Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT4150 Plasticity Theory
Wednesday 23 August 2005, 14:00 – 17:00 hours

Problem 1

A frame consists of two columns and two roof beams (Figure 1). The columns are rigidly connected to the roof beam and to the foundation. The frame is loaded by three forces F. The columns and beams have different yield contours (Figure 2). The following relation exists between the plastic moment M_p and the plastic normal force N_p.

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

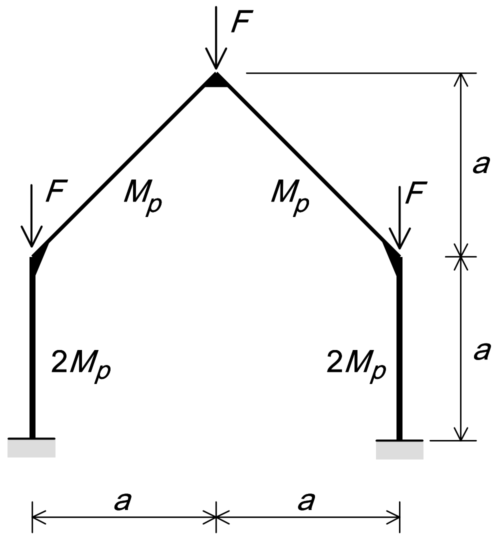


Figure 1. Frame

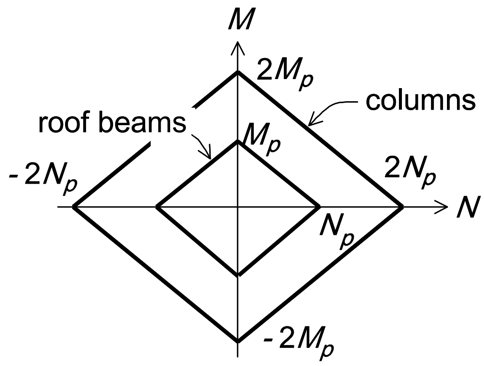


Figure 2. Yield contour

a Assume $\beta \to \infty$. Determine the collapse load F for each possible mechanism. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 points)

b Assume $\beta \to \infty$. Draw the bending moment diagram and the normal force diagram for the whole structure at the moment of collapse. (1 point)

c Assume $\beta = 40$. Choose one of the following problems (You need not do both).

– Determine the largest lower-bound for F.

– Determine the smallest upper-bound for F.

Use the decisive mechanism of problem **1a**. If you choose the upper-bound you only need to write down the equations. You do not need to simplify or solve the equations (2 points).

Problem 2

A square plate is fixed on part of one edge (Figure 3). The plate carries an evenly distributed load q [kN/m²]. The plate is homogeneous. The yield moment in the top of the plate in the x direction is $3m_p$. The yield moment in the bottom of the plate in the x direction is m_p. The yield moment in the top of the plate in the y direction is m_p [kNm/m].

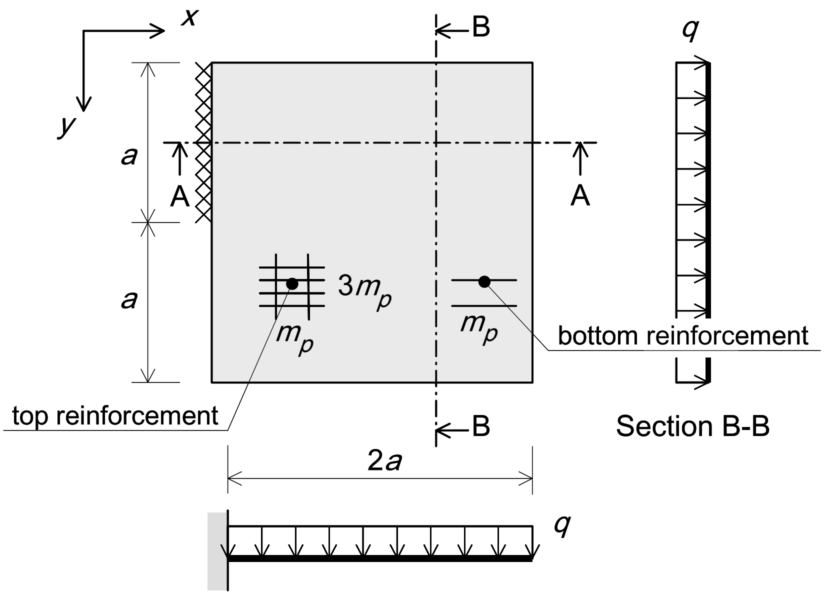


Figure 3. A square plate

a We consider the yield line patterns of Figure 4. Which of these patterns give kinematically possible mechanisms (1 point).

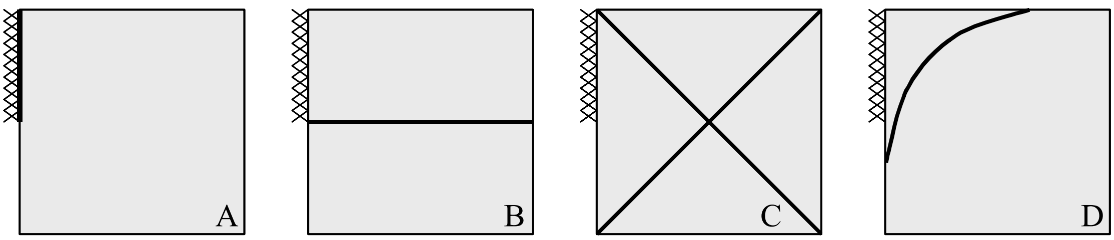


Figure 4. Yield line patterns of problem 2a

b We consider the yield line pattern of Figure 5. Determine an upper bound for q expressed in m_p and a (1.5 points).

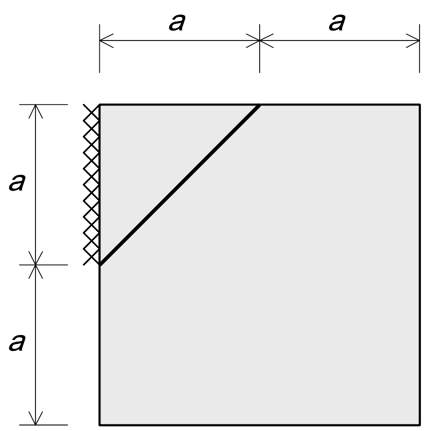


Figure 5. Yield line pattern of problem 2b

c Determine the largest <u>lower-bound</u> for q using torsion free beams ($m_{xy} = 0$) in the x direction and y direction (1.5 points).

Problem 3

a The virtual work in a plastic hinge equals moment times virtual rotation plus normal force times virtual extension.

$$E = M\theta + Nu$$

Proof that this is equal to

$$E = M_p\theta$$

for a rhombic yield contour (Figure 6) and associated flow (= normality) (0.5 points).

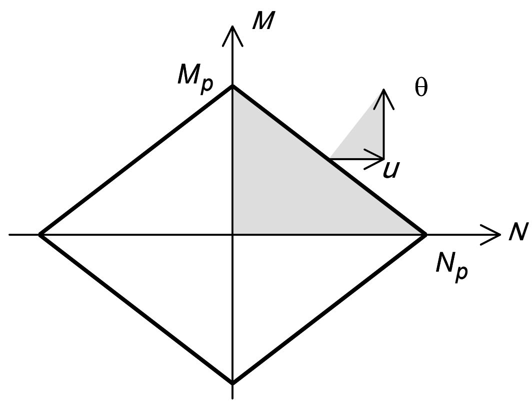


Figure 6. Yield contour of a plastic hinge

b A solution according to elasticity theory is ... Select one of the following answers (0.5 point).

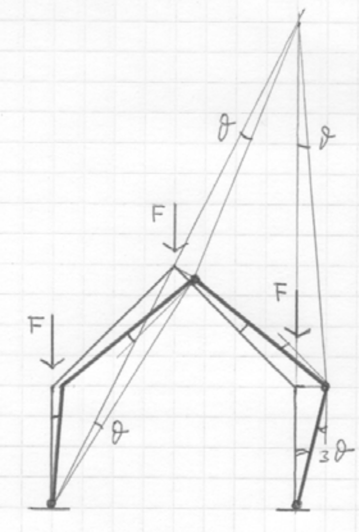
A a lower bound solution because it is an equilibrium system.
B a lower bound solution because it might not have sufficient ductility.
C an upper bound solution because there is no yielding.
D an upper bound solution because the internal energy is equal to the work by the load.

c A finite element program can plot the following quantity over the surface of plates

$$\max\left(\frac{m_{xy}^2}{(m_{px} - m_{xx})(m_{py} - m_{yy})}, \frac{m_{xy}^2}{(m'_{px} + m_{xx})(m'_{py} + m_{yy})}\right),$$

which should be smaller than or equal to one. Which equation in the lecture books can be used to derive this unity check? (You do not need to do the derivation.) (0.5 point)

Answer to Problem 1a

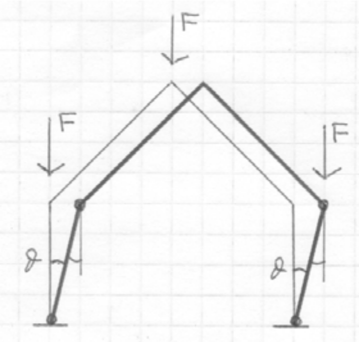


$$A = F\vartheta a$$

$$E = 2M_p\vartheta + M_p(\vartheta+\vartheta) + M_p(\vartheta+3\vartheta) + 2M_p 3\vartheta$$

$$= 14 M_p \vartheta$$

$$A = E \Rightarrow \quad F = 14\frac{M_p}{a}$$

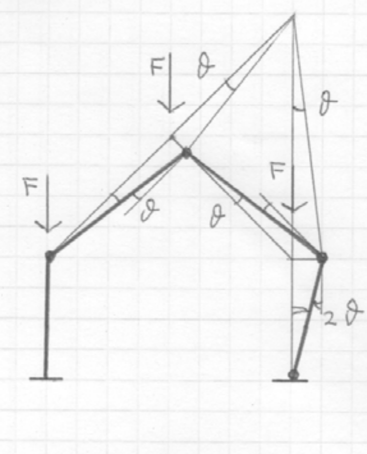


$$A = F\,0$$

$$E = 2M_p\vartheta + M_p\vartheta + M_p\vartheta + 2M_p\vartheta$$

$$= 6M_p\vartheta$$

$$A = E \Rightarrow \quad F = \infty$$



$$A = F\vartheta a$$

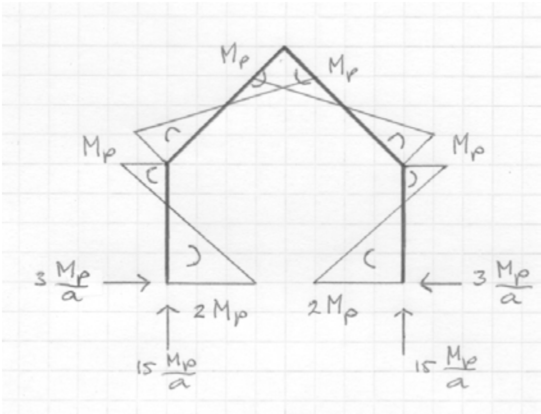
$$E = M_p\vartheta + M_p(\vartheta+\vartheta) + M_p(\vartheta+2\vartheta) + 2M_p 2\vartheta$$

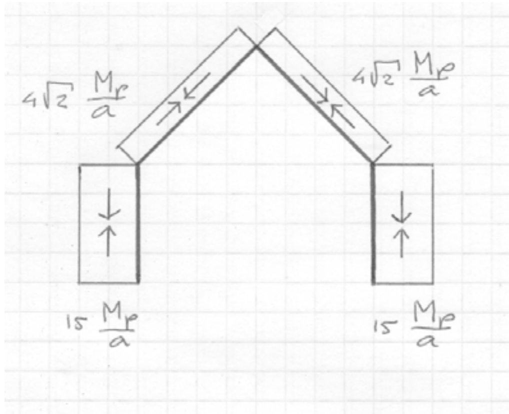
$$= 10 M_p\vartheta$$

$$A = E \Rightarrow \quad \boxed{F = 10\frac{M_p}{a}}$$

decisive

Answer to Problem 1b





Answer to problem 1c Lower-bound

The columns are more overloaded than the roof beams ($15 > 2 \times 4\sqrt{2}$). Therefore, we consider the plastic hinges at the bottom of the frame and reduce the loading with a factor α such that the yield contour is touched. Also the moments and the normal forces are reduced with α. In a section with a moment $\alpha 2M_p$ we can allow a normal force $(1-\alpha)2N_p$.

$$N = \alpha 15 \frac{M_p}{a} = (1-\alpha)2N_p = (1-\alpha)2\beta \frac{M_p}{a} .$$

Consequently,

$$\alpha 15 = (1-\alpha)2\beta$$

and

$$\alpha = \frac{2\beta}{15+2\beta} .$$

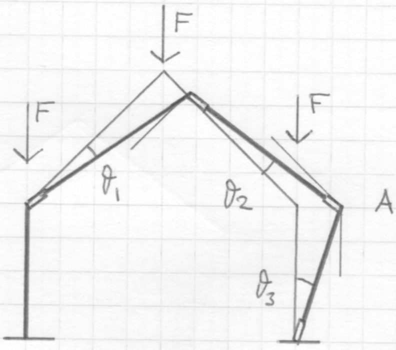
Since $\beta = 40$ we find

$$\alpha = \tfrac{80}{95} .$$

Therefore, a lower bound of the collapse load is

$$F = \alpha 10 \frac{M_p}{a} = \tfrac{80}{95} 10 \frac{M_p}{a} = \tfrac{800}{95} \frac{M_p}{a} = 8.42 \frac{M_p}{a}$$

Answer to Problem 1c Upper-bound



$u = \vartheta \frac{a}{\beta}$

horizontal displacement of A →

$$-\vartheta_1 \frac{a}{\beta}\frac{1}{\sqrt{2}} + \vartheta_1 a - (\vartheta_1+\vartheta_2)\frac{a}{\beta}\frac{1}{\sqrt{2}} + \vartheta_2 a - (\vartheta_2+\vartheta_3)\frac{a}{\beta}\frac{1}{\sqrt{2}} = \vartheta_3 a$$

vertical displacement of A ↓

$$\vartheta_1 \frac{a}{\beta}\frac{1}{\sqrt{2}} + \vartheta_1 a - (\vartheta_1+\vartheta_2)\frac{a}{\beta}\frac{1}{\sqrt{2}} - \vartheta_2 a - (\vartheta_2+\vartheta_3)\frac{a}{\beta}\frac{1}{\sqrt{2}} = \vartheta_3 \frac{a}{\beta}$$

$$A = F0 + F\left(\vartheta_1 \frac{a}{\beta}\frac{1}{\sqrt{2}} + \vartheta_1 a\right) + F\vartheta_3\frac{a}{\beta}$$

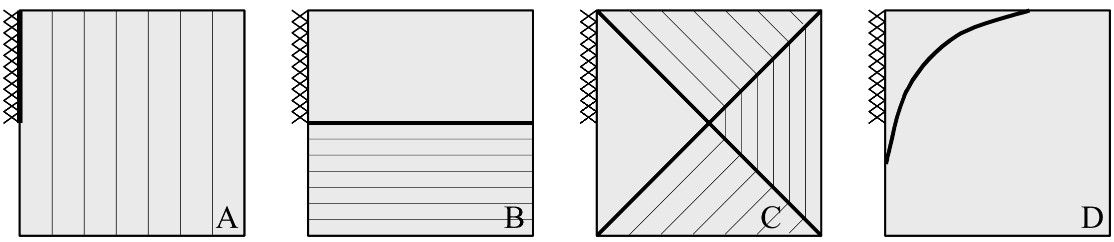
$$E = M_p\vartheta_1 + M_p(\vartheta_1+\vartheta_2) + M_p(\vartheta_2+\vartheta_3) + 2M_p\vartheta_3$$

$$A = E$$

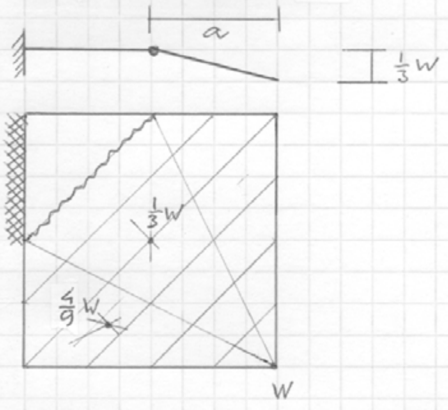
the solution is $\vartheta_1 = 1{,}12\,\vartheta_2$, $\vartheta_3 = 2{,}01\,\vartheta_2$ $F = 8{,}62\,\frac{M_p}{a}$

Answer to Problem 2a

Kinematically possible are patterns A, B, C. The figure below shows the altitude lines of the deformed mechanisms.



Answer to Problem 2b



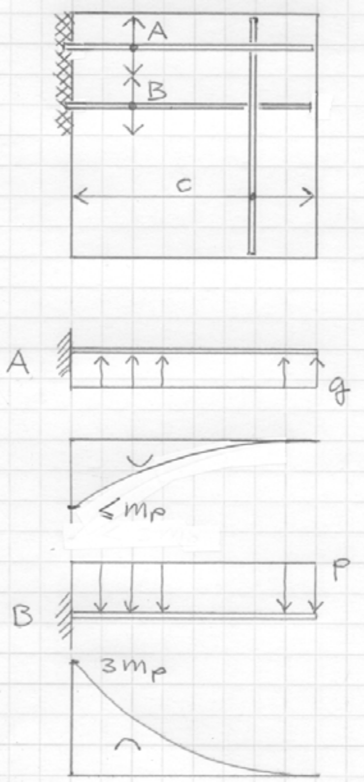
$$\frac{0+1+\frac{1}{3}}{3}=\frac{4}{9}$$

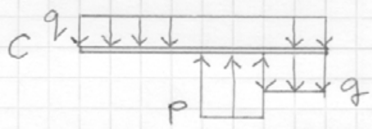
$$A=2\left(\tfrac{1}{2}a\,2a\,q\,\tfrac{4}{9}w\right)+\tfrac{1}{2}a\sqrt{2}\,\tfrac{3}{4}2a\sqrt{2}\,q\,\tfrac{1}{3}w=\tfrac{25}{18}a^2qw$$

$$E=3m_p\,a\,\frac{\frac{1}{2}w}{a}+m_p\,a\,\frac{\frac{1}{3}w}{a}=\tfrac{4}{3}m_pw$$

$$A=E\Rightarrow\quad q=\frac{24}{25}\frac{m_p}{a^2}$$

Answer to Problem 2c





$$\begin{cases} q\,2a\,a-p\,\tfrac{1}{2}a\,\tfrac{3}{4}a+g\,\tfrac{1}{2}a\,\tfrac{1}{4}a=0\\ q\,2a-p\,\tfrac{1}{2}a+g\,\tfrac{1}{2}a=0\end{cases}$$

$$\begin{cases} p=6q\\ g=2q\end{cases}$$

$$3m_p=p\,2a\,a=12qa^2\Rightarrow$$

$$\boxed{q=\frac{1}{4}\frac{m_p}{a^2}}$$

check beam A

$$m_p \overset{?}{\geq} q\,2a\,a = 4qa^2 = m_p \quad \text{O.K.}$$

check beam C

$$m_p \overset{?}{\geq} q\,\frac{5}{4}a\,\frac{5}{8}a - p\,\frac{1}{4}a\,\frac{1}{8}a = q\,\frac{19}{32}a^2 = \frac{19}{128}\,m_p \quad \text{O.K.}$$

Answer to Problem 3a

$$E = M\theta + Nu = (1-\alpha)M_p\theta + \alpha N_p u = (1-\alpha)M_p\theta + \alpha\frac{\theta}{u}M_p u = M_p\theta \quad \text{Q.E.D.}$$

Answer to Problem 3b

A …a lower bound solution because it is an equilibrium system.

Answer to Problem 3c

Equation (12.13) in the plate lecture book.