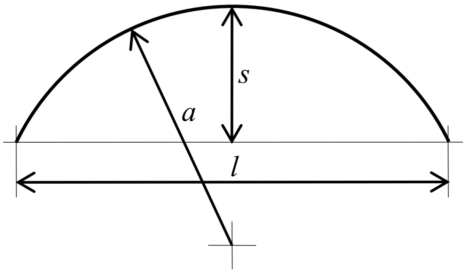
Sagitta

The height of an arch is called the *rise* or the *sagitta* (pronounce with emphasis on “git”) (Latin for arrow). When the sagitta s and the span l are known, we can calculate the radius a of a circular arch.

$$a = \frac{1}{2}s + \frac{1}{8}\frac{l^2}{s}$$



For example the dome of the palazzetto dello sport (fig. 1)(p. 163) has a span of 58.5 m, and sagitta of 20.9 m. The radius is

$$a = \frac{20.9}{2} + \frac{58.5^2}{8 \times 20.9} = 30.9 \text{ m}.$$

Radius/thickness

The palazzetto dello sport (p. 163) has ribs which are 330 mm thick. The shell between the ribs is 120 mm thick. The ratio radius/thickness is

$$\frac{a}{t} = \frac{30.9}{0.12} = 260.$$

When we include the ribs the ratio is

$$\frac{a}{t} = \frac{30.9}{0.33} = 94.$$

Figure 1. Palazzetto dello sport in Rome [www.galinsky.com]

Table 1 shows this ratio for several shell structures. Clearly, a large ratio shows that little material is used. For example, if your design has a ratio $a / t = 500$, it is really efficient.

Objective

The objective of these notes is to predict the behaviour of shell structures. After completing the course you can answer the following questions about your shell designs. Will it deflect too much? Will it yield? Will it crack or break? Will it vibrate annoyingly? Will it buckle? Will it be safe? What causes this and how can I improve it?

Exercise: Psychologists say that a person or animal needs an objective in order to determine how to look at something. For example, when you are tired, a chair is a thing-to-sit-on and when you need to replace a light bulb, a chair is a thing-to-stand-on. Rephrase the former sentence using the words "engineer", "model", "predict".

Table 1. Dimensions of shell structures

structure	location, year, architect	geometry	dimensions	radius a	thickness t	ratio a / t
chicken egg	150 10^6 BC	surface of revolution	60 mm length	20 mm minimum	0.2–0.4 mm	100
Treasury of Atreus (p. 4)	Μυκηνες Greece 1100 BC	surface of revolution	14.5 m diameter	16 m	≈ 0.8 m	20
Pantheon (p. 14)	Rome 126 AD	hemisphere	43.4 m diameter	21.7 m	1.2 m at the top	18
Viking ship Oseberg (p. 109)	Tønsberg Norway 820 AD	ellipsoid part	21.58 m long 5.10 m wide			
Duomo di Firenze (p. 42)	Italy 1420 Brunelleschi	octagonal dome	44 m diameter	22 m		
St. Paul's Cathedral (p. 43)	London 1675 Wren	cone and hemisphere	35 m diameter	15.25 m		
Jena planetarium [1]	Germany 1925 Bauersfeld	hemisphere	25 m diameter	12.5 m	60 mm	200
Algeciras market hall [1]	Spain 1934 Torroja	spherical cap on 8 supports	47.6 m diameter	44.1 m	90 mm	490
beer can (p. 143)	1935	cylinder	66 mm diameter	33 mm	0.08 mm	410
Hibbing water filter plant [1]	Minnesota 1939 Tedesko	ellipsoid of revolution	45.7 m diameter	47.24–5.33 m	900–150 mm	35–525
Bryn Mawr factory [1]	Pennsylvania 1947	elpar on a rect. plan	19.6 x 25.3 m	25.0–32.9 m	90 mm	300–400
Kresge Auditorium (p. 115)	Cambridge 1955 Saarinen	segment of a sphere on 3 points	48.0 m between supports	33.0 m	90 mm	370
Kaneohe Foodland [1]	Hawaii 1957 Bradshaw	intersection of 2 tori on 4 supports	39.0 x 39.0 m between supports	39.0–78.0 m	76–178 mm	500–1000
Palazzetto dello sport (p. 63)	Rome 1957 Nervi	spherical cap with ribs	58.5 m diameter	30.9 m	0.12 m shell 0.33 m ribs	260 or 94
CNIT (p. 149)	Paris 1957 Esquilan	intersection of 3 cylinders on 3 supports	219 m between supports	89.9–420.0 m	1.91–2.74 m total 0.06 – 0.12 m outer layers	47–153
Zeckendorf Plaza (p. 102)	Denver, USA 1958 Tedesko	4 hypars	40 x 34 m height 8.5 m	40 m	76 mm	528
Ferrybridge cooling towers (p. 155)	Ferrybridge UK 1960	hyperboloid	height 115 m	44 m	130 mm repaired … mm	350
Paaskerk (p. 127)	Amstelveen 1963 Van Asbeck	hypar on 2 points	25 x 25 m height 10.3 m	31 m		
Tucker gym (p. 135)	Henrico USA 1965 Hanson	4 hypars	47 x 49 m height 4.6 m	127 m	90 mm	1400

Deitingen petrol station (p. 116)	Switserland 1968 Isler	segment of a sphere on 3 points	span 31.6 m height 11.5 m	52 m	90 mm	580
Saturn V (p. 76)	Houston USA 1965-1975	cylinders and stiffeners	height 111 m	5 m		
oil tanker (p. 146)	~1970	all curvatures with stiffeners	length 300 m width 30 m		20 mm	
Savill building (p. 22)	Windsor UK 2005 Howells	freeform	length 98 m width 24 m	143 m	300 mm	41
Sillogue water tower (p. 27)	Dublin 2007 Collins	surface of revolution	height 39 m top diameter 38 m	24.8 m	786 mm	32

Summary

Shell structures display four phenomena that are different from other structures. These phenomena are listed below. An engineer working with shell structures needs to understand these.

- Arches are thick because pressure lines (p. 6) need go through the middle third (p. 7). Shells are thin because hoop forces (p. 13) push and pull the pressure lines into the middle third.
- Large moments occur in supported edges. This is called edge disturbance (p. 14, 71). It happens because the deformed shell needs to connect to the undeformed support.
- Shells with special curvatures and particular supports behave like flat plates. This is called in-extensional deformation (p. 109)
- Small shape imperfections often cause a large reduction of the buckling load. This is called imperfection sensitivity (p. 142).

Corbel arch

When piling blocks we can shift each block a little compared to the previous one. In this way we can make an arch without formwork (fig. 2). This arch is called a *corbel arch*. It can be analysed best starting from the top. The top block needs to be supported below its centre of gravity. Therefore, it can be shifted up to half its length *c*. The top two blocks need to be supported in their centre of gravity too. Therefore, they can be shifted up to one-fourth of *c*. The shifts produce a row of fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$... The shape of the arch is described by

$$x = nb, \quad y = \sum_{\eta=1}^{n} \frac{c}{2\eta}.$$

Where *b* and *c* are the block height and length. If *x* goes to infinity then *y* goes to infinity. So, there is no theoretical restriction to the span that can be created in this way. However, for large spans and small blocks the arch will become extremely high.

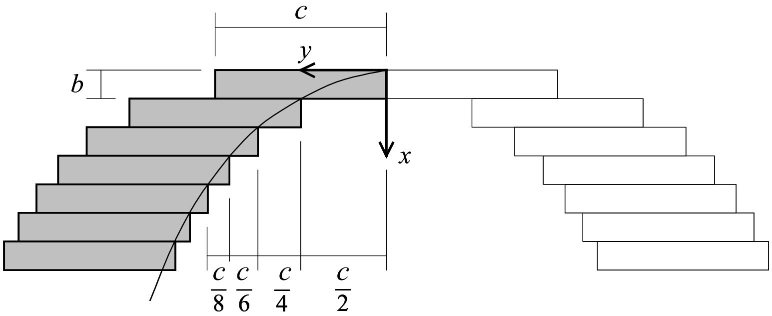


Figure 2. Pile of shifted blocks

Corbel dome

The concept of a corbel arch (p. 3) can be used for building domes too. The following program computes the coordinates x and y. In the derivation was used that the top block has a small angle.

```
x:=0: y:=0: M:=0: A:=0:
for n from 1 to 100 do
  M:=M+2/3*((y+a)^3-y^3):
  A:=A+(y+a)^2-y^2:
  x:=n*b:
  y:=M/A:
end do;
```

Treasury of Atreus

In ancient Greece was a civilisation called Mycenaean (pronounce my-se-nee-an with emphasis on my). It flourished for 500 years until 1100 BC.[1] The Mycenaeans buried their kings in corbel dome tombs (p. 4). Some still exist. One is called the treasury of Atreus (fig. 3, 4). It is located in the ancient city of Μυκηνες (pronounce me-kee-ness with emphasis on kee). It has a span of 14.5 m, a radius of curvature of 16 m and a thickness of approximately 0.8 m. Therefore, $a/t = 20$.

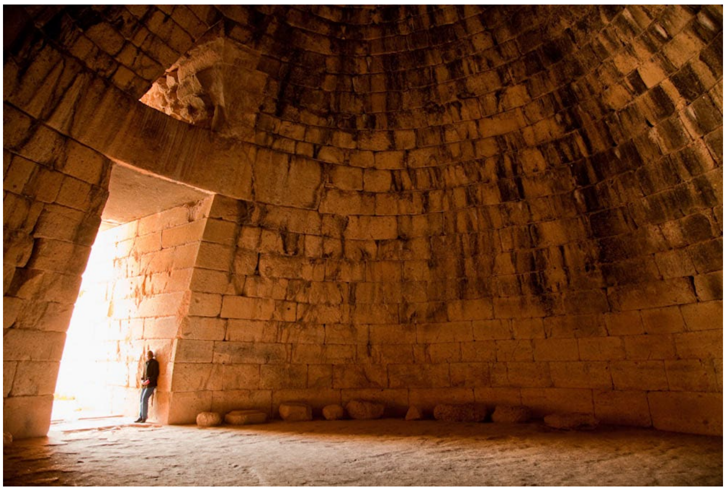


Figure 3. Interior of the treasury of Atreus [gjclarthistory.blogspot.com]

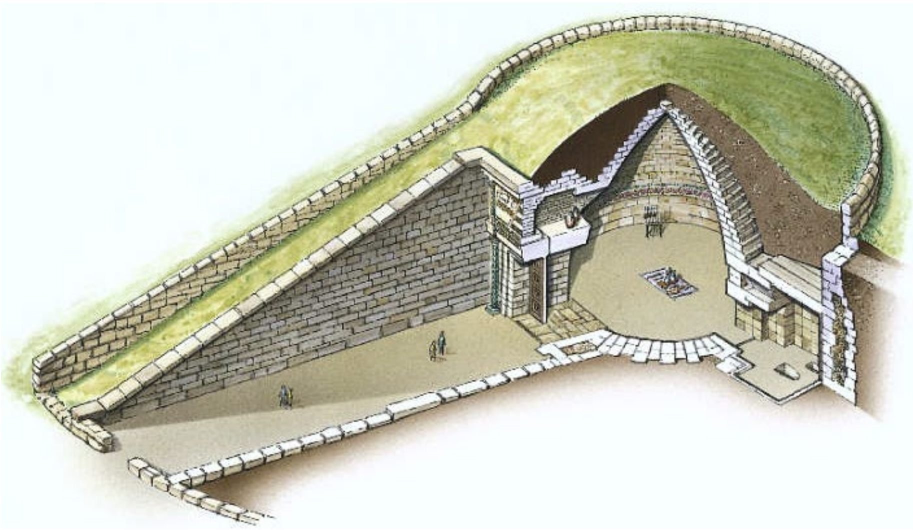


Figure 4. Structure of the treasury of Atreus [gjclarthistory.blogspot.com]

[1] The following dates provide a time frame: Around 2560 BC the oldest of the three large pyramids close to Cairo was build. In 753 BC the city of Rome was founded [Wikipedia].

Cables and arches
In 1664, Robert Hooke was curator of experiments of the Royal Society of London. He took his job very seriously and every week he showed an interesting experiment to the members of this society, which included Isaac Newton.[2] The members were enthusiastic about the experiments and published scientific papers on them. Often they forgot to mention that it was Hooke's idea they had started with. He became rather tired of this, therefore, he kept some discoveries to himself. He formulated them in Latin and published the mixed up letters [2]. One went like this.

abcccddeeeeefggiiiiiiiillmmmmnnnnnooprrsssttttttuuuuuuuux.

When Hooke died in 1703, the executor of his will gave the solution to this anagram.

Ut pendet continuum flexile, sic stabit contiguum rigidum inversum.

which can be translated as,

As hangs a flexible cable, so inverted, stand the touching pieces of an arch.

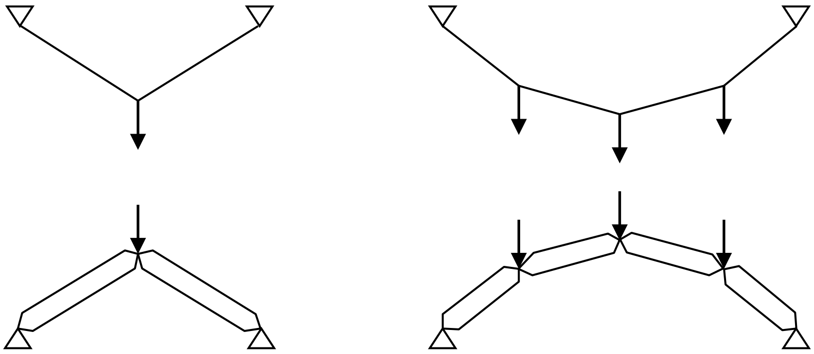


Figure 5. Hooke's discovery

Though not telling the world, it is likely that Hooke shared this discovery with his best friend Christopher Wren, who designed St Paul's Cathedral (p. 43) and supervised its construction (1669–1708).

Catenary or **funicular**
A chain hanging between two points will adopt a shape that is called *catenary* (emphasise ca) or *funicular* (emphasise ni) (fig. 6).

$$y = \frac{T}{q}\left(\cosh\frac{qx}{T} - \cosh\frac{ql}{2T}\right)$$

T is the horizontal support reaction and q is the self-weight of the chain per unit length. This shape is the solution to the differential equation

$$T\frac{d^2y}{dx^2} = q\sqrt{1+\left(\frac{dy}{dx}\right)^2} \ ,$$

and the boundary conditions

[2] Robert Hooke also encouraged Isaac Newton to use his mathematical expertise on the motions of the planets. Newton discovered his laws around 1684 [Wikipedia].

$$\begin{aligned} x &= -\tfrac{1}{2}l \quad \rightarrow \quad y = 0 \\ x &= \tfrac{1}{2}l \quad \rightarrow \quad y = 0 \end{aligned}\;.$$

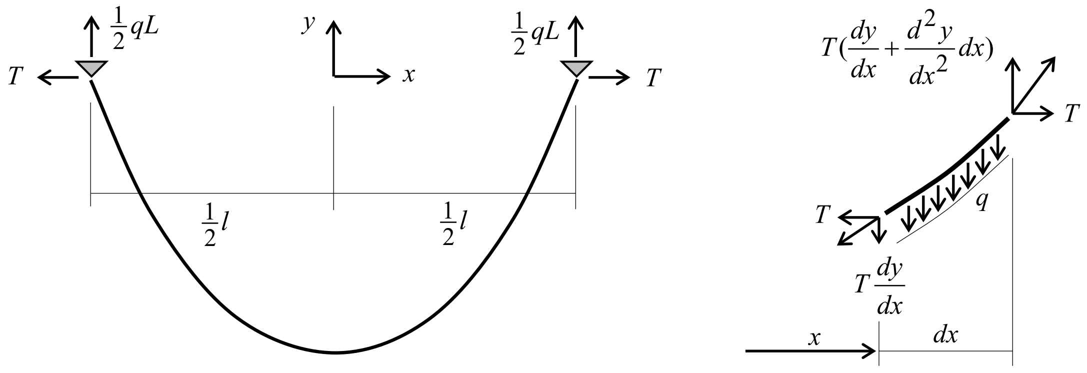


Figure 6. Catenary, T/q = 4 m, l = 14 m

The chain length is

$$L = \frac{2T}{q}\sinh\frac{q\,l}{2T}\,.$$

Challenging exercise: In 1690, Jakob Bernoulli wrote the following question in the journal *Acta Eruditorum*. What is the shape of a hanging chain? (Translated from Latin.) This problem had not been solved before. He got the right answer from three people; Gottfried Leibniz, Christiaan Huygens and his brother Johann Bernoulli [3]. (You can find these names in your history book too.) If you can derive the chain differential equation and solve it, you might be just as smart as they were.

Pressure line

In the analysis of an arch it is common to draw the pressure line for dead load. The procedure is demonstrated in an example (fig. 7) for a uniformly distributed vertical load. We first divide the distributed load into concentrated loads. Then we draw the loads head to tail in a *Magnitude plan*. We select a pole O somewhere to the left of the loads. We draw the rays Oa through Og (fig. 7, green lines). We proceed to draw the green curve in a *Line of action plan*. For this we start at the left support and draw a line parallel to ray Oa until we cross the line of action of force P_1. Next we draw a line parallel to ray Ob and so forth. The position of the pole O determines the shape of the pressure line. We make adjustments to the pole to design the shape. When you have done this a few times, you know what adjustments to make.

An arch constructed to follow a pressure line will carry loads P1 through P6 in pure axial compression. Often the pressure line is called funicular (p. 5). However, the shape is more like a parabola. In fact, if we would divide the uniformly distributed load in an infinite number of very small concentrated loads, the result would be a perfect parabola.

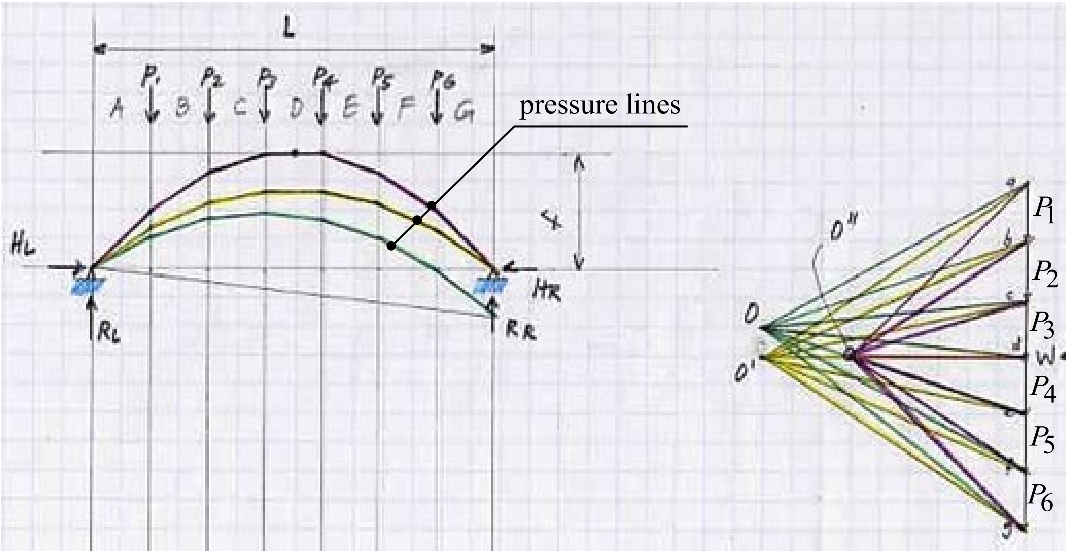


Figure 7. Line of action plan *Magnitude plan*

Exercise: In figure 7, line Oa is a vector that represents a force. Lines Oa, Ob and P1 can be rearranged into a parallelogram of forces. Draw this parallelogram of forces in the line of action plan. Do you see that the magnitude plan is a clever rearrangement of all parallelograms of forces in the line of action plan?

Exercise: In figure 7, suppose that $P_1 = P_2 = .. = P_6 = 10$ kN. What is the largest force in an arch that follows the O'' (purple) pressure line?

Middle third rule

There is no tensile stress in a rectangular cross-section, if the resulting force F is within the middle third of the thickness (fig. 8). F causes a normal force $N = F$ and a moment $M = F\,e$, where e is the eccentricity. There is no tension when e is smaller than $\frac{1}{6}t$. Since e is equal to M / N there is no tension when $-\frac{1}{6}t \le \frac{M}{N} \le \frac{1}{6}t$, which is called the *middle third rule*.

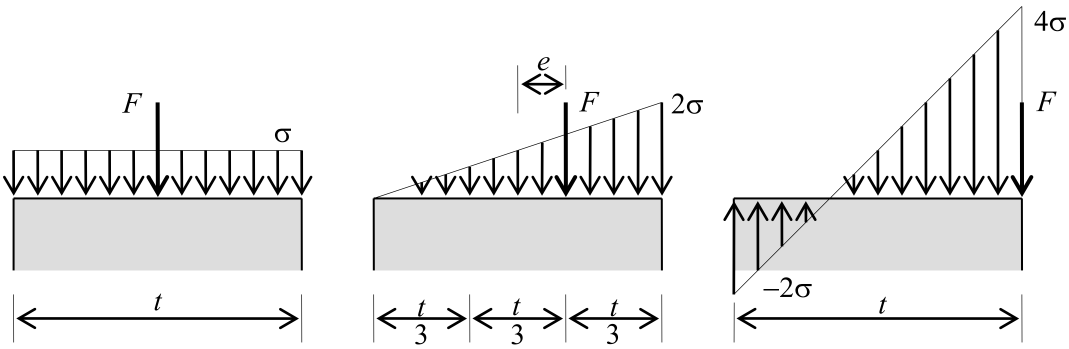


Figure 8. Stress distribution due to an eccentric normal force

Using the pressure line (p. 6) and the middle third rule we can design an arch which has no tensile stresses.

Optimal arch

Suppose we want to build an arch with as little material as possible. The arch has a span l and carries an evenly distributed line load q. The sagitta of this optimal arch is about 40% of its span. To be exact, the shape of this arch is a parabola with a ratio sagitta to span of $\sqrt{3}$ to 4 (fig. 9). The material volume of this arch is

$$V = \frac{q\,l^2}{\sqrt{3} f},$$

where f is the material compressive strength. The *abutment force* (horizontal component of the support reaction) is

$$R_h = \frac{1}{6}\sqrt{3} q l \approx 0.29 q l$$

These results are mathematically exact, however, self-weight of the arch and buckling have been neglected (See derivation in appendix 1).

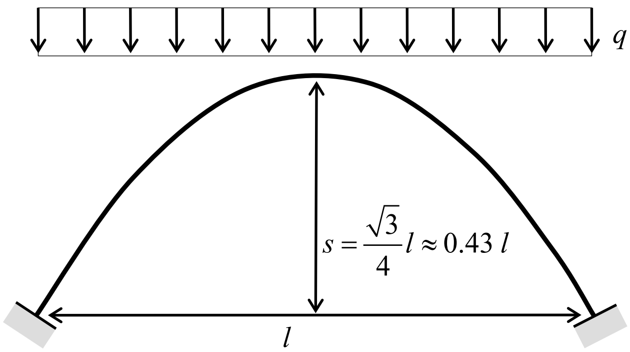


Figure 9. Optimal arch proportions

Barlow's formula

A cylindrical shell with a radius a [m] is loaded by a uniformly distributed force p [kN/m²] (fig. 10). The normal force n [kN/m] in the shell wall is

$$n = p\,a.$$

This equation is called Barlow's formula.[3] For the derivation we replace the load by compressed water. Subsequently, we cut the shell and water in halves (fig. 11). In the cut the water pressure is p and the shell forces are n. Vertical equilibrium gives $n + n = p\,2a$, which simplifies to $n = p\,a$.   Q.E.D.

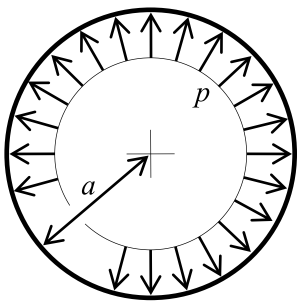


Figure 10. Cylindrical shell loaded by an evenly distributed force

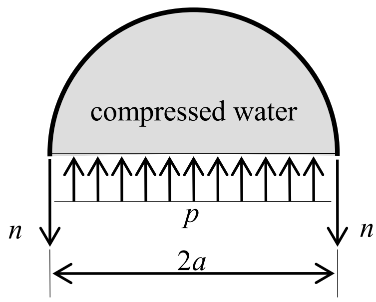


Figure 11. Derivation of Barlow's formula

Exercise: Show that the normal force n [kN/m] in a pressurised spherical shell is $n = \frac{1}{2} p\,a$.

[3] Peter Barlow (1776–1862) was an English scientist interested in steam engine kettles [Wikipedia].

Drafting spline
A spline is a flexible strip of metal, wood or plastic. Designers use it for drawing curved lines (fig. 12). For example when designing and building boats a spline is an indispensible tool. The spline is fixed in position by weights. Traditionally, the weights have a whale shape and they are made of lead. Often they are called ducks.

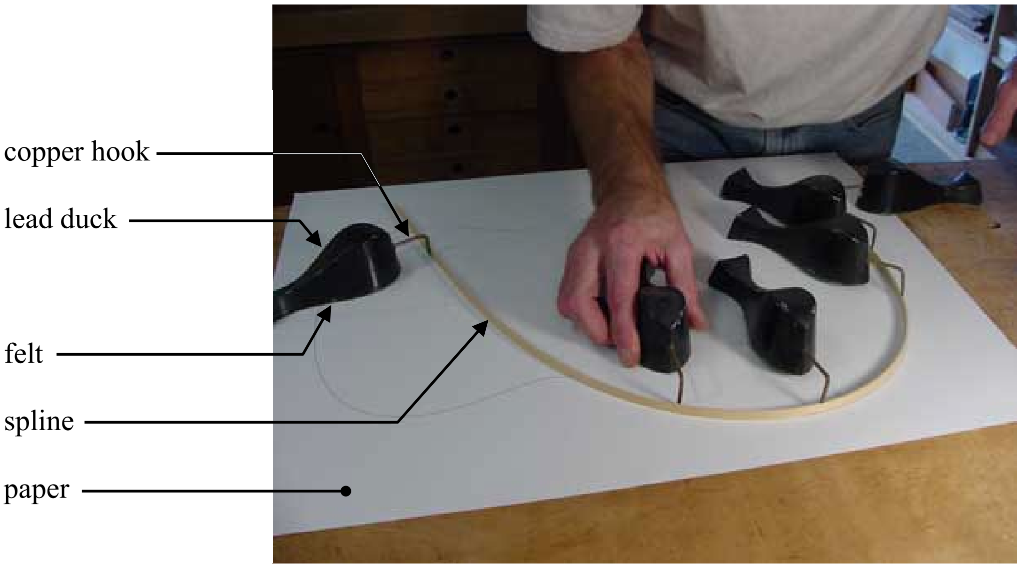


Figure 12. Spline and ducks for drawing smooth lines [Rain Noe, www.core77.com]

B-spline
In the earliest CAD programs we could draw straight lines only.[4] Every line had a begin point and an end point. This was soon extended with poly lines (plines) which also had intermediate points. It is faster to enter one pline instead of many lines. This was extended with splines. A spline is a curved line that goes smoothly through a number of points (see drafting spline p. 9). The problem with mathematically produced splines is that often loops occur which is not what we want (fig. 13). Therefore, a new line was introduced called basis spline (B-spline). Its mathematical definition is a number of smooth curves that are added. A B-spline goes through a begin point and an end point but it does not go through the intermediate points (fig. 13). The intermediate points are called control points. We can move these points on the computer screen and the B-spline follows smoothly. It acts as attached to the control points by invisible rubber bands.

NURBS
NURBS stands for Non Uniform Rational B-Spline. It is a mathematical way of defining surfaces. It was developed in the sixties to model car bodies (fig. 14). NURBS surfaces are generalizations of B-splines (p. 9). A NURBS surface is determined by an order, weighted control points and knots. We can see it as a black box in which the just mentioned data is input and any 3D point of the surface is output. Our software uses this black box to plot a surface. NURBSes are always deformed squares. They are organised in square patches which can be deformed and attached to each other (fig. 15). We can change the shape by moving the control points on the computer screen.

[4] The first version of AutoCAD was released in 1982. It run on the IBM Personal Computer which was developed in 1981. The IBM Personal Computer was one of the first computers that ordinary people could afford. It was priced at $1565 [Wikipedia]. Assuming 2.5% inflation, to date it would cost $1565 \times 1.025^{(2023-1981)} = \$4415$.

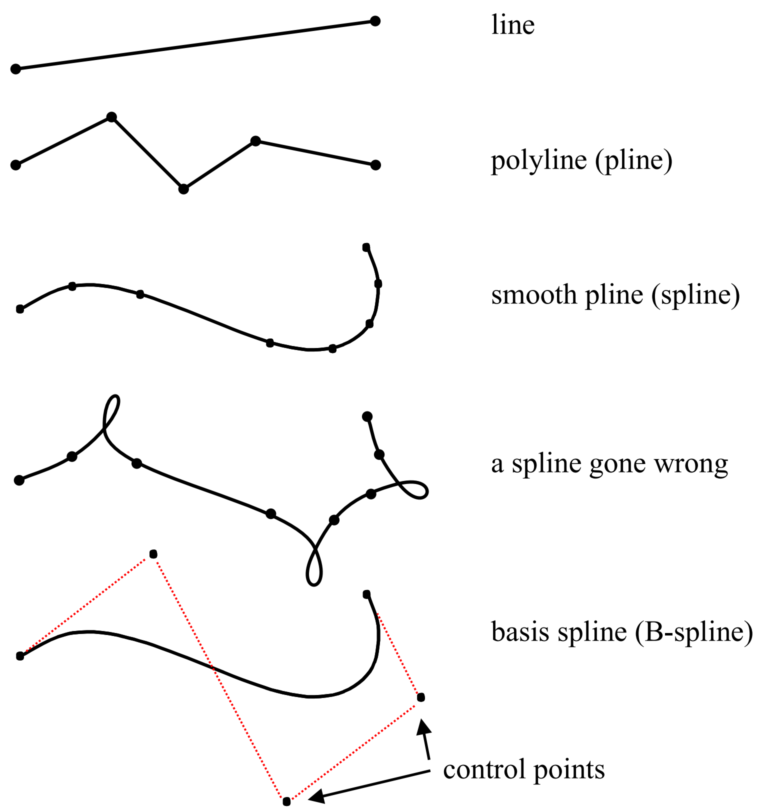


Figure 13. Types of line

Figure 14. Chrysler 1960 [www.carnut.com]

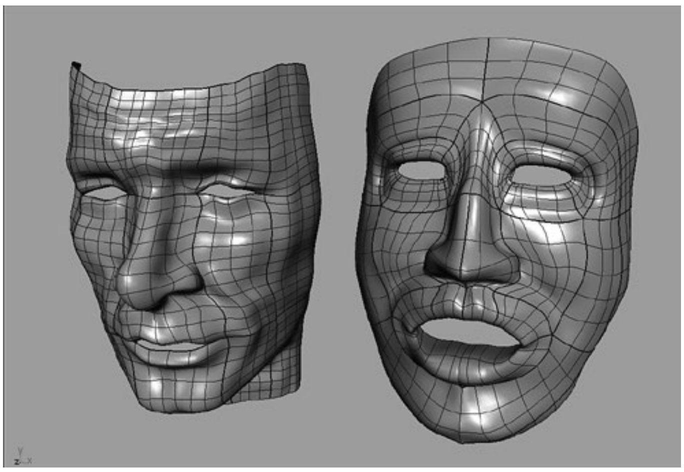


Figure 15. Faces made of NURBSes. The thin lines are NURBS edges. The thick lines are patch edges. Control points are not shown. [www.maya.com]

Continuity
Surfaces can be connected with different levels of continuity: C_0 continuity means that the surfaces are just connected, C_1 continuity means that also the tangents of the two surfaces at the connection line are the same. It can be recognized as not kinky. C_2 continuity means that also the curvatures of the two surfaces are the same at the connection line. It can be visually recognized as very smooth.
Higher orders of continuity are also possible. C_3 continuity means that also the third derivative of the surface shape in the direction perpendicular to the connection line is the same at either side of the connection line. If a shell has less than C_2 continuity then stress concentrations will occur at the connection line. Such a stress concentration is called edge disturbance (p. 14, 71).

Exercise: What is the level of continuity of the shape of a drafting spline? (p. 9)

Zebra analysis
People look fat in a convex mirror and slim in a concave mirror. Apparently, the curvature determines the width that we see. A neon light ceiling consists of parallel lines of neon light tubes. This light reflects of a car that is parked underneath. The car surface curvature determines the width of the tubes that we see. Car designers use this to inspect the continuity of a prototype car body. Any abrupt change in curvature will show as an abrupt change in tube width. The computer equivalent of this inspection is called *zebra analysis*.

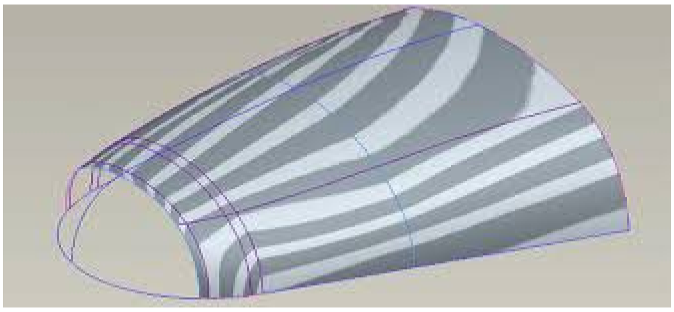


Figure 16. Simulated reflection of neon light tubes […]

Finite element mesh
A complicated shell structure needs to be analysed using a finite element program (ANSYS, DIANA, Mark, etc.). To this end the shell surface needs to be subdivided in shell finite elements (p. 82) which are triangular or quadrilateral. This subdivision is called finite element mesh. CAD software (Maya, Rhinoceros, etc) can transform a NURBS (p. 9) mesh into a finite element mesh and export it to a file. The finite element program can read this file.
The size of the finite elements is very important for the accuracy of the analyses. We need to carefully determine and adjust the element size in each part of a shell.

NURBS finite elements
Scientists are developing finite elements that look like NURBSes (p. 9). The advantage of these elements is that there is no need to transform CAD model meshes into finite element meshes (p. 12). Both meshes are the same. In the future this can save us a lot of time. However, it seems that this development is overtaken by another development. CAD programs start using polygon meshes (p. 11) instead of NURBSes. These meshes may be used directly in finite element analyses.

Polygon meshes
The problem with NURBSes (p. 9) is that they have so many control points. For example, if we have modeled Mickey Mouse and we want to make him smile we need to move more than 20 control points. This is especially impractical for animations. Therefore, CAD programs

also provide polygon meshes (fig. 17). Every part of a polygon mesh consist of a polygon, for example, a triangle, a square, a pentagon. The advantage is that we can work quickly with a rough polygon model. The mesh is automatically smoothened during rendering to any level of continuity (p. 11).

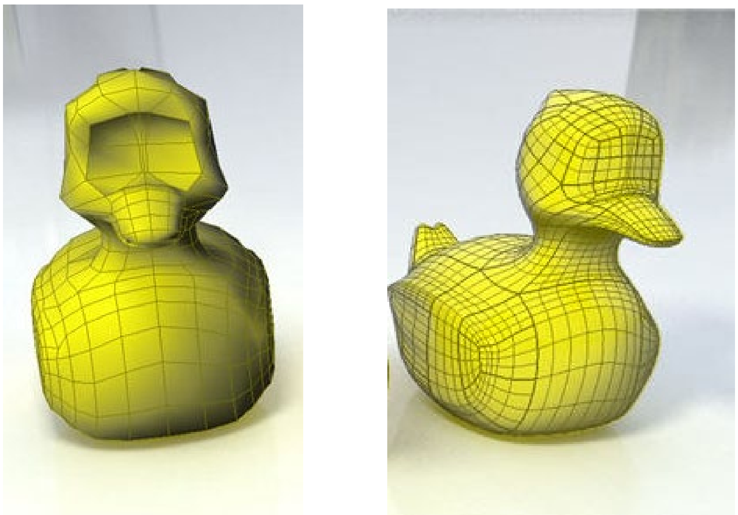


Figure 17. Polygon mesh and NURBS mesh [...]

Section forces and moments

Consider a small part of a shell structure and cut away the rest. If there were stresses in the cuts they are replaced by forces per unit length [N/m] and moments per unit length [Nm/m] (fig. 18). The membrane forces are n_{xx}, n_{yy} and $\frac{1}{2}(n_{xy}+n_{yx})$. The first two are the normal forces and the third is the in-plane shear force. The moments are m_{xx}, m_{yy} and m_{xy}. The first two are the bending moments and the third is the torsion moment. The out-of plane shear forces are v_x and v_y.

In a tent structure only membrane forces occur. Therefore, $m_{xx}=m_{yy}=m_{xy}=v_x=v_y=0$. In addition, the tent fabric can only be tensioned. Therefore, $n_1 \geq 0,\ n_2 \geq 0$, where n_1 and n_2 are the principal membrane forces (p. 98).

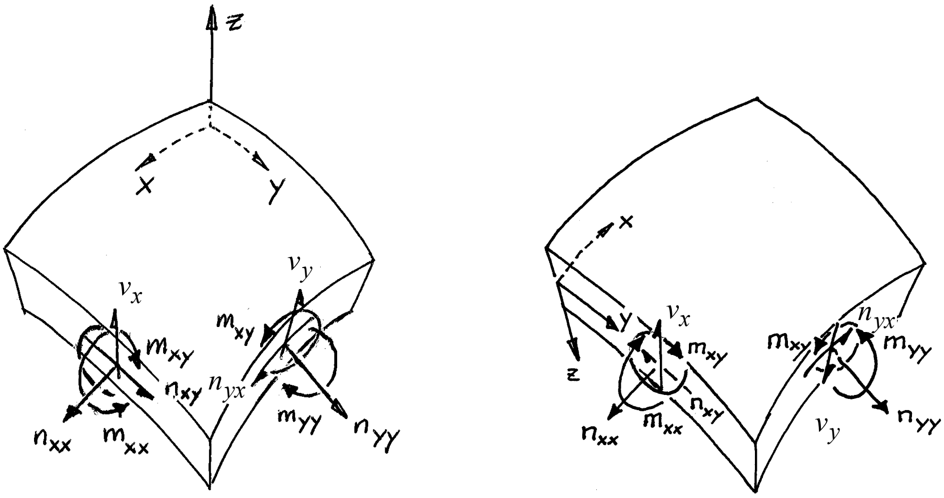


Figure 18. Positive section forces and moments in shell parts

Definition of membrane forces, moments and shear forces

In thin shells the membrane forces, the moments and the shear forces are defined in the same way as in plates.

$$n_{xx} = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{xx} dz \qquad n_{yy} = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{yy} dz \qquad \tfrac{1}{2}(n_{xy} + n_{yx}) = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{xy} dz$$

$$m_{xx} = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{xx} z dz \qquad m_{yy} = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{yy} z dz \qquad m_{xy} = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{xy} z dz$$

$$v_x = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{xz} dz \qquad v_y = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \sigma_{yz} dz$$

For thick shells the definitions are somewhat different (appendix 8).

Thickness

A shell has a small thickness t compared to other dimensions such as width, span and radius a. The following classification is used.

- *Very thick shell* ($a / t < 5$): needs to be modelled three-dimensionally; structurally it is not a shell
- *Thick shell* ($5 < a / t < 30$): membrane forces, out of plane moments and out of plane shear forces occur; all associated deformations need to be included in modelling its structural behaviour
- *Thin shell* ($30 < a / t < 4000$): membrane forces and out of plane bending moments occur; out of plane shear forces occur, however, shear deformation is negligible; bending stresses vary linearly over the shell thickness
- *Membrane* ($4000 < a / t$): membrane forces carry all loading; out of plane bending moments and compressive forces are negligible; for example a tent

Shell force flow

Brick or stone arches are thick (p. 13) because the pressure line (p. 6) needs to go through the middle third (p. 7) for all load combinations. Shell structures are often thin. This is possible due to *hoop forces* (fig. 19). The hoop forces push and pull the pressure line into the middle third for any distributed loading. In other words, a well-designed shell does not need moments to carry load.

In the bottom of a spherical dome the hoop forces are tension (for quantification see p. 38). If this dome is made of brick or stone it needs horizontal steel reinforcement, but not much.

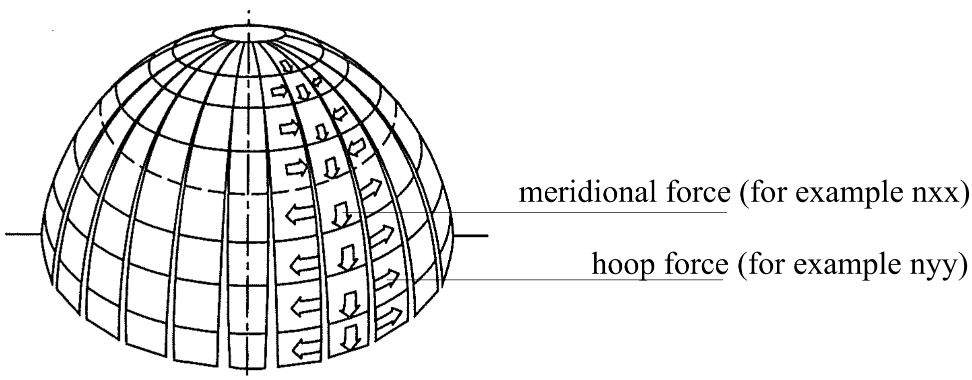


Figure 19. Forces in a spherical dome due to self-weight

Exercise: The designer of the Hagia Sophia found an even better solution for the tension hoop forces: He put windows at the locations where tension would have occurred. Which part of the Hagia Sophia dome can be classified as a shell and which part as arches?

Pantheon
The pantheon has been built in the year 126 AD in Rome as a Roman temple (fig. 20). Since the year 609 it is a catholic church. The concrete of the dome top is made of light weight aggregate called pumice (fig. 21). The hole in the roof is called oculus. The name of the designer is unknown. The construction method is unknown. It has been well maintained through the centuries, which shows that people have always considered it a very special structure. You should go there one day and see it with your own eyes.

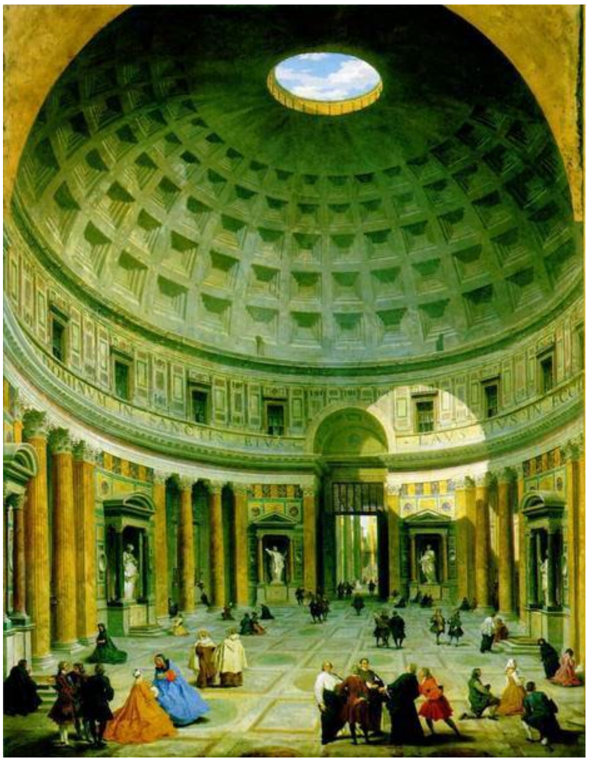


Figure 20. Pantheon painting by Panini in 1734 [National Gallery of Art, Washington D.C.]

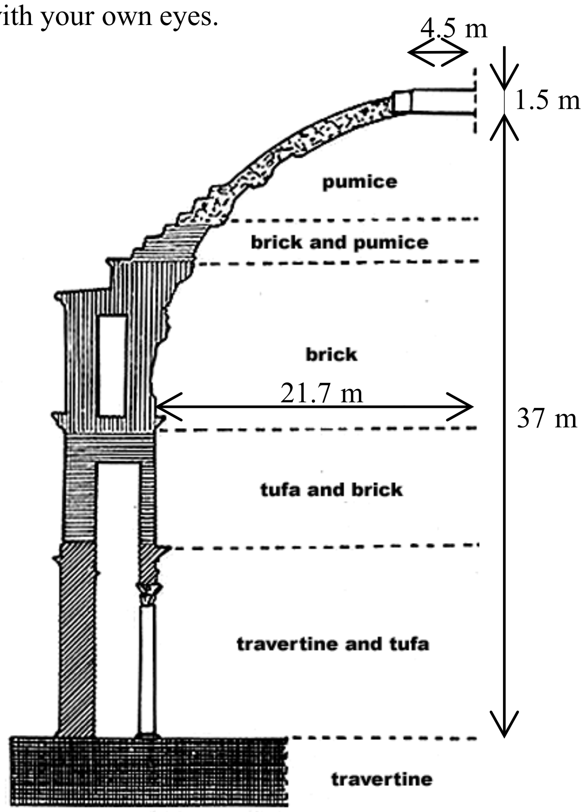


Figure 21. Pantheon cross-section [engineeringrome.org]

Edge disturbance
In a well-designed shell with distributed loads and roller supports the moments are very small (see shell force flow p. 13). However, rollers are expensive and do not resist wind, therefore, shell edges are often fixed. This causes a phenomena typical for thin shell structures: the *edge disturbance*.

Let's explain it by an experiment of thought. A dome loaded is by self-weight and supported by rollers. The membrane forces change the shape of the dome (fig. 22). This deformation is small – much smaller than the deformation of a similar size plate, truss or frame structure – but it does occur. Subsequently, we remove the rollers, push the dome edge back and fix it to the foundation (fig 23). In the process we have curved the shell wall. This curving occurs over a small width because the thin shell wall has little bending stiffness.

From the curvature we deduce that moments occur. The moment is large in the edge. The moment moves into the shell like a wave that dampens quickly. Of course, *wave* is not the right word because this wave does not move. It is called *edge disturbance*. It occurs where ever a shell edge is fixed or pinned to something solid. (see also generalised edge disturbance p. 71)

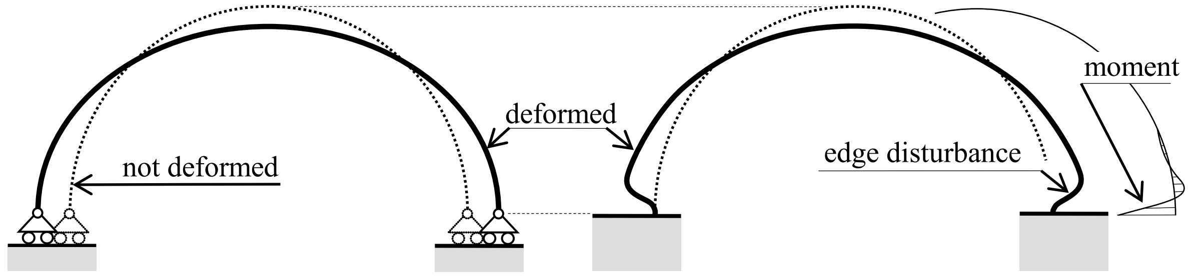


Figure 22. Dome with roller support *Figure 23. Dome with fixed support*

Compatibility moment

The moments in a well-designed thin shell do not carry load. All load in the shell is carried by the membrane forces (see shell force flow p. 13). The shell moment is caused by the deformation necessary for the parts to stay connected (see edge disturbance p. 14). Such a moment is called *compatibility moment*.

Comparison of an arch and a dome

Figure 24 shows two moment distributions. On the left-hand side is shown an arch shaped as a horse shoe fixed at the foundation and loaded by self-weight. On the right-hand side is shown a cross-section of a spherical dome also fixed at the foundation and also loaded by self-weight. (This dome could protect an airport radar from rain and wind.) The left hand distribution has been obtained by solving the differential equation. The right hand distribution has been obtained by linear elastic finite element analysis. The left and right moment distributions are in the same directions and can be compared.

We observe that the arch has moments everywhere and the dome has moments in its edge only. The shell moment demonstrates the shell force flow (p. 13) and the edge disturbance (p. 14). The arch and the shell behave very differently.

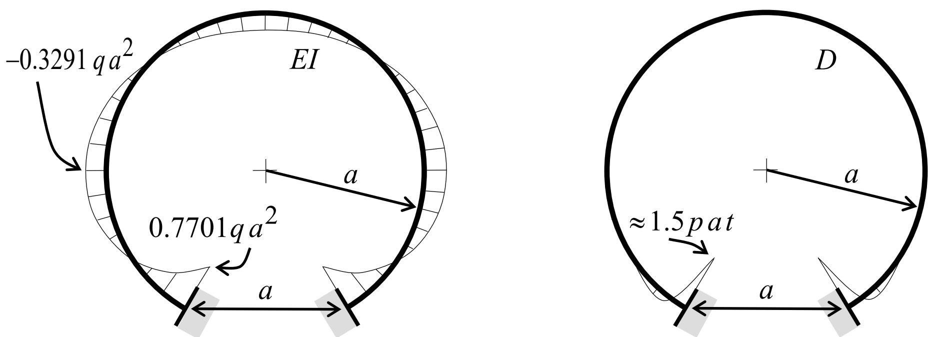


Figure 24. Linear elastic moment distributions due to self-weight in (left) a circular arch and (right) a spherical dome. Symbol a is the radius, t is the thickness, q [N/m] and p [N/m²] are self-weight. The dome result is computed for $a = 20$ m, $t = 0.05$ m, $E = 3{\cdot}10^{10}$ N/m², $\nu = 0$, $p = 1500$ N/m². The plotted dome moment is in the same direction as that of the arch.

Exercise: If you plot the arch moment in figure 24 upside down you see the pressure line (p. 6). Can you explain this?

Plastic deformation in shell edges
Figure 24 left shows the equation of the arch peak moment. The thickness *t* does not occur in this equation, while it does occur in the equation of the dome moment. When we double the thickness, self-weight will double, the arch moment will double and the dome moment will increase by a factor four. When we divide moment by section modulus we obtain stress. Doubling the thickness halves the arch bending stress but the dome bending stress stays the same.

For shell design this means that we often have to accept plastic deformation in supported shell edges. Steel edges yield. Reinforced concrete edges crack. Extra attention is required for fatigue and durability of shell edges.

Exercise: Consider live load instead of self-weight. What happens if we make a dome thicker? Do the stresses become larger, smaller or do they stay the same? Compare this to a plate.

Form finding
A tent needs to be in tension everywhere otherwise the fabric would wrinkle. Therefore, the first step in tent design is to determine a shape that satisfies this condition. This is called form finding. The designer specifies the support points and prestressing and the computer determines a tent shape that is in equilibrium everywhere.

Some architects would like to reverse this procedure and directly specify the shape while the computer would find the required prestress. In theory this is possible, however, it is not supported by any software because the optimisation to find a suitable prestressing is very time consuming [5].

In contrast, shells do not need form finding. They can be designed as any frame structure: 1) choose shape, thickness, supports and loading, 2) compute the stresses, 3) check the stresses and improve the design. Repeat this until satisfied.

Fully stressed dome
Consider a dome loaded by self-weight only. The shape and thickness are such that everywhere in the dome the maximum compressive stress occurs (fig. 25). The compressive stress is both in the meridional direction and in the hoop direction (p. 13). This dome is called a fully stressed dome because everywhere the material is loaded to its full capacity.

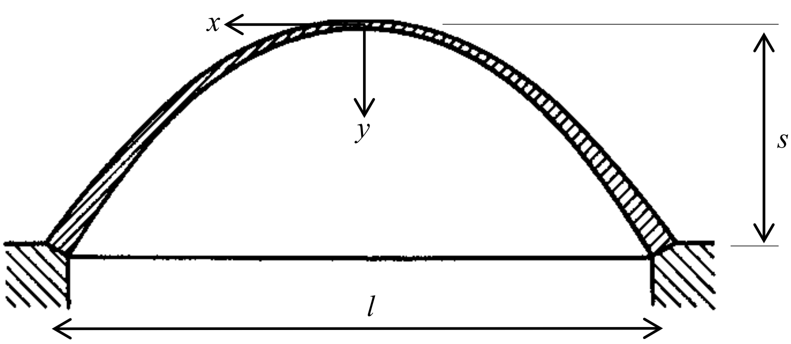


Figure 25. Cross-section of a fully stressed dome [6] (The proportions are exaggerated)

The shape of a fully stressed dome cannot be described by any mathematical function [6]. The following program can be used for calculating the dome shape. The thickness of a fully stressed dome is undetermined. (Any extra thickness gives both more load and more strength which compensate each other.) The program starts at the dome top with a specified thickness and stress. For every step in *x* a value *y* and a new thickness are determined.

```
t:=200:         # mm      top thickness
f:=4:           # N/mm2   compressive stress
rho:=2350e-9:   # kg/mm3  specific mass
g:=9.8:         # m/s2    gravitational acceleration
dx:=1:          # mm      horizontal step size
alpha:=0.1:     # rad     horizontal angle, has no influence on the results

x:=0: y:=0: V:=t*1/2*dx/2*alpha*dx/2*rho*g: H:=f*t*dx/2*alpha:
for i from 1 to 200000 do
  N:=sqrt(V^2+H^2):
  t:=N/(f*alpha*(x+dx/2)):
  dy:=V/H*dx:
  ds:=sqrt(dx^2+dy^2):
  x:=x+dx:
  y:=y+dy:
  V:=V+t*ds*alpha*x*rho*g:
  H:=H+f*t*ds*alpha:
end do:
```

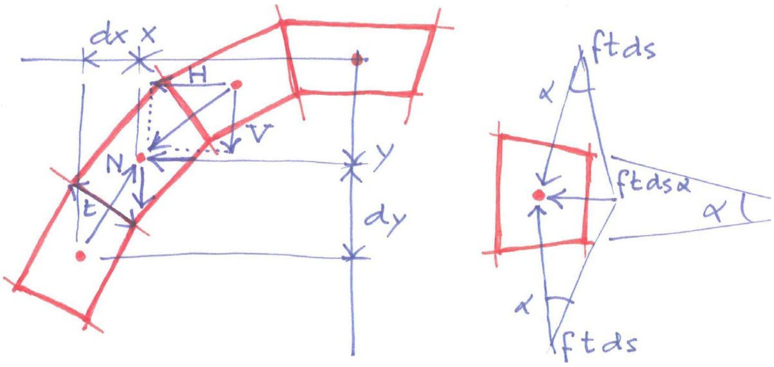


Figure 26. Derivation of the fully stressed dome program

Approximation of the fully stressed dome
For realistic material values the computed shape of a fully stressed dome (p. 16) can be approximated accurately by the formula

$$y = \frac{\rho g x^2}{4\sigma},$$

where ρ is the mass density, g is the gravitational acceleration and σ is the stress.
For example, a fully stressed masonry dome with a compressive strength of 4 N/mm² and a span of 100 m has a sagitta (p. 1) of

$$y = \frac{2000 \times 10 \times 50^2}{4 \times 4 \cdot 10^6} = 3.13 \text{ m}$$

Note that this is a very shallow dome. The above program also shows that the dome thickness is everywhere almost the same.

Buckling of the fully stressed dome
Buckling of a dome occurs at a stress of $0.1 E t / a$, therefore, $\sigma \le 0.1 E t / a$ (see buckling p. 140). The radius of curvature of the fully stressed dome top is (see line curvature p. 20)

$$a = \frac{1}{\frac{d^2 y}{dx^2}} = \frac{2\sigma}{\rho g}$$

Substitution of *a* in the buckling stress equation gives a condition for the dome thickness

$$t \geq \frac{20\sigma^2}{\rho g E}.$$

For example, the fully stressed masonry dome with Young's modulus 10000 N/mm² needs a thickness

$$t \geq \frac{20 \times (4 \cdot 10^6)^2}{2000 \times 10 \times 10000 \cdot 10^6} = 1.6 \text{ m}$$

$$\frac{a}{t} = \frac{E}{10\sigma} = \frac{10000}{10 \times 4} = 250$$

Optimal dome
Suppose we want to build a dome with a span *l* that carries its weight with as little material as possible. We call this an *optimal dome*. An optimal dome is not a fully stressed dome (p. 16). The cause is that a larger sagitta (p. 1) will give smaller stresses and a much smaller thickness, which results in less material.

The sagitta of an optimal dome is about 30% of its span. To be exact, a spherical cap of constant thickness has the optimal ratio sagitta to span of $\sqrt{3}$ to 6 (fig. 27) (derivation in appendix 2). [5]

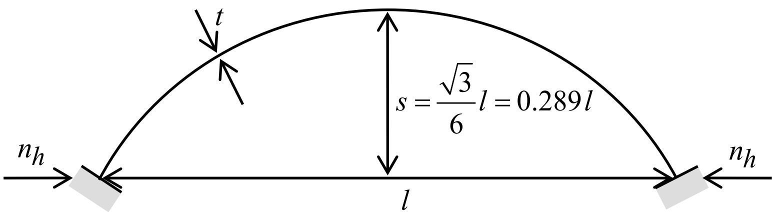


Figure 27. Proportions of an optimal spherical dome

The thickness of the spherical optimal dome is

$$t = \frac{20}{9} \frac{\rho g l^2}{E}$$

Applied to the masonry dome example above we find,

$$t = \frac{20}{9} \frac{2000 \times 10 \times 100^2}{10000 \cdot 10^6} = 0.044 \text{ m}$$

[5] Kris Riemens showed in his bachelor project at Delft University that other shapes can be more optimal than the spherical cap [7]. In addition, a varying thickness can reduce the amount of material by 15% compared to a constant thickness dome. Therefore, the exact optimal dome has not been found as yet.

This is much thinner than the fully stressed dome. A 44 mm thick masonry dome with a span of 100 m has never been build. We need to keep in mind that this dome would just carry its self-weight. Nonetheless, the equations show that great shell structures are possible.

$$a = \tfrac{1}{2}s + \tfrac{1}{8}\frac{l^2}{s} = \frac{0.3\times 100}{2} + \frac{100^2}{8\times 0.3\times 100} = 57 \text{ m} \qquad \frac{a}{t} = \frac{57}{0.044} = 1295$$

Exercise: Consider a glass dome covering a city. What thickness is needed? What thickness is needed on the Moon? Can this Moon dome be pressurised with Earth atmosphere?

Global coordinate system
Shell shapes can be described in a global Cartesian coordinate system $\overline{x}$, $\overline{y}$, $\overline{z}$. For example half a sphere is described by

$$\overline{z} = \sqrt{a^2 - \overline{x}^2 - \overline{y}^2}, \quad -\sqrt{a^2 - \overline{x}^2} \le \overline{y} \le \sqrt{a^2 - \overline{x}^2}, \quad -a \le \overline{x} \le a .$$

Local coordinate system
Consider a point on a shell surface. We introduce a positive Cartesian coordinate system in this point (fig. 28). The *z* direction is perpendicular to the surface. The *x* and *y* direction are tangent to the surface. The right-hand-rule is used to determine which axis is *x* and which is *y* (fig. 29).

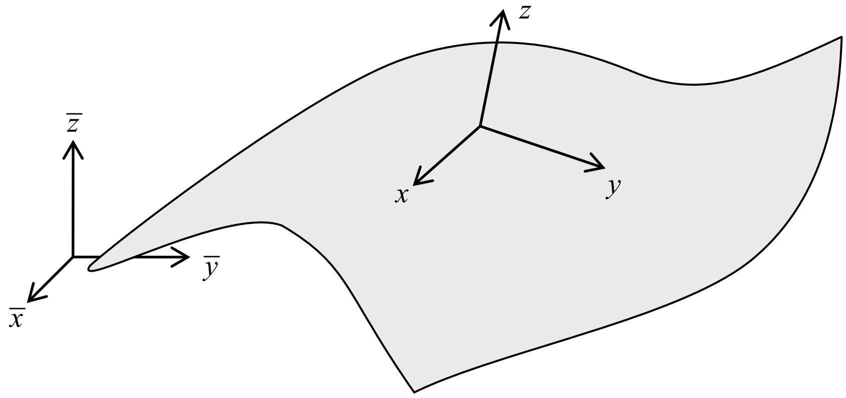


Figure 28. Global and local coordinate system

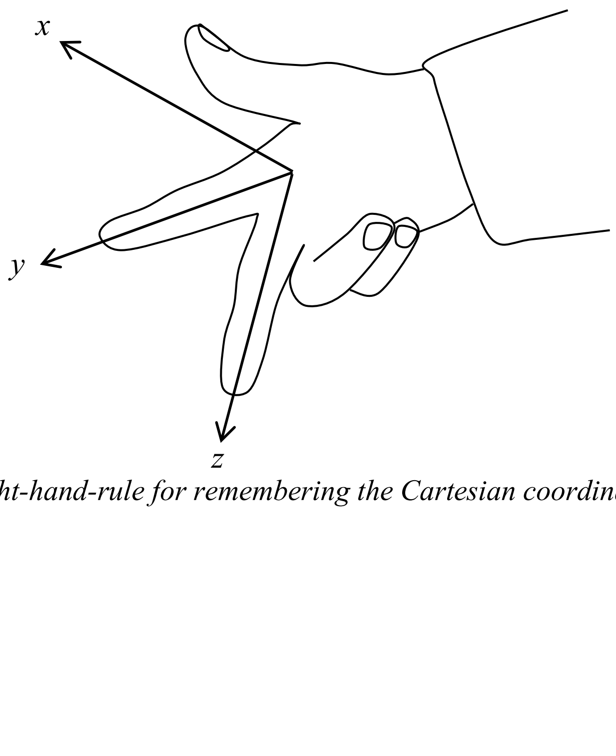


Figure 29. Right-hand-rule for remembering the Cartesian coordinate system

Line curvature
Consider a curved line on a flat sheet of paper (fig. 30). At any point of the curve there is a best approximating circle that touches the curve. The middle point of this circle is constructed by drawing two lines perpendicular to the curve at either side of the considered point. The reciprocal of the circle radius a is the curvature k at this point of the curve. The circle may lie above the curve or below the curve. We can choose to give the curvature a positive sign if the circle lies above the curve and negative sign if the circle lies below the curve. This is known as *signed curvature*. The Latin name of a best approximating circle is *circulus osculans*, which can be translated as *kissing circle*.

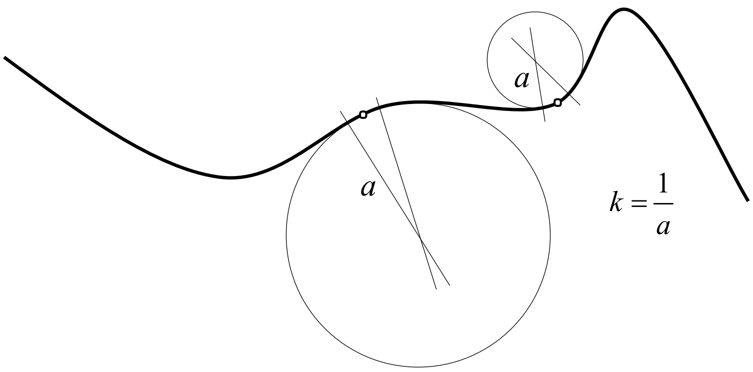


Figure 30. Curvature of a line

Exercise: Choose a local coordinate system x, y on a curve and show that $k = \pm \frac{d^2 y}{dx^2}$

Surface curvature
Curvature is also defined for surfaces. We start with a point on the surface and draw in this point a vector z that is normal to the surface (fig. 31). Subsequently, we draw any plane through this normal vector. This normal plane intersects the surface in a curved line. The curvature of this line is referred to as normal section curvature k. If the circle lies at the positive side of the z axis the normal section curvature is positive. If the circle lies at the negative side of the z axis the normal section curvature is negative. The direction of the z axis can be chosen freely (pointing inward or outward).

The z axis is part of a local coordinate system (p. 19). When the normal plane includes the x direction vector the curvature is k_{xx}. When the plane includes the y direction vector the curvature is k_{yy}. These curvatures can be calculated by

$$k_{xx} = \frac{\partial^2 z}{\partial x^2}, \qquad k_{yy} = \frac{\partial^2 z}{\partial y^2}.$$

The twist of the surface k_{xy} is defined as

$$k_{xy} = \frac{\partial^2 z}{\partial x \partial y}.$$

These formulas are valid for the local coordinate system. In the global coordinate system (p. 19) the formulas for the curvature are

$$k_{\overline{xx}} = \frac{\dfrac{\partial^2 \overline{z}}{\partial \overline{x}^2}}{\left(1 + \dfrac{\partial \overline{z}}{\partial \overline{x}}^2\right)^{\frac{3}{2}}}, \qquad k_{\overline{yy}} = \frac{\dfrac{\partial^2 \overline{z}}{\partial \overline{y}^2}}{\left(1 + \dfrac{\partial \overline{z}}{\partial \overline{y}}^2\right)^{\frac{3}{2}}}.$$

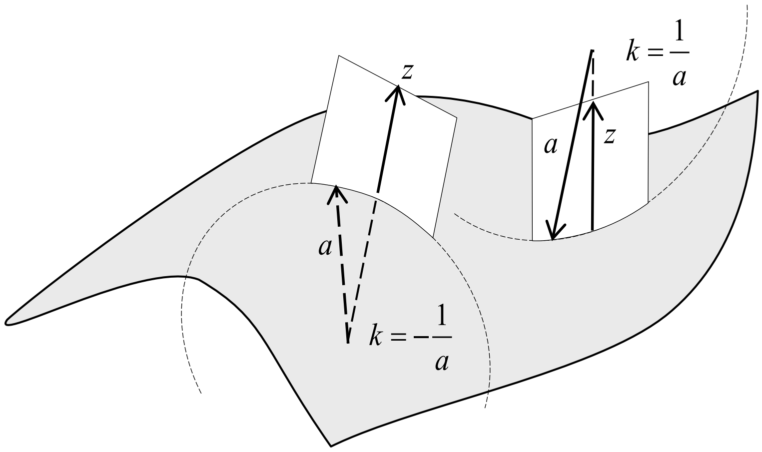


Figure 31. Normal section curvature

Note that these curvatures are not the same as the curvatures of the deformation of a flat plate. The latter curvatures are defined as

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}, \qquad \rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y},$$

where w is the deflection perpendicular to the plate.

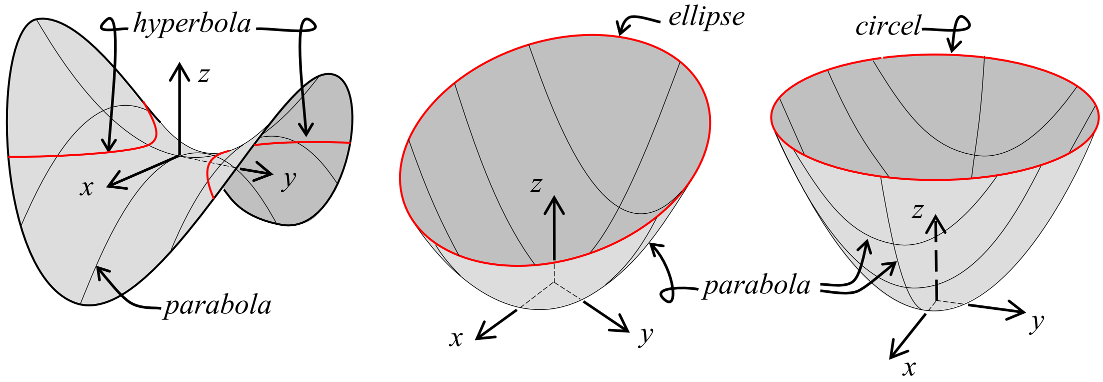
Paraboloid

A surface can be approximated around a point on the surface by

$$z = \tfrac{1}{2} k_{xx} x^2 + k_{xy} xy + \tfrac{1}{2} k_{yy} y^2 .$$

Exercise: Check this approximation by substitution in the definitions of curvature and twist.

The above function is called paraboloid. If the principal curvatures (p. 22) have opposite signs it is a hyperbolical paraboloid (hypar). If the principal curvatures have the same sign it is an elliptical paraboloid (elpar). If the principal curvatures are the same, it is a circular paraboloid (fig. 32).



Hyperbolical paraboloid (hypar) *Elliptical paraboloid (elpar)* *Circular paraboloid*
Figure 32. Types of paraboloid

Principal curvatures
In a point of a surface many normal planes are possible. If we consider all of them and compute the normal section curvatures then there will be a minimum value k_2 and a maximum value k_1. These minimum and maximum values are the *principal curvatures* at this point.

$$k_1 = \tfrac{1}{2}(k_{xx} + k_{yy}) + \sqrt{\tfrac{1}{4}\left(k_{xx} - k_{yy}\right)^2 + k_{xy}^2}$$

$$k_2 = \tfrac{1}{2}(k_{xx} + k_{yy}) - \sqrt{\tfrac{1}{4}\left(k_{xx} - k_{yy}\right)^2 + k_{xy}^2}$$

The directions in which the minimum and maximum occur are perpendicular. In fact, curvature is a second order tensor (p. 97) and can be plotted using Mohr's circle (for a proof see appendix 3).

Savill building
Savill garden is close to Windsor castle in England. Its visitors centre has a *timber grid shell* roof (fig. 33). The roof was built in 2005 using timber from the forest of Winsor castle. The roof dimensions are; length 98 m, width 24 m, height 10 m. The structural thickness is 300 mm.

$$a = \tfrac{1}{2}s + \tfrac{1}{8}\frac{l^2}{s} = \frac{10}{2} + \frac{24^2}{8 \times 10} = 12.2 \qquad \frac{a}{t} = \frac{12.2}{0.3} = 41$$

The laths are made of larch with a strength of 24 N/mm^2. The roof is closed by two layers of plywood panels each 12 mm thick (fig. 34). This plywood is part of the load carrying system. The weather proofing consist of aluminium plates. On top of this, a cladding of oak has been applied. The roof has a steel tubular edge beam. Next to the edge beam the laths are strengthened by laminated veneer lumber (LVL), which is bolted to the edge beam (fig. 33). The roof is expected to deflect 200 mm under extreme snow and wind loading [8].

Project manager:	Ridge & Partners LLP
Architect:	Glenn Howells Architects
Structural engineers:	Engineers Haskins Robinson Waters
	Buro Happold
Main contractor:	William Verry LLP
Carpenters:	The Green Oak Carpentry Co Ltd
Falsework supplier:	PERI
Owner:	Crown Estate
Costs:	£ 5.3 million

The building won several awards including one from the Institution of Structural Engineers in the United Kingdom. Before construction of Savill building the garden had approximately 80 000 visitors a year. After construction the garden attracts approximately 400 000 visitors a year.[6]

Figure 33. Savill building [...]

Gaussian curvature

The Gaussian[7] curvature of a surface in a point is the product of the principal curvatures in this point $k_G = k_1 k_2$. It can be shown that also $k_G = k_{xx} k_{yy} - k_{xy}^2$. The Gaussian curvature is independent of how we choose the directions of the local coordinate system (p. 19). A positive value means the surface is bowl-like (fig. 34). A negative value means the surface is saddle-like. A zero value means the surface is flat in at least one direction (plates, cylinders, and cones have zero Gaussian curvature).

[6] Statement by deputy ranger P. Everett in a Youtube movie of 22 September 2007: http://www.youtube.com/watch?v=3xNdVDAoI5U

[7] Carl Gauß (1777-1855) was director of the observatory of Göttingen, Germany … and a brilliant mathematician. The German letter "ß" is pronounced "s".

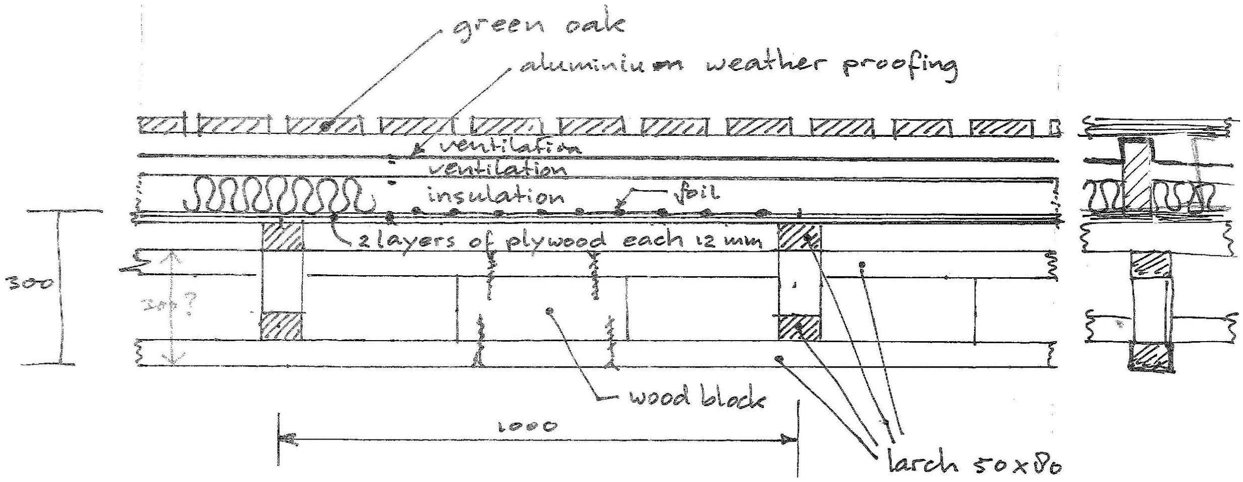


Figure 34. Cross-section of the Savill building roof

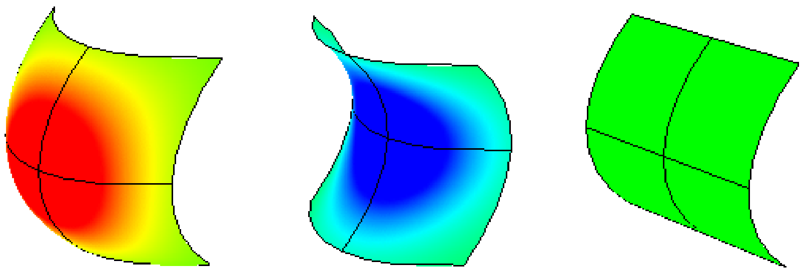


Figure 35. Gaussian curvature (contour plot)

A surface having everywhere a positive Gaussian curvature is *synclastic*. A surface having everywhere a negative Gaussian curvature is *anticlastic*. Tents need to be anticlastic and pre-tensioned in order not to wrinkle. Some surfaces have a Gaussian curvature that is everywhere the same. Examples are a plane, a cylinder, a cone, a sphere, and a tractricoid (p. 26).

The Gaussian curvature is important for the deflection of a shell due to a point load. A large Gaussian curvature (in absolute value) gives a small deflection. The Gaussian curvature is also important for the membrane stresses in a shell. Membrane stresses occur when the Gaussian curvature changes during loading (see theorema egregium p. 113).

Mean curvature
The mean curvature of a surface in a point is half the sum of the principal curvatures in this point $k_m = \frac{1}{2}(k_1 + k_2)$. It can be shown that also $k_m = \frac{1}{2}(k_{xx} + k_{yy})$. The mean curvature is independent of how we choose the local coordinate system (p. 19) except for the direction of the z axis. If the direction of the z axis is changed from outward to inward than the sign of the mean curvature changes too. For this reason CAD programs often plot the absolute value of the mean curvature.

An example of a surface with zero mean curvature is a soap film (p. 46). In a soap film there is tension, which is the same in all directions and all positions, which makes it a fully stressed design (p. 16). This property is used in form finding (p. 16) of tent structures.

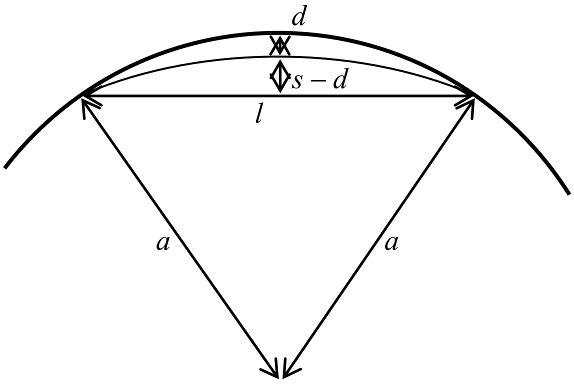
Exercise: A shell has a shape imperfection with magnitude d, length l and width l. Derive the following relations between the perfect and imperfect ($'$) curvatures. Assume that $d, s \ll l$.

$$k_1' = k_1 - 8\frac{d}{l^2}$$

$$k_2' = k_2 - 8\frac{d}{l^2}$$

$$k_m' = k_m - 8\frac{d}{l^2}$$

$$k_G' = k_G - 16\frac{d}{l^2}k_m$$



Note that the mean curvature is important for the change in the Gaussian curvature.
For example, adding a small imperfection to a shell that has zero mean curvature leads to no change in the Gaussian curvature.

Orthogonal parameterisation

A sphere can be described by $\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = a^2$. Another way of describing a sphere is

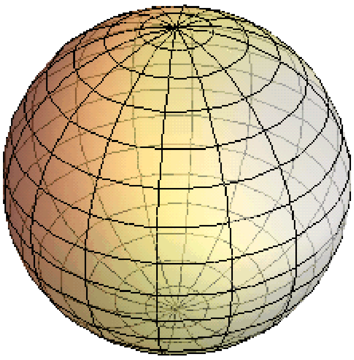
$$\bar{x} = a \sin u \cos v$$
$$\bar{y} = a \sin u \sin v$$
$$\bar{z} = a \cos u \qquad 0 \le u \le \pi \qquad 0 < v \le 2\pi$$

This is called a *parameterisation*. The parameters are *u* and *v*. There are many ways to parameterise a sphere and this is just one of them. When *u* has some constant value and *v* is varied then a line is drawn on the surface (fig. 36). The other way around, when *v* has some constant value and *u* is varied then another line is drawn on the surface. In shell analysis we choose the lines *u* = constant and the lines *v* = constant perpendicular to each other. This is called an *orthogonal* parameterisation.

Other surfaces can be parameterised too, for example catenoids (p. 26) and tractricoids (p. 26). Unfortunately, for some surfaces an orthogonal parameterisation is not available, for example there is no orthogonal parameterisation available for a paraboloid (p. 21, 102, 128). It can be easily checked whether a parameterisation is orthogonal. In this case the following equation is true.

$$\frac{\partial \bar{x}}{\partial u}\frac{\partial \bar{x}}{\partial v} + \frac{\partial \bar{y}}{\partial u}\frac{\partial \bar{y}}{\partial v} + \frac{\partial \bar{z}}{\partial u}\frac{\partial \bar{z}}{\partial v} = 0$$

The proof is simple. If we change *u* a bit, then $\bar{x}$, $\bar{y}$ and $\bar{z}$ change a bit. These $\bar{x}$, $\bar{y}$, $\bar{z}$ bits form a small vector. If we change *v* a bit, another small vector is formed. These two vectors must be perpendicular, so their dot product must be zero. Q.E.D.



$$k_{xx} = \frac{-1}{a}$$
$$k_{yy} = \frac{-1}{a}$$
$$k_{xy} = 0$$
$$\alpha_x = a$$
$$\alpha_y = a \sin u$$

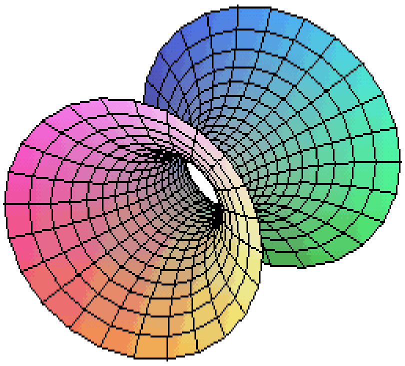
Figure 36. Parameter lines on a sphere. (α_x and α_y will be explained later.)

Exercise: I live at the location $u = 0.66029$, $v = 0.07995$. Where do you live?

Catenoid
A catenoid is formed by rotating a catenary (p. 5) around an axis (fig. 37). It can be parameterised by

$$\bar{x} = a u$$
$$\bar{y} = a \cosh u \sin v$$
$$\bar{z} = a \cosh u \cos v \qquad -\infty < u < \infty \qquad 0 \le v < 2\pi$$

The mean curvature (p. 24) is zero everywhere. The Gaussian curvature (p. 23) varies over the surface.



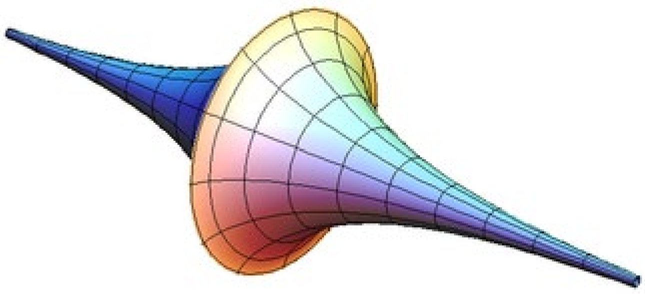
$$k_{xx} = \frac{1}{a \cosh^2 u}$$
$$k_{yy} = \frac{-1}{a \cosh^2 u}$$
$$k_{xy} = 0$$
$$\alpha_x = a \cosh u$$
$$\alpha_y = a \cosh u$$

Figure 37. Parameter lines on a catenoid

Tractricoid
A tractricoid (fig. 38) can be parameterised by

$$\bar{x} = a(\cos u + \ln \tan \frac{u}{2})$$
$$\bar{y} = a \sin u \sin v$$
$$\bar{z} = a \sin u \cos v \qquad 0 < u < \pi \qquad 0 \le v < 2\pi$$

Its volume is $\frac{4}{3}\pi a^3$ and its surface area is $4\pi a^2$, which are the same as those of a sphere. It has a constant negative Gaussian curvature $k_G = -a^{-2}$ (p. 23). Note that a sphere has a constant positive Gaussian curvature $k_G = a^{-2}$. The mean curvature (p. 24) varies over the surface of a tractricoid.



$$k_{xx} = \frac{|\tan u|}{a}$$
$$k_{yy} = \frac{-1}{a|\tan u|}$$
$$k_{xy} = 0$$
$$\alpha_x = \frac{a}{|\tan u|}$$
$$\alpha_y = a \sin u$$

Figure 38. Parameter lines on a tractricoid

Interpretation
We can interpret a parameterisation as the deformation of a rectangular sheet into a curved shell (fig. 39).

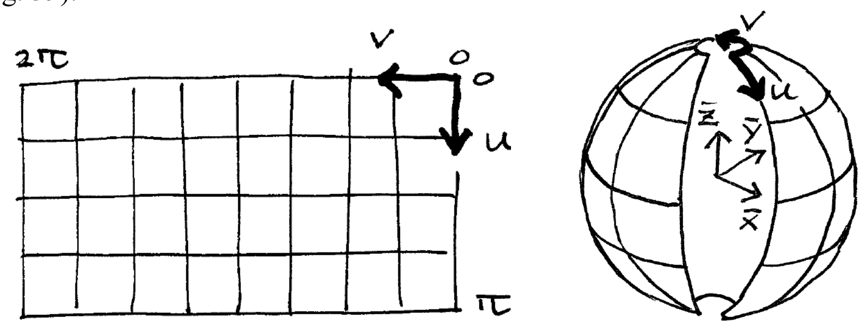


Figure 39. Deformation of a rectangular sheet

Exercise: In the above drawing, the local *z* axis is not shown. It can be deduced. In what direction is it? Into or out of the page? Inwards or outwards of the shell?

Sillogue water tower
Sillogue (pronounce silok) water tower stands close to Dublin airport in Ireland (fig. 40, 41, 42). Its shape is based on efficiency and aesthetics. (Water towers need a wide top diameter to obtain small fluctuations in water pressure when water is taken out and refilled.) It received the 2007 Irish Concrete Award for the best infrastructural project. It was honourably mentioned in the European Concrete Award 2008.

Height: 39 m
Top diameter: 38 m
Thickness: 786 mm
Steel formwork: 6300 m^2
Reinforcing steel: 580 tonnes
Concrete volume: 4950 m^3
External painting: 3700 m^2
Capacity: 5000 m^3
Engineers: McCarthy Hyder Consultants
Architects: Michael Collins and Associates
Contractor: John Cradock Ltd.
Formwork: Rund-Stahl-Bau, Austria

Figure 40. Sillogue water tower [Dublin City Council Image Gallery]

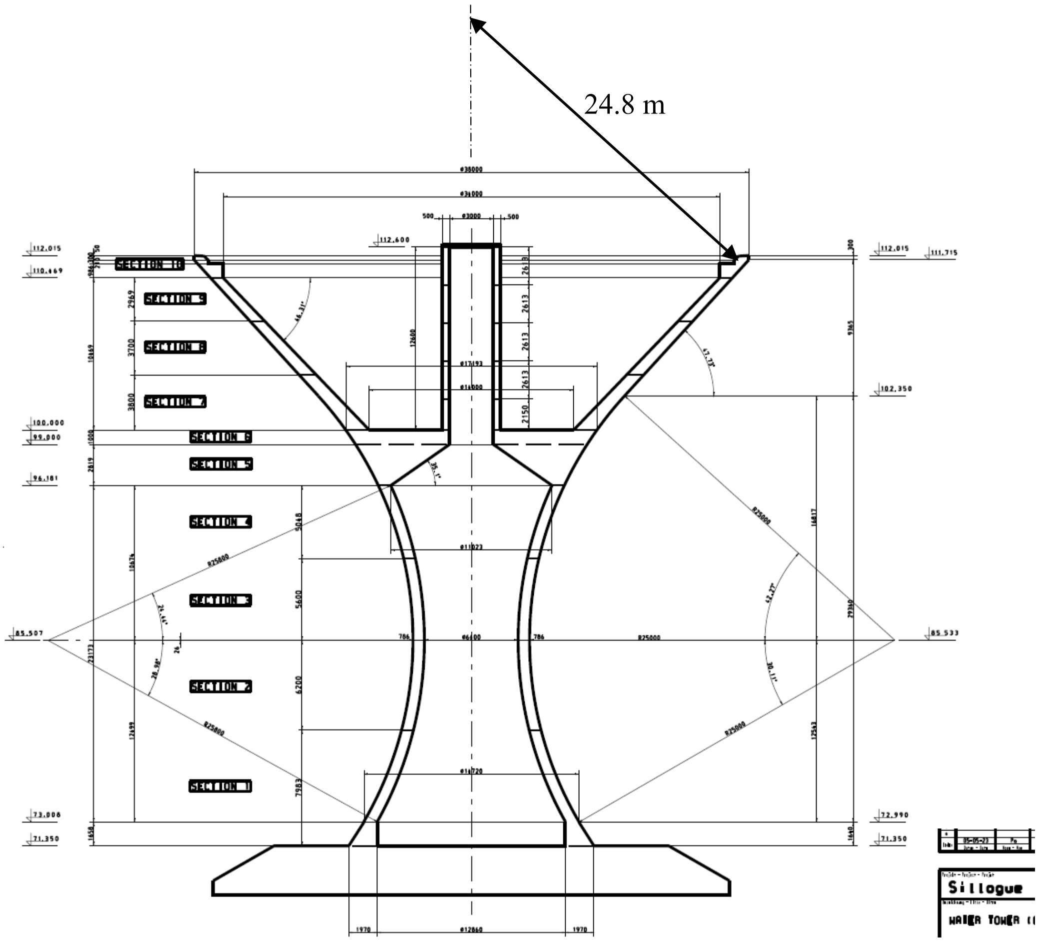


Figure 41. Cross-section of Sillogue water tower [Rund-Stahl-Bau]

Figure 42. Sillogue water tower under construction [Rund-Stahl-Bau]

To calculate the slenderness we measure the radius of curvature from the drawing. This is a line from the centre line of the tower perpendicular to the cone edge (fig. 41). The shell thickness is 0.786 m. Consequently, the slenderness is a / t = 24.8 / 0.786 = 32. This is a very small value in comparison to other shell structures (see table 1 p. 2). This suggests that the shell of Sillogue water tower could have been much thinner.

Exercise: Explain the radius of curvature of the water tower. Make a paper model or use your visual imagination. Note that the latter is a very powerful tool.

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8 Savil building online … retrieved …

Appendix 1. Optimal sagitta of an arch

An arch with a sagitta of about 40% of the span needs the least material. This appendix presents the proof.

For an evenly distributed load q [N/m] the arch has a parabolic shape (fig. 1).

$$y = s\left(1+2\frac{x}{l}\right)\left(1-2\frac{x}{l}\right), \qquad (1)$$

where l is the span and s is the sagitta.

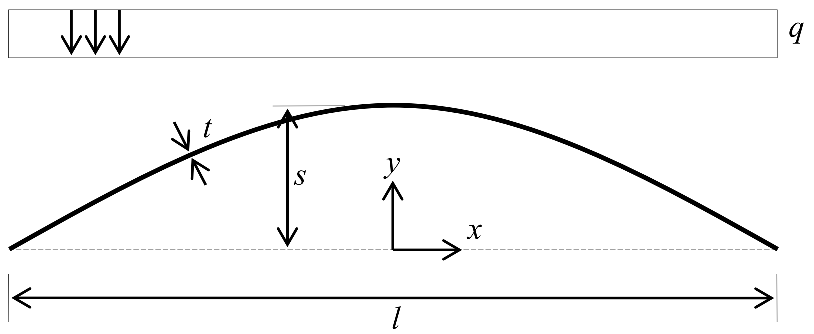


Figure 1. Parabolic arch

The volume of the arch is

$$Vol = \int_{x=-\frac{1}{2}l}^{\frac{1}{2}l} t\, w\, dz \,, \qquad (2)$$

where $t = t\,(x)$ is the thickness, w is the width and dz is a small distance along the arch. The thickness t is related to the axial force $N = N(x)$.

$$t\, w\, f = N\,, \qquad (3)$$

where f is the compressive strength of the material. The axial force N in the arch has a vertical component V and a horizontal component H (fig 2.).

$$\frac{N}{V} = \frac{dz}{dy} \qquad (4)$$

This is valid for $x < 0$. The vertical components V need to be in equilibrium with the loading q (fig. 2).

$$V = -x\, q \qquad (5)$$

This is valid for $x < 0$.

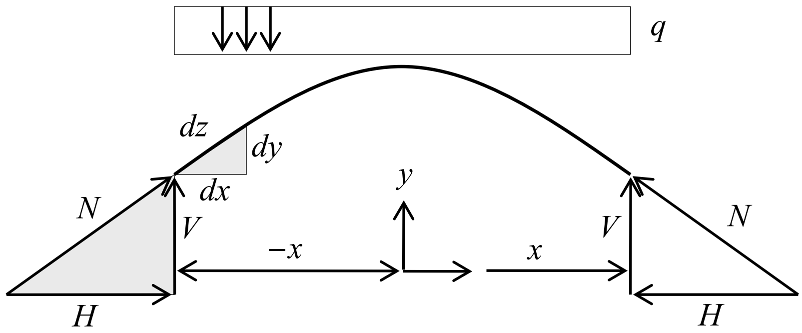


Figure 2. Section forces

Substitution of equations 3 to 5 in equation 2 gives

$$Vol = 2 \int_{x=-\frac{1}{2}l}^{0} \frac{-xq}{f} \frac{dz^2}{dxdy} dx \tag{6}$$

Using the Pythagorean theorem $dz^2 = dx^2 + dy^2$ we obtain

$$\frac{dz^2}{dxdy} = \frac{1}{\frac{dy}{dx}} + \frac{dy}{dx} . \tag{7}$$

Substitution of equations 1 and 7 in equation 6 and evaluation of the integral gives

$$Vol = ql \frac{16s^2 + 3l^2}{24 f s} . \tag{8}$$

For the minimum volume it holds

$$\frac{dVol}{ds} = 0 , \tag{9}$$

from which s can be solved.

$$s = \frac{\sqrt{3}}{4} l \approx 0.4l \tag{10}$$

Q.E.D.

Appendix 2. Optimal sagitta of a dome

A dome with a sagitta of about 30% of the span needs the least material. This appendix presents the proof.

The shape is assumed to be a spherical cap (fig. 3).

$$y = -\sqrt{a^2 - x^2} .$$

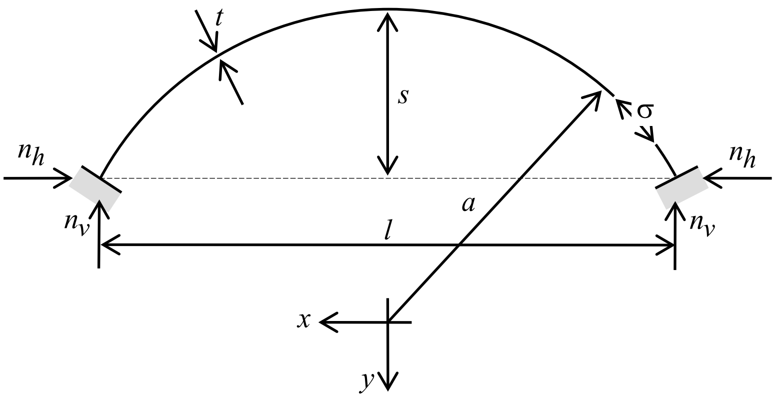


Figure 3. Dome dimensions and coordinate system

The radius of curvature is

$$a = \frac{s}{2} + \frac{l^2}{8s} .$$

The dome surface area is

$$A = \int_{\varphi=0}^{2\pi} \int_{x=0}^{\frac{1}{2}l} \sqrt{dx^2 + dy^2}\, x\, d\varphi = 2\pi \int_{x=0}^{\frac{1}{2}l} \sqrt{1 + (\frac{dy}{dx})^2}\, x\, dx = \pi a \left(2a - \sqrt{4a^2 - l^2} \right). \quad (1)$$

We assume the thickness t to be constant. The vertical support reaction is

$$n_v = \frac{A \rho g t}{\pi l} ,$$

Where ρ is the specific mass, g is the gravitational acceleration. The horizontal support reaction is

$$n_h = n_v \left. \frac{dx}{dy} \right|_{x=\frac{1}{2}l} = a \rho g t \left(\frac{2a}{\sqrt{4a^2 - l^2}} - 1 \right) .$$

The meridional stress in the dome foot is

$$\sigma = \frac{1}{t} \sqrt{n_v^2 + n_h^2} = 2 \frac{a^2 \rho g}{l^2} (2a - \sqrt{4a^2 - l^2}) . \quad (2)$$

The hoop stress in the dome foot is smaller than the meridional stress. The stress in the dome top is

$$\lim_{l \downarrow 0} \sigma = \tfrac{1}{2} a \rho g$$

We assume that the dome is fixed at the support. The thickness for which the dome almost buckles is [8] (see buckling p. 140)

$$\sigma_{cr} = \frac{1}{C\sqrt{3(1-\nu^2)}} \frac{E t}{a} \quad \Rightarrow \quad t = C\sqrt{3(1-\nu^2)}\, \frac{\sigma a}{E}, \tag{3}$$

where $1/C$ is the knockdown factor for including imperfections. The material volume V of the dome is found by substituting (1), (2) and (3) in

$$V = A\, t,$$

which can be evaluated to

$$V = \left(2a - \sqrt{4a^2 - l^2} \right)^2 \frac{a^4}{l^2} \frac{2\pi\rho g}{E} C\sqrt{3(1-\nu^2)}\,.$$

This can be rewritten in dimensionless quantities

$$\frac{VE}{2\pi\rho g\, l^4 C\sqrt{3(1-\nu^2)}} = \left(2\frac{a}{l} - \sqrt{4\frac{a^2}{l^2} - 1} \right)^2 \frac{a^4}{l^4}, \text{ where } \frac{a}{l} = \tfrac{1}{2}\frac{s}{l} + \tfrac{1}{8}\frac{l}{s}.$$

Figure 30 shows the dimensionless material volume as a function of $\frac{s}{l}$.

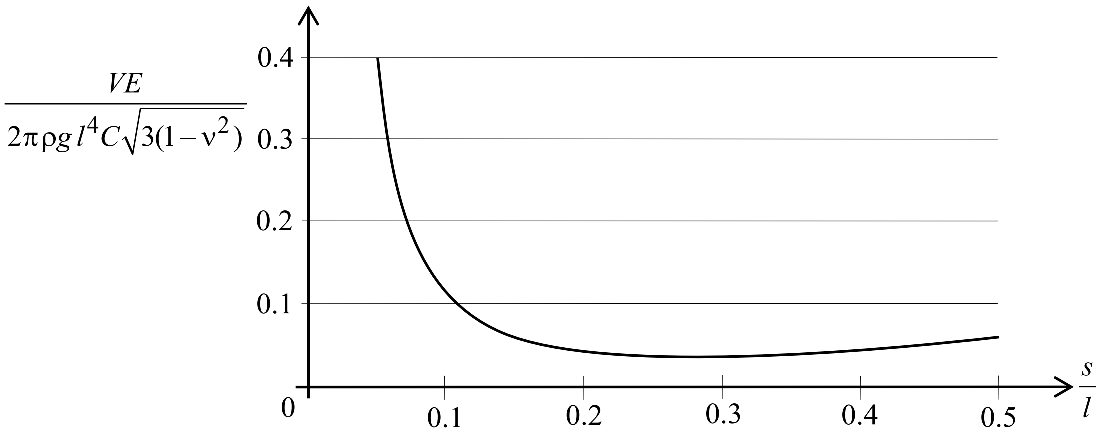


Figure 4. Material volume V as a function of sagitta s

The roots of $\frac{dV}{ds}$ are $s = -\frac{\sqrt{3}}{2} l,\ -\frac{\sqrt{3}}{6} l,\ \frac{\sqrt{3}}{6} l,\ \frac{\sqrt{3}}{2} l$.

[8] Thin domes almost always buckle before yielding or crushing. It can be shown that for yielding or crushing to occur due to self-weight the span l needs to exceed 1 km.

Therefore, the minimum material volume occurs at $\frac{s}{l} = \frac{\sqrt{3}}{6} \approx 0.3$. Q.E.D.

The optimal $\frac{s}{l}$ value does not depend on the material E, ν, ρ, it does not depend on the span l, it does not depend on the imperfections C and it does not depend on the gravity g (earth or moon).

The thickness at minimum volume is $t = \frac{2}{9} C\sqrt{3(1-\nu^2)}\,\frac{\rho g l^2}{E}$.

Since $C\sqrt{3(1-\nu^2)} \approx 6\sqrt{3(1-0.27^2)} = 10$,

the thickness is approximately $t = \frac{20}{9}\frac{\rho g l^2}{E}$.

The thickness can be written as $\frac{Et}{2\rho g l^2} C\sqrt{3(1-\nu^2)} = \frac{a^3}{l^3}\left(2\frac{a}{l} - \sqrt{4\frac{a^2}{l^2} - 1}\right)$.

Figure 31 shows the dimensionless thickness as function of $\frac{s}{l}$. For $\frac{s}{l}$ values larger than 0.3 the thickness does not change much.

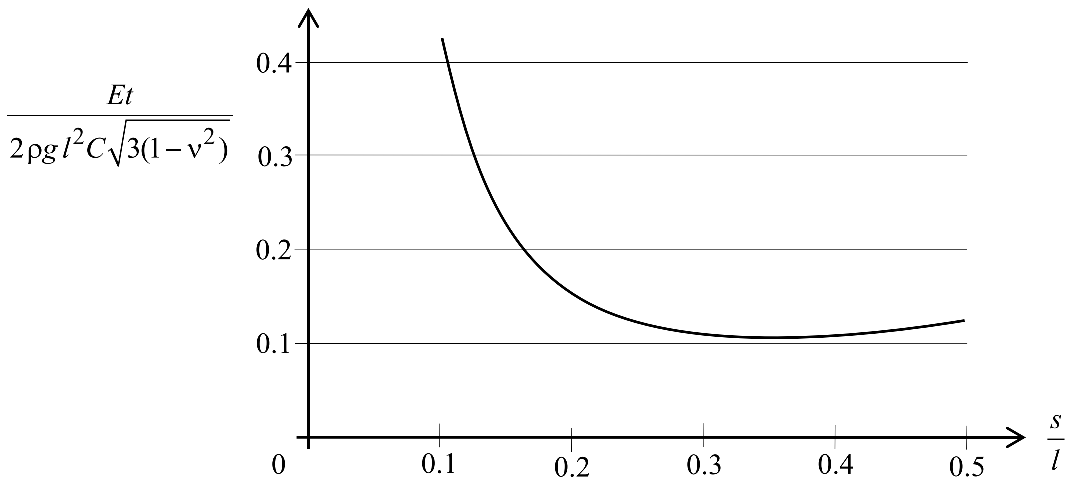


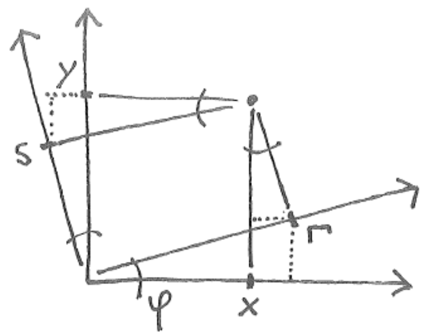
Figure 5. Thickness t as a function of the sagitta s

In this derivation it is assumed that the thickness is everywhere the same. However, the stress in the top is 25% smaller than in the foot of the dome. Therefore, the top can be 25% thinner. A varying thickness would give a somewhat different optimum sagitta.

The horizontal support reaction of the optimal dome is evaluated to $n_h = \frac{1}{3} l \rho g t$.

Appendix 3: Curvature tensor

This appendix proves that curvature is a tensor. Consider a point on a shell middle surface. In this point are a local coordinate system *x*, *y*, *z* and a rotated local coordinate system *r*, *s*, *z*. A point (*x, y*) can be expressed in (*r*, *s*) by



$$x = r\cos\varphi - s\sin\varphi$$
$$y = r\sin\varphi + s\cos\varphi$$

The shell middle surface can be described by (see page 20)

$$z = \tfrac{1}{2}k_{xx}x^2 + k_{xy}xy + \tfrac{1}{2}k_{yy}y^2$$

Only second order terms are included because higher order terms are much smaller close to the origin of the local coordinate system. Substitution of the former into the latter gives

$$z = \tfrac{1}{2}k_{xx}(r\cos\varphi - s\sin\varphi)^2 + k_{xy}(r\cos\varphi - s\sin\varphi)(r\sin\varphi + s\cos\varphi) + \tfrac{1}{2}k_{yy}(r\sin\varphi + s\cos\varphi)^2$$

The definition of curvature is (see page 20)

$$k_{rr} = \frac{\partial^2 z}{\partial r^2}, \quad k_{ss} = \frac{\partial^2 z}{\partial s^2}, \quad k_{rs} = \frac{\partial^2 z}{\partial r \partial s}$$

Substitution of the former into the latter gives

$$k_{rr} = k_{xx}\cos^2\varphi + k_{yy}\sin^2\varphi + k_{xy}2\sin\varphi\cos\varphi$$
$$k_{ss} = k_{xx}\sin^2\varphi + k_{yy}\cos^2\varphi - k_{xy}2\sin\varphi\cos\varphi$$
$$k_{rs} = (k_{yy} - k_{xx})\sin\varphi\cos\varphi + k_{xy}(\cos^2\varphi - \sin^2\varphi)$$

A quantity that can be transformed to another coordinate system by these equations is by definition a tensor (dimensions 2, rank 2). Q.E.D.

The transformation equations can be rewritten as

$$k_{rr} = \tfrac{1}{2}(k_{xx} + k_{yy}) + \tfrac{1}{2}(k_{xx} - k_{yy})\cos 2\varphi + k_{xy}\sin 2\varphi$$
$$k_{ss} = \tfrac{1}{2}(k_{xx} + k_{yy}) - \tfrac{1}{2}(k_{xx} - k_{yy})\cos 2\varphi - k_{xy}\sin 2\varphi$$
$$k_{rs} = \qquad\qquad -\tfrac{1}{2}(k_{xx} - k_{yy})\sin 2\varphi + k_{xy}\cos 2\varphi$$

and as

$$\begin{bmatrix} k_{rr} & k_{rs} \\ k_{rs} & k_{ss} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

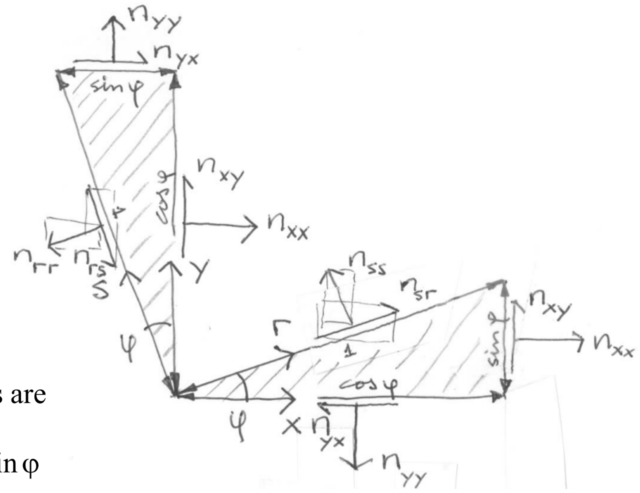
and as

$$k_{ij} = \sum_{m=x,y} \sum_{n=x,y} k_{mn}\, t_{mi}\, t_{nj} \qquad i = r, s \quad j = r, s$$

The equations can be plotted by Mohr's circle.

Appendix 13: Membrane force tensor

This appendix shows that shell membrane forces can be transformed to another coordinate system in almost the same way as regular tensors. Consider a local coordinate system x, y, z and a rotated local coordinate system r, s, z. Consider two small triangular shell parts.



The equilibrium equations of these parts are

$$n_{rr} 1 \cos\varphi - n_{rs} 1 \sin\varphi = n_{xx} \cos\varphi + n_{yx} \sin\varphi$$
$$n_{rs} 1 \cos\varphi + n_{rr} 1 \sin\varphi = n_{yy} \sin\varphi + n_{xy} \cos\varphi$$
$$n_{sr} 1 \sin\varphi + n_{ss} 1 \cos\varphi = n_{yy} \cos\varphi - n_{xy} \sin\varphi$$
$$n_{ss} 1 \sin\varphi - n_{sr} 1 \cos\varphi = n_{xx} \sin\varphi - n_{yx} \cos\varphi$$

This can be written as

$$n_{rr} = n_{xx} \cos^2\varphi + n_{yy} \sin^2\varphi + (n_{xy} + n_{yx}) \sin\varphi \cos\varphi$$
$$n_{ss} = n_{xx} \sin^2\varphi + n_{yy} \cos^2\varphi - (n_{xy} + n_{yx}) \sin\varphi \cos\varphi$$
$$n_{rs} = (n_{yy} - n_{xx}) \sin\varphi \cos\varphi + n_{xy} \cos^2\varphi - n_{yx} \sin^2\varphi$$
$$n_{sr} = (n_{yy} - n_{xx}) \sin\varphi \cos\varphi + n_{yx} \cos^2\varphi - n_{xy} \sin^2\varphi$$

and as

$$n_{rr} = \tfrac{1}{2}(n_{xx} + n_{yy}) + \tfrac{1}{2}(n_{xx} - n_{yy}) \cos 2\varphi + \tfrac{1}{2}(n_{xy} + n_{yx}) \sin 2\varphi$$
$$n_{ss} = \tfrac{1}{2}(n_{xx} + n_{yy}) - \tfrac{1}{2}(n_{xx} - n_{yy}) \cos 2\varphi - \tfrac{1}{2}(n_{xy} + n_{yx}) \sin 2\varphi$$
$$n_{rs} = \tfrac{1}{2}(n_{xy} - n_{yx}) - \tfrac{1}{2}(n_{xx} - n_{yy}) \sin 2\varphi + \tfrac{1}{2}(n_{xy} + n_{yx}) \cos 2\varphi$$
$$n_{sr} = -\tfrac{1}{2}(n_{xy} - n_{yx}) - \tfrac{1}{2}(n_{xx} - n_{yy}) \sin 2\varphi + \tfrac{1}{2}(n_{xy} + n_{yx}) \cos 2\varphi$$

and as

$$\begin{bmatrix} n_{rr} & n_{rs} \\ n_{sr} & n_{ss} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} n_{xx} & n_{xy} \\ n_{yx} & n_{yy} \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

and as

$$n_{ij} = \sum_{m=x,y} \sum_{n=x,y} n_{mn}\, t_{mi}\, t_{nj} \qquad i = r, s \quad j = r, s$$