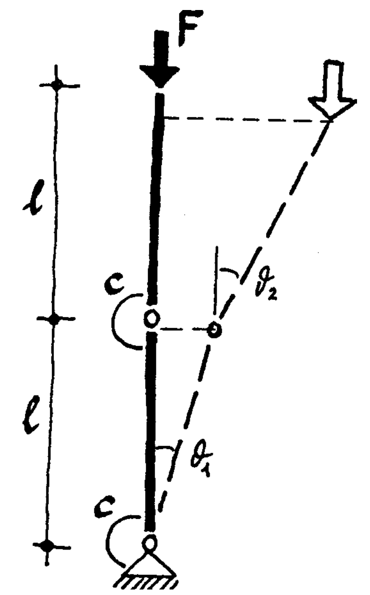


Figure 1. Pendulum system

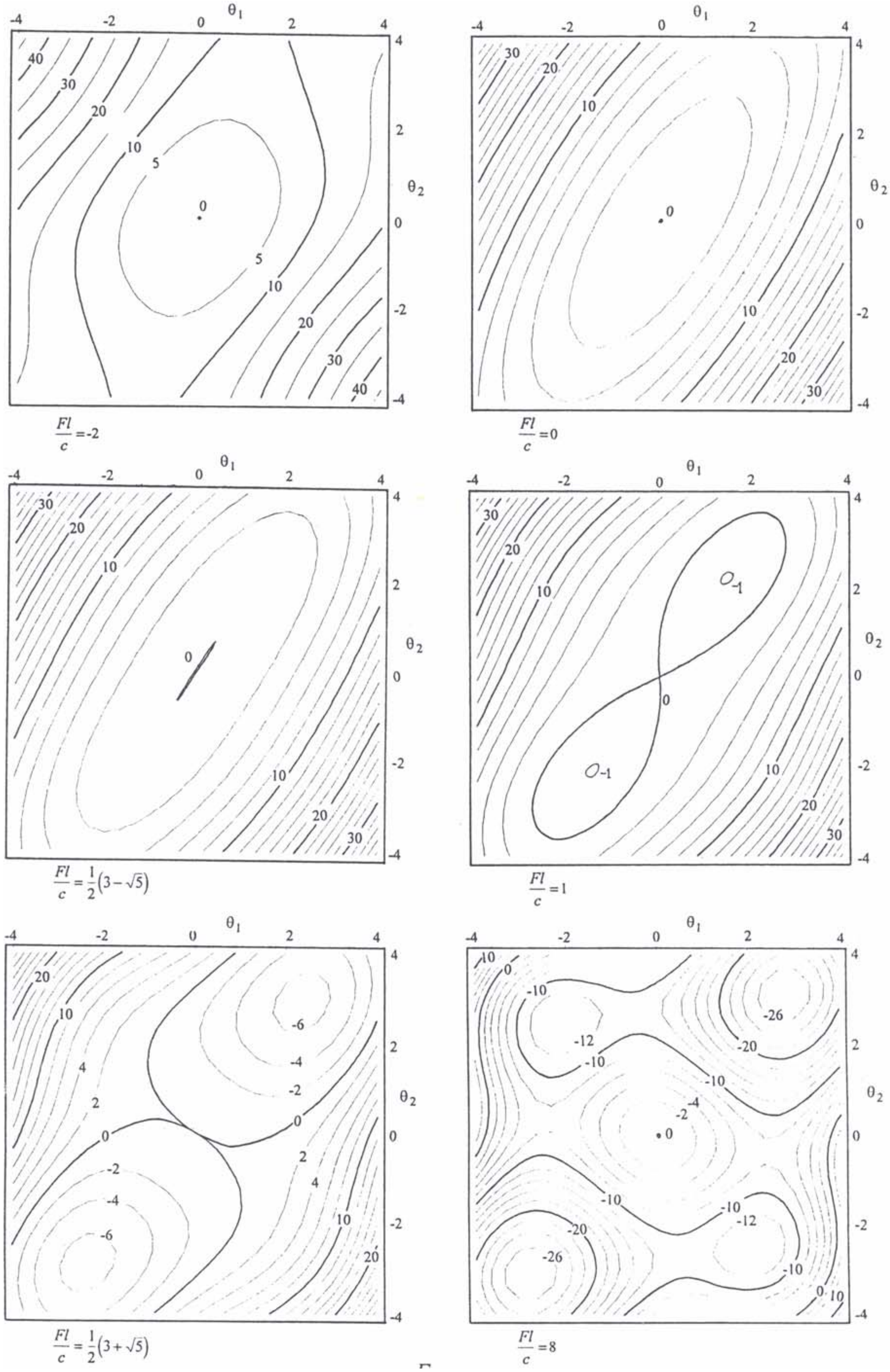
a Express the vertical displacement of the top in θ_1 and θ_2.

b Formulate the exact expression of the potential energy of the system

c Derive the equilibrium equations from the potential energy

d Question **c** gives two coupled non-linear equations. Several approximation methods exist to solve these equations. However, these will not be used. Instead we use a graphical method to gain insight in the system behaviour. The next page shows the potential energy as formulated in question **b** for several magnitudes of loading.

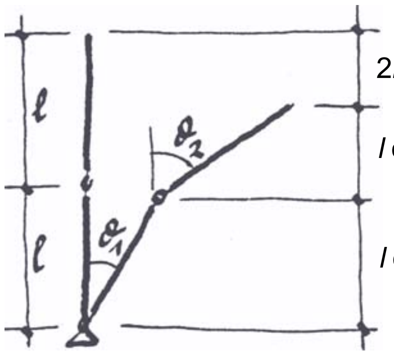
Draw the equilibrium states of the system in the graphs for each loading.



The potential energy $\frac{E_{pot}}{c}$ of the system for several loadings $\frac{Fl}{c}$.

Answers

a Top Displacement



b Potential Energy

$$E_{pot} = \tfrac{1}{2} c\, \theta_1^2 + \tfrac{1}{2} c(\theta_2 - \theta_1)^2 - Fl(2 - \cos\theta_2 - \cos\theta_1)$$

c Equilibrium Equations

$$\frac{\partial E_{pot}}{\partial \theta_1} = c\, \theta_1 - c(\theta_2 - \theta_1) - Fl\sin\theta_1 = 0$$

$$\frac{\partial E_{pot}}{\partial \theta_2} = c(\theta_2 - \theta_1) - Fl\sin\theta_2 = 0$$

d The system assumes the state in which the potential energy is minimal. We can see the graphs as altitude maps. In the deepest point of a valley the system is in equilibrium (see page 4).

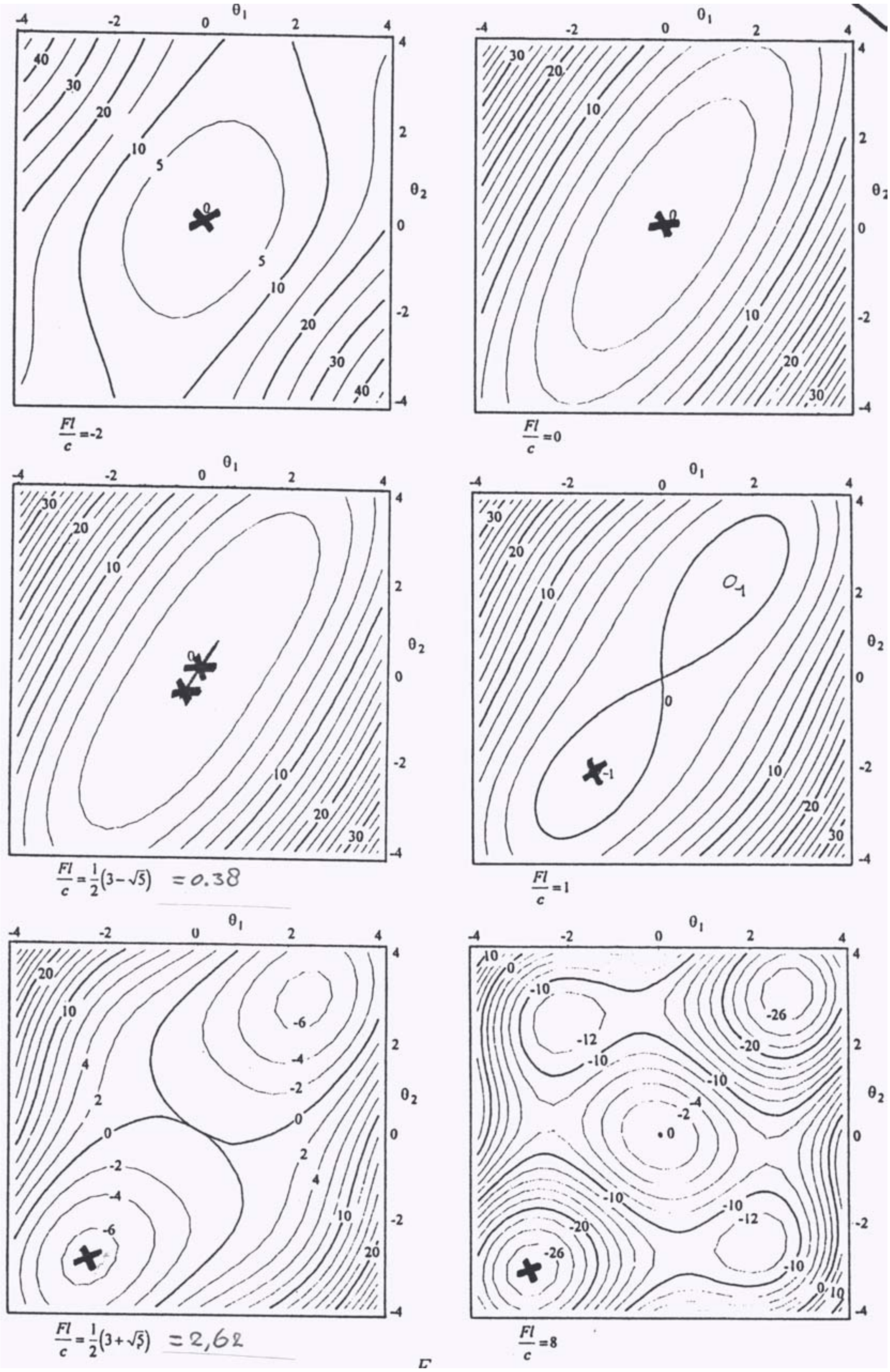
At loading $\frac{Fl}{c} = -2$ and $\frac{Fl}{c} = 0$ we can clearly see a deepest point.

At loading $\frac{Fl}{c} = \frac{1}{2}(3 - \sqrt{5})$ multiple equilibrium states are possible. The system buckles.

At loading $\frac{Fl}{c} = 1$ two valleys occur. Which of these the system will choose is accidental.

At loading $\frac{Fl}{c} = 8$ four minima occur. The equilibrium state depends on the state at a smaller load.

To understand the dynamic behaviour we can visualise the system as a ball that rolls over the surface due to its self-weight.



Note that θ_1 and θ_2 have the unit radians. Therefore the deformation at equilibrium is very large in some of the graphs.