Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT4150 Plasticity Theory
Wednesday 9 June 2004, 9:00 – 12:00 hours

Problem 1

A frame consists of two columns and a roof beam (Figure 1). The left hand column is rigidly connected to the roof beam and the foundation. The right hand column is connected with hinges to the roof beam and the foundation. The frame is loaded by five forces λF . The columns and beam have different yield contours (Figure 2). The following relation exists between the plastic moment M_p and the plastic normal force N_p .

$$N_p = \beta \frac{M_p}{a}$$

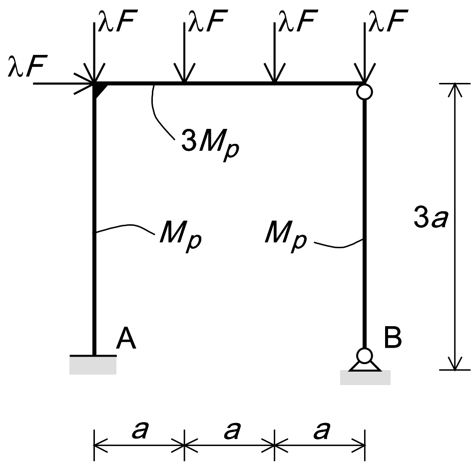


Figure 1. Frame

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

a Assume $\beta \to \infty$. Determine the collapse load λF for every possible mechanism. Write the collapse loads as functions of M_p and *a*. Which mechanism is decisive? What is the corresponding collapse load? (1.5 points)

b Assume $\beta \to \infty$. Draw the bending moment and normal force diagram for the whole structure at the moment of collapse (1 point).

c Assume $\beta = 2$. Choose one of the following problems (You need not do both). Use the decisive mechanism of problem **1a** (2.5 points).

– Determine the largest lower-bound for λF.

– Determine the smallest upper-bound for λF.

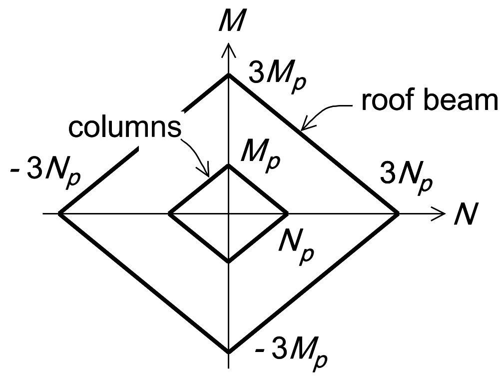


Figure 2. Yield contour

Problem 2

A square plate is simply supported at parts of the edges (Figure 3). One edge of the plate carries a evenly distributed load λf [kN/m]. The plate is homogeneous. The yield moment in the x direction is m_p. The yield moment in the y direction is $3\,m_p$ [kNm/m].

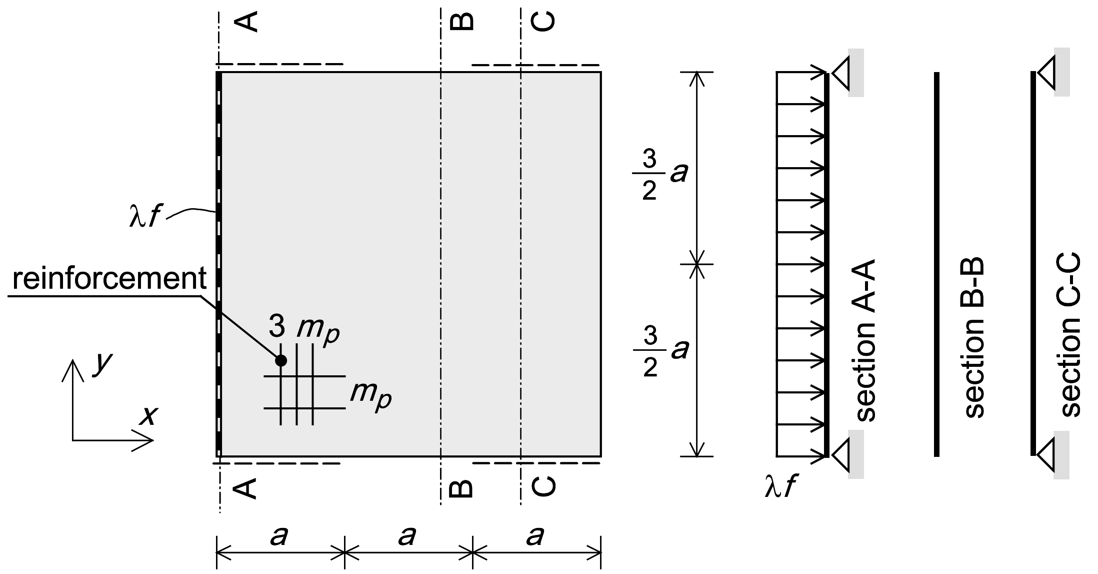


Figure 3. Simply supported square plate

a We consider the yield line patterns of Figure 4 and Figure 5. Which of these patterns give kinematically possible mechanisms (2 points).

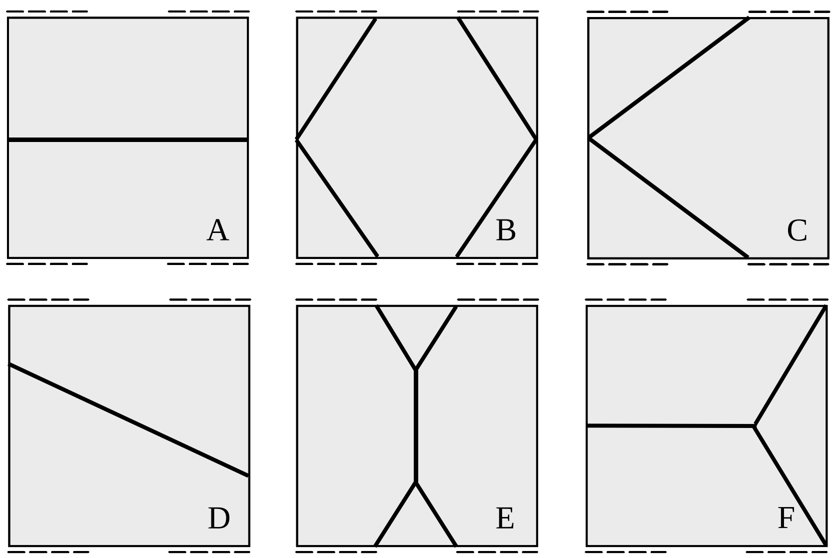


Figure 4. Yield line patterns of problem 2a (see also Figure 5)

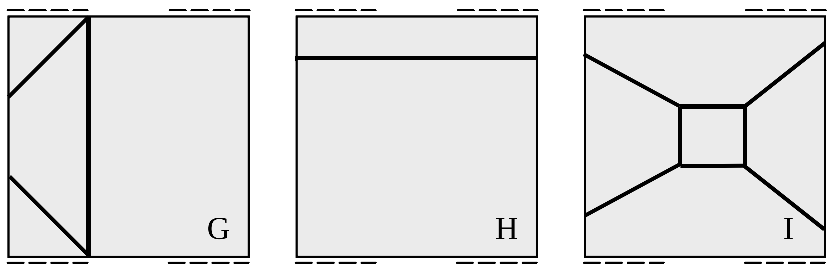


Figure 5. Yield line patterns of problem 2a

b We consider the yield line pattern of Figure 6. Determine an <u>upper bound</u> for λf expressed in m_p and a (1 point).

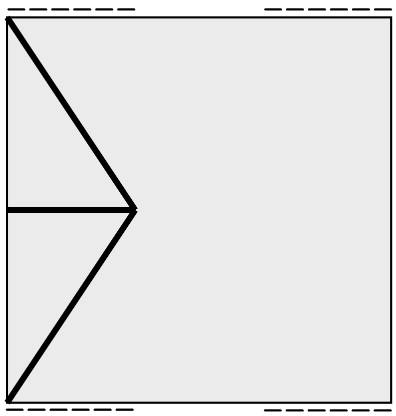
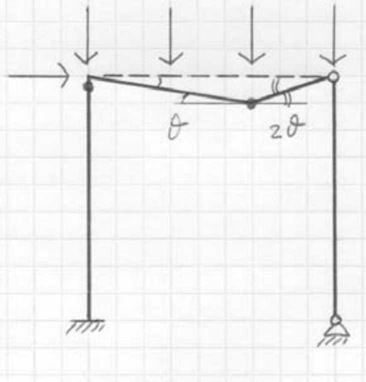


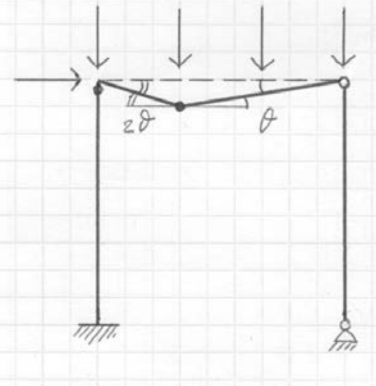
Figure 6. Yield line pattern of problem 2b

c Determine the largest <u>lower-bound</u> for λf using torsion free beams ($m_{xy} = 0$) in the x direction and y direction (2 points).

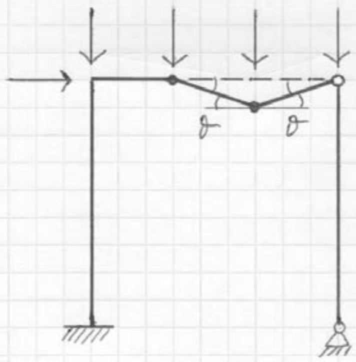
Answer to Problem 1a



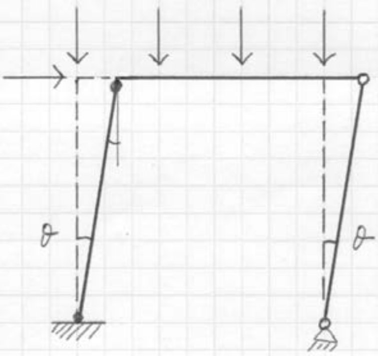
$$A = \lambda F \vartheta a + \lambda F \vartheta 2a$$
$$= 3 \lambda F \vartheta a$$
$$E = M_p \vartheta + 3 M_p (\vartheta + 2\vartheta)$$
$$= 10 M_p \vartheta$$
$$\lambda F = \frac{10}{3} \frac{M_p}{a}$$



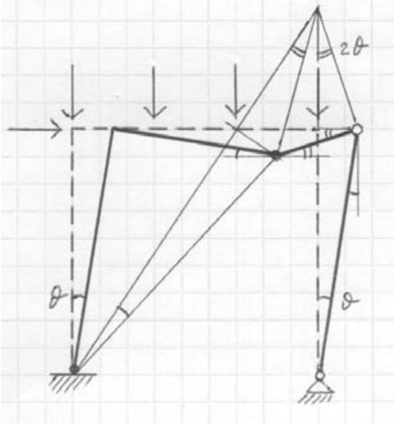
$$A = \lambda F 2\vartheta a + \lambda F \vartheta a$$
$$= 3 \lambda F \vartheta a$$
$$E = M_p 2\vartheta + 3 M_p (2\vartheta + \vartheta)$$
$$= 11 M_p \vartheta$$
$$\lambda F = \frac{11}{3} \frac{M_p}{a}$$



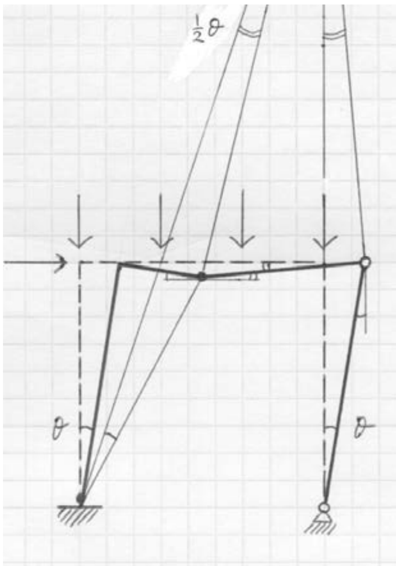
$$A = \lambda F \vartheta a$$
$$E = 3 M_p (\vartheta + \vartheta) + 3 M_p \vartheta$$
$$\lambda F = 9 \frac{M_p}{a}$$



$$A = \lambda F \vartheta 3a$$
$$E = M_p \vartheta + M_p \vartheta$$
$$\boxed{\lambda F = \frac{2}{3} \frac{M_p}{a}}$$ decisive

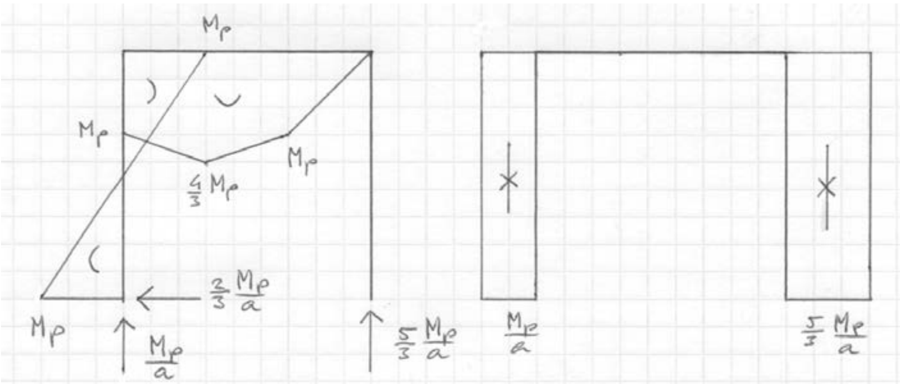


$$A = \lambda F \vartheta 3a + \lambda F \vartheta a + \lambda F \vartheta 2a$$
$$= 6 \lambda F \vartheta a$$
$$E = M_p \vartheta + 3M_p(\vartheta + 2\vartheta)$$
$$= 10\, M_p \vartheta$$
$$\lambda F = \frac{5}{3}\frac{M_p}{a}$$



$$A = \lambda F \vartheta 3a + \lambda F \vartheta a + \lambda F \tfrac{1}{2}\vartheta a$$
$$= \frac{9}{2}\lambda F \vartheta a$$
$$E = M_p \vartheta + 3M_p(\vartheta + \tfrac{1}{2}\vartheta)$$
$$= \frac{11}{2} M_p \vartheta$$
$$\lambda F = \frac{11}{9}\frac{M_p}{a}$$

Answer to Problem 1b



Answer to problem 1c Lower-bound

We reduce the loading with a factor α. Also the moments and the normal forces are reduced with α. In a section with a moment αM_p we can allow a normal force $(1-\alpha)N_p$. Therefore, at the bottom and top of the left hand column we have

$$N = \alpha \frac{M_p}{a} = (1-\alpha) N_p = (1-\alpha)\beta \frac{M_p}{a} .$$

Consequently,

$$\alpha = (1-\alpha)\beta$$

and

$$\alpha = \frac{\beta}{1+\beta} .$$

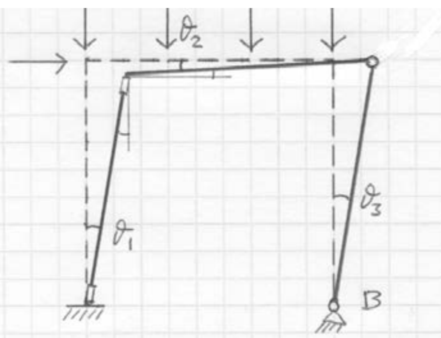
Since $\beta = 2$ we find

$$\alpha = \tfrac{2}{3} .$$

The collapse load is

$$\lambda F = \alpha \frac{2}{3} \frac{M_p}{a} = \frac{4}{9} \frac{M_p}{a}$$

Answer to Problem 1c Upper-bound



$u = \vartheta \frac{a}{2}$

horizontal displacement of point B →

$\vartheta_1 3a - \vartheta_3 3a = 0$

vertical displacement of point B ↓

$\vartheta_1 \frac{a}{2} + (\vartheta_1 + \vartheta_2)\frac{a}{2} - \vartheta_2 3a = 0$

$$\begin{cases} \vartheta_1 = \vartheta_3 \\ \vartheta_2 = \frac{2}{5}\vartheta_3 \end{cases}$$

$A = \lambda F \vartheta_1 3a + \lambda F \vartheta_2 3a + \lambda F \vartheta_2 2a + \lambda F \vartheta_2 a$

$\quad = \frac{27}{5} \lambda F \vartheta_3 a$

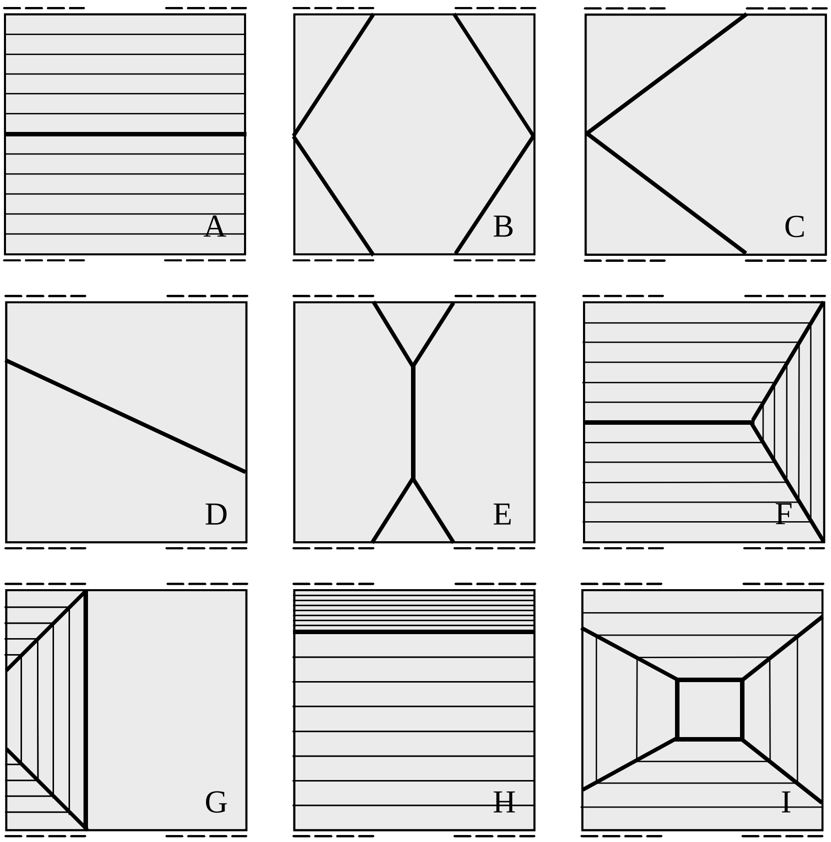
$E = M_p \vartheta_1 + M_p(\vartheta_1 + \vartheta_2)$

$\quad = \frac{12}{5} M_p \vartheta_3$

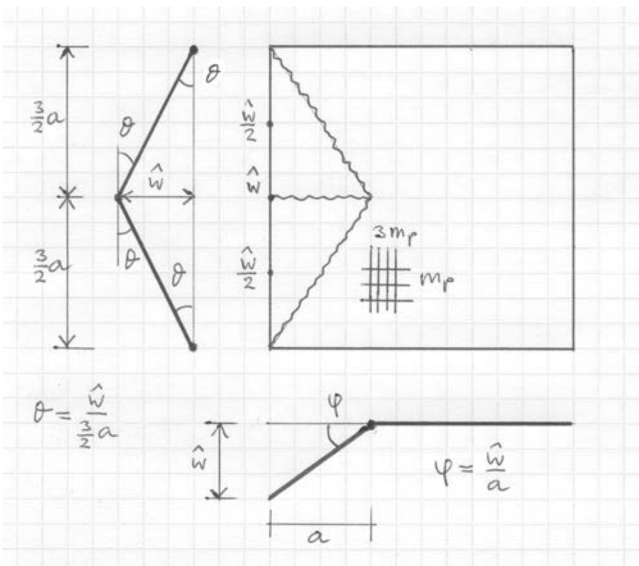
$\lambda F = \frac{4}{9} \frac{M_p}{a}$

Answer to Problem 2a

Kinematically possible are patterns A, F, G, H and I. The figure below shows the altitude lines of the deformed mechanisms.

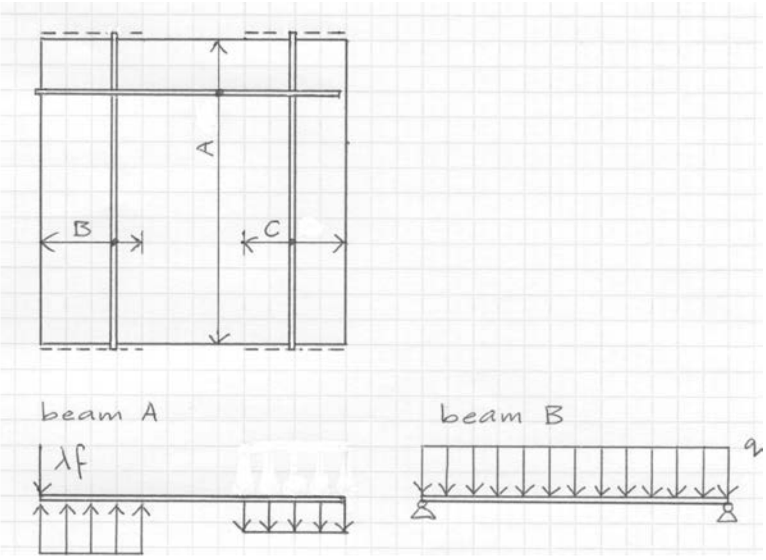


Answer to Problem 2b



$$E = 3m_p\, a\, 2\vartheta + 3m_p\, a\, \vartheta + 3m_p\, a\, \vartheta + m_p \tfrac{3}{2} a\, \varphi + m_p \tfrac{3}{2} a\, \varphi$$
$$= 12\, m_p\, a\vartheta + 3\, m_p\, a\varphi$$
$$= 12\, m_p\, a \frac{\hat{w}}{\frac{3}{2}a} + 3\, m_p\, a \frac{\hat{w}}{a} = 11\, m_p\, \hat{w}$$
$$A = \lambda f\, \tfrac{3}{2} a\, \tfrac{\hat{w}}{2} + \lambda f\, \tfrac{3}{2} a \tfrac{\hat{w}}{2} = \lambda f\, a\, \hat{w}\, \tfrac{3}{2}$$
$$\lambda f = \frac{22}{3} \frac{m_p}{a}$$

Answer to Problem 2c



$b \qquad \lambda f = b q \Rightarrow b = \dfrac{\lambda f}{q}$

$$3\, m_p = \tfrac{1}{8} q (3a)^2$$
$$q = \frac{8}{3} \frac{m_p}{a^2}$$

vert. equilibrium

$\lambda f - qa + pa = 0$

mom. equilibrium

$pa \tfrac{5}{2} \cancel{a} - qa \tfrac{1}{2} \cancel{a} = 0$

$$\lambda f = qa - pa$$
$$pa = \tfrac{1}{5} qa$$
$$\lambda f = \tfrac{4}{5} qa = \tfrac{4}{5}\, \tfrac{8}{3}\, \frac{m_p}{a^2} a$$
$$\boxed{\lambda f = \frac{32}{15} \frac{m_p}{a}}$$

check

$$m = b\lambda f - q b \tfrac{1}{2} b$$
$$= \frac{\lambda f^2}{q} - \tfrac{1}{2} q \left(\frac{\lambda f}{q}\right)^2$$
$$= \tfrac{1}{2} \frac{\lambda f^2}{q} = \tfrac{1}{2} \frac{\left(\frac{32}{15} \frac{m_p}{a}\right)^2}{\frac{8}{3} \frac{m_p}{a^2}} = \frac{64}{75} m_p < m_p \quad \text{okay}$$