Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your <u>name</u> and <u>study number</u> at the top right-hand of your work.

Exam CT4150 Plasticity Theory
Wednesday 21 June 2006, 9:00 – 12:00 hours

Problem 1

A frame consists of a column with a strength M_p and a beam with a strength $2M_p$ (Fig. 1). The column is rigidly connected to the beam and to the foundation. The frame is loaded by two forces F and $10F$. The following relation exists between the plastic moment M_p and the plastic normal force N_p (Fig. 2).

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

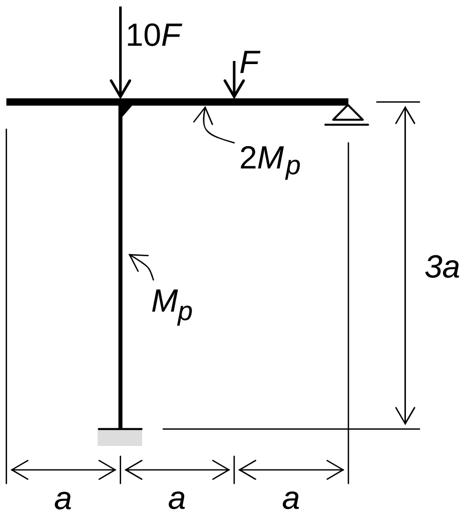


Figure 1. Frame

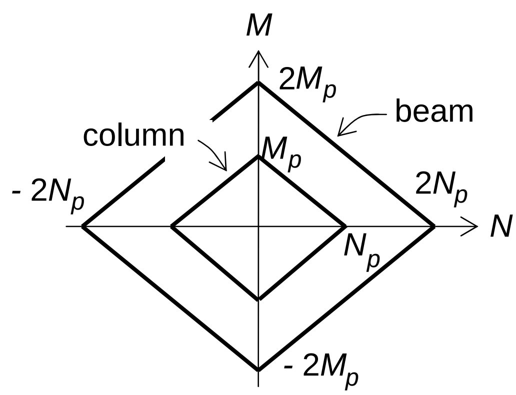


Figure 2. Yield contour

a Assume $\beta \to \infty$. Determine the collapse load F for each possible mechanism. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 points)

b Assume $\beta \to \infty$. Draw the bending moment diagram and the normal force diagram for the structure at the moment of collapse. (1 point)

c Assume $\beta = 42$. Choose one of the following problems (You need not do both).

– Determine the largest <u>lower-bound</u> for F.

– Determine the smallest <u>upper-bound</u> for F.

Use the decisive mechanism of problem **1a** (2 points).

Problem 2

A plate with an opening is simply supported at four edges (Figure 3). The plate carries an evenly distributed load q [kN/m²]. The plate is homogeneous. Reinforcing bars in the bottom of the plate provide a positive yield moment $2m_p$ [kNm/m] in the y direction and m_p in the x direction. The top face of the plate is not reinforced.

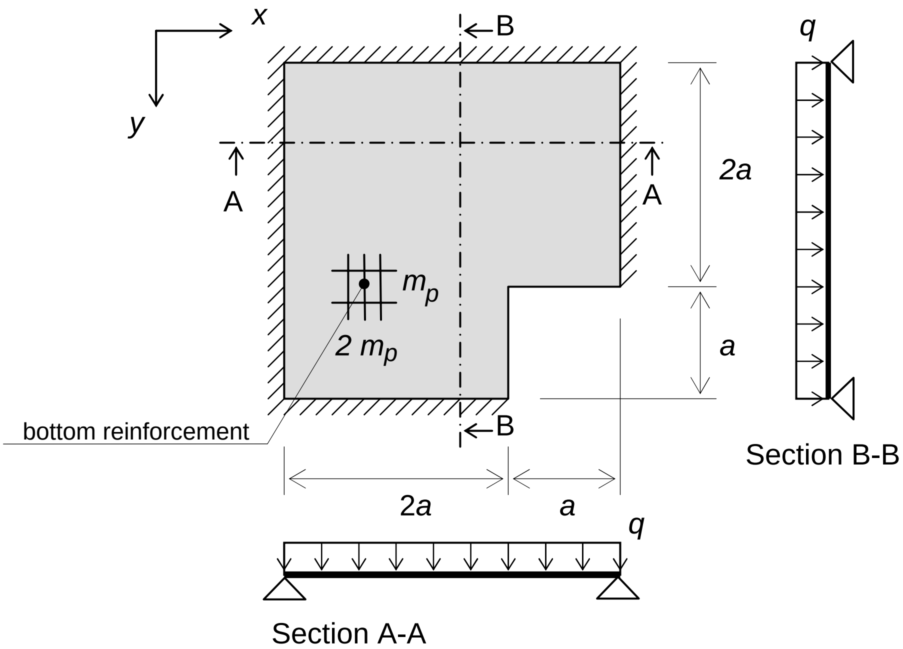


Figure 3. Square plate with an opening

a We consider the yield line patterns of Figure 4. Which of these patterns give kinematically possible mechanisms (1 point).

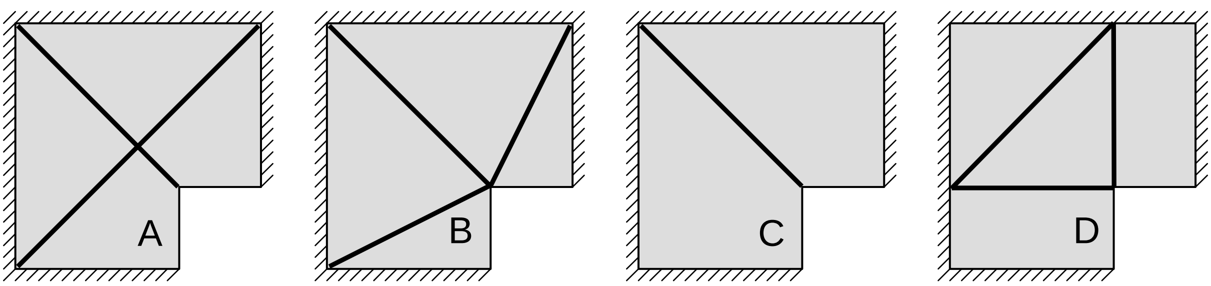


Figure 4. Yield line patterns of problem 2a

b We consider the yield line pattern of Figure 5. Determine an <u>upper bound</u> for q expressed in m_p and a (1.5 points).

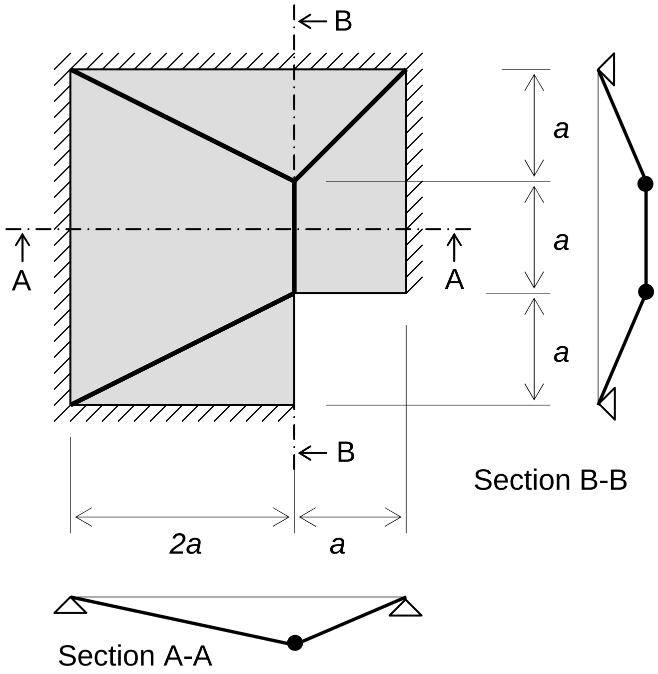


Figure 5. Yield line pattern of problem 2b

c Determine the largest <u>lower-bound</u> for q using torsion free beams ($m_{xy} = 0$) in the x direction and y direction (1.5 points).

Problem 3

a Explain shortly why plasticity is closely related to optimisation (0.5 points).

b Steel tubular joints are often analysed using an upper bound method and Rankine's yield condition (Fig. 6). However, Von Mises' yield condition would be more appropriate. Why is Rankine used? (0.5 points)

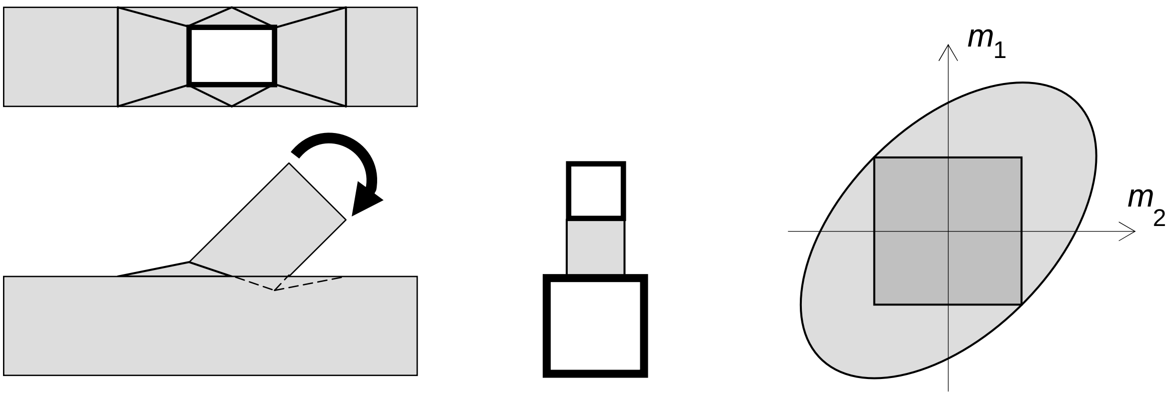
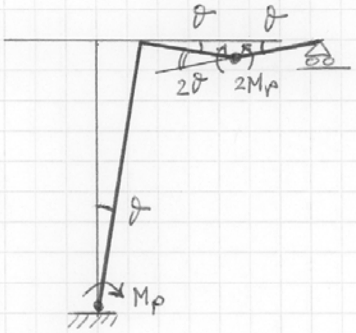


Figure 6. Yield lines of a tubular joint (left), Yield contours of Von Mises and Rankine (right)

c How can plastic analysis be used to estimate crack widths in a concrete structure in the serviceability limit state? (0.5 point)

Answer to Problem 1a

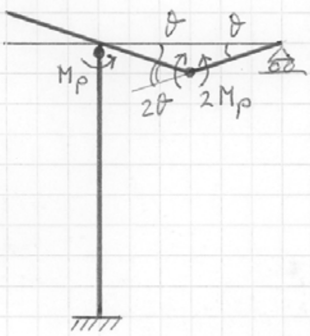


$$A = Fa\vartheta + 10F0$$

$$E = M_p\vartheta + 2M_p\,2\vartheta = 5M_p\vartheta$$

$$A = E \Rightarrow F = 5\frac{M_p}{a}$$

also decisive

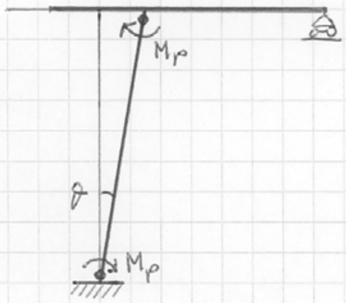


$$A = Fa\vartheta$$

$$E = M_p\vartheta + 2M_p\,2\vartheta = 5M_p\vartheta$$

$$A = E \Rightarrow \boxed{F = 5\frac{M_p}{a}}$$

decisive

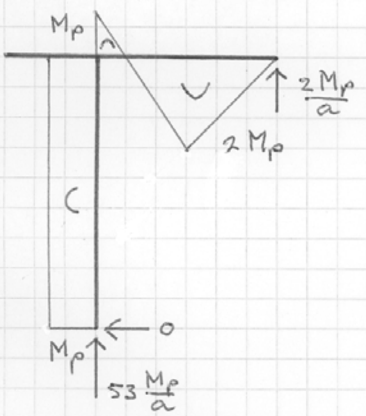


$$A = F0$$

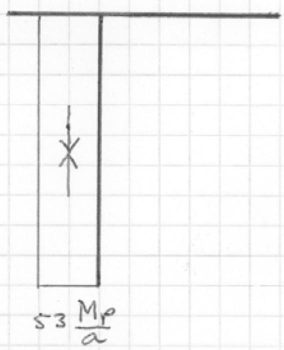
$$E = M_p\vartheta + M_p\vartheta = 2M_p\vartheta$$

$$A = E \Rightarrow F = \frac{2M_p\vartheta}{0} = \infty$$

Answer to Problem 1b

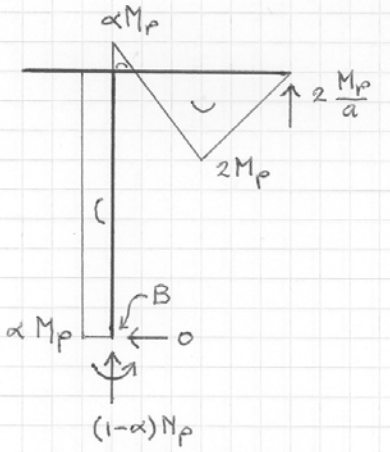


moment distribution



normal force distribution

Answer to problem 1c Lower-bound



$$N_p = \beta \frac{M_p}{a}$$

vert. equilibrium ↓

$$F + 10F - 2\frac{M_p}{a} - (1-\alpha) N_p = 0$$

mom. equilibrium ↻ B

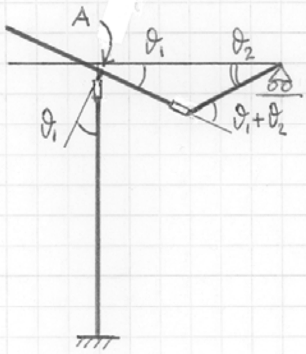
$$\alpha M_p = Fa - 2\frac{M_p}{a} 2a$$

from these we solve

$$\alpha = \frac{\beta-42}{\beta+11} = 0$$

$$F = \frac{2+5\beta}{11+\beta}\frac{M_p}{a} = 4\frac{M_p}{a}$$

Answer to Problem 1c Upper-bound



$$u = \frac{a}{\beta}\vartheta$$

deflection of point A ↓

$$\frac{a}{\beta}\vartheta_1 = \vartheta_2 a - \vartheta_1 a$$

$$\vartheta_1\left(\frac{a}{\beta}+a\right) = \vartheta_2 a$$

$$\vartheta_1 = \frac{1}{\frac{1}{\beta}+1}\vartheta_2 = \frac{\beta}{1+\beta}\vartheta_2 = \frac{42}{43}\vartheta_2$$

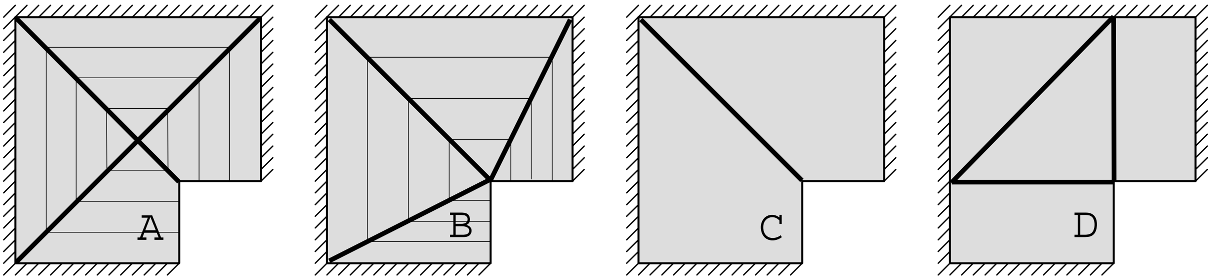
$$E = M_p\,\vartheta_1 + 2M_p(\vartheta_1+\vartheta_2) = M_p\vartheta_2\left(\frac{42}{43} + 2\left(\frac{42}{43}+1\right)\right) = M_p\,\vartheta_2\,\frac{212}{43}$$

$$A = F\vartheta_2 a + 10F\frac{a}{\beta}\vartheta_1 = Fa\vartheta_2\left(1+\frac{10}{42}\frac{42}{43}\right) = Fa\vartheta_2\frac{53}{43}$$

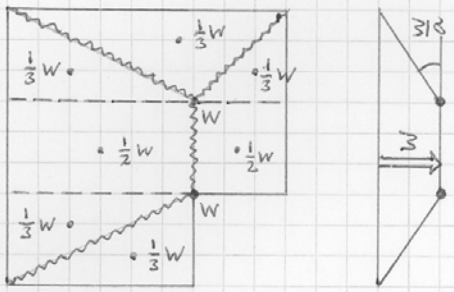
$$E = A \Rightarrow F = \frac{212}{43}\frac{43}{53}\frac{M_p}{a} = 4\frac{M_p}{a}$$

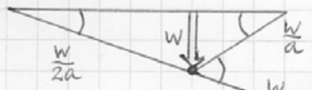
Answer to Problem 2a

Kinematically possible are patterns A, B. The figure below shows the altitude lines of the deformed mechanisms.



Answer to Problem 2b





$$\frac{w}{2a}+\frac{w}{a}=\frac{3}{2}\frac{w}{a}$$

$$E = m_p \frac{3}{2}\frac{w}{a} a + 2\left(m_p \frac{w}{2a} a + 2 m_p \frac{w}{a} 2a\right) + \left(m_p \frac{w}{a} a + 2 m_p \frac{w}{a} a\right)$$

$$= m_p w \left(\frac{3}{2} + 2\left(\frac{1}{2}+4\right) + 1 + 2\right) = m_p w \frac{27}{2}$$

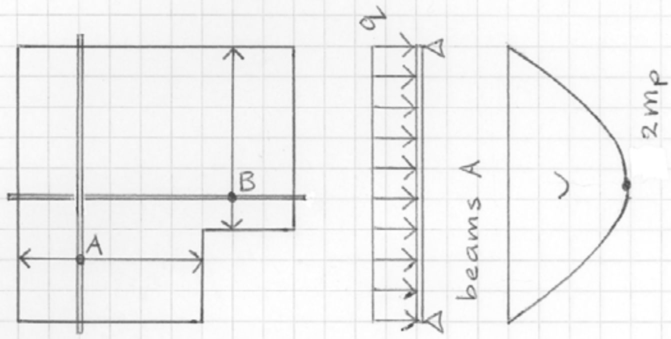
$$A = q \frac{2a\,a}{2}\frac{1}{3}w + q \frac{2a\,a}{2}\frac{1}{3}w + q\, 2a\,a \frac{1}{2}w + q \frac{2a\,a}{2}\frac{1}{3}w +$$

$$+ q \frac{3a\,a}{2}\frac{1}{3}w + q \frac{a\,a}{2}\frac{1}{3}w + q\, a\,a \frac{1}{2}w$$

$$= q a^2 w \left(\frac{1}{3}+\frac{1}{3}+1+\frac{1}{3}+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}\right) = q a^2 w \frac{19}{6}$$

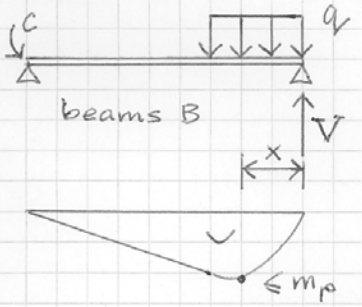
$$E = A \Rightarrow q = \frac{27}{2}\frac{6}{19}\frac{m_p}{a^2} = \frac{81}{19}\frac{m_p}{a^2}$$

Answer to Problem 2c



$$\frac{1}{8} q (3a)^2 = 2 m_p$$

$$\boxed{q = \frac{16}{9} \frac{m_p}{a^2}}$$



check beams B

mom. around C

$$q a \tfrac{5}{2} a - V 3a = 0 \Rightarrow V = \tfrac{5}{6} q a$$

vert. equilibrium ↓

$$q x - V = 0 \Rightarrow x = \frac{V}{q} = \frac{5}{6} a$$

mom. max.

$$m = V x - q x \tfrac{1}{2} x = \tfrac{5}{6} q a \tfrac{5}{6} a - q \tfrac{5}{6} a \tfrac{1}{2} \tfrac{5}{6} a$$

$$= \frac{25}{72} q a^2 = \frac{25}{72} \frac{16}{9} m_p = 0.62\, m_p < m_p$$

okay

Answer to Problem 3a

In plastic analysis we are looking for the smallest of all upper bounds or the largest of all lower bounds. Finding the smallest or largest is optimisation.

Answer to Problem 3b

Rankine is used because Von Mises would be too difficult. The bending moment in a section perpendicular to the yield line would affect the plastic moment in the yield line.

Answer to Problem 3c

Plastic analysis cannot be used to calculate crack widths. It is only suitable for the ultimate limit state.