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Computation of optimal concrete reinforcement in three dimensions

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ABSTRACT: A method is proposed for determining the required reinforcement based on stresses that have been computed by the finite element method using volume elements. Included are, multiple load combinations, compression reinforcement, confinement reinforcement and crack control. The method is illustrated by several stress examples and a structural example.

1 INTRODUCTION

Many computer programs for structural analysis have post processing functionality for designing reinforcement and performing code compliance checks. For example the moments and normal forces computed with shell elements can be used to determine the required reinforcement based on the Eurocode design rules [1, 2, 3]. However, for finite element models containing volume elements the codes do not provide design rules. Software companies that are developing structural analysis programs are in the process of extending their program capabilities with volume elements. Consequently, also the algorithms for computing reinforcement requirements need to be extended for use with volume elements.

In 1983, Smirnov pointed out the importance of this problem for design of reinforced concrete in hydroelectric power plants [4]. In 1985, Andreasen and Nielsen derived formulas for the optimal reinforcement for three-dimensional stress states [5]. They also designed a flow chart for determining which formula to use. In 1994, Kamezawa et al. proposed and tested several formulas for three-dimensional reinforcement design [6]. In 2002 and 2003, Foster, Marti and Mojsilović published two thorough studies on the subject [7, 8]. In 2008, Hoogenboom and de Boer used analytical and numerical methods for computing three-dimensional reinforcement requirements [9].

This paper continues on this path. An algorithm is proposed for computing the optimal reinforcement for multiple load combinations. The load combinations are related to the ultimate limit state or the serviceability limit state. Not only tension reinforcement is considered but also compression reinforcement and confinement reinforcement. A maximum crack width is imposed for the serviceability limit state. Numerical results are compared to analytical results of elementary stress states. The algorithm has been implemented in a finite element program. A structural example is included.

2 PROBLEM FORMULATION

It is assumed that reinforcing bars are present in the x, y and z direction only (Fig. 1, App. 2). The reinforcement ratios are ρ_x, ρ_y and ρ_z, respectively. The smallest amount of reinforcement is obtained when the volume reinforcement ratio is minimised.

$$\text{Minimise } \rho_x + \rho_y + \rho_z \qquad (1)$$

Clearly, the reinforcement ratios need to be positive which gives a constraint on the solution.

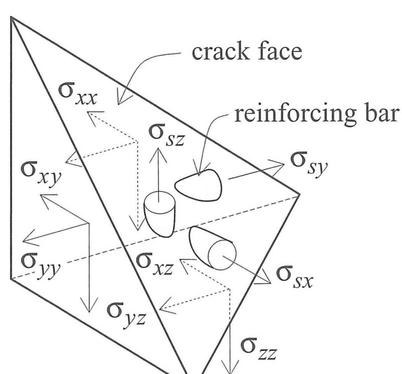

Figure 1. Elementary part of reinforced concrete. Shown are a crack and the reinforcing bars that bridge this crack.

$$\rho_x \geq 0$$
$$\rho_y \geq 0 \tag{2}$$
$$\rho_z \geq 0$$

For load combinations related to the ultimate limit state the stresses σ_{sx}, σ_{sy}, σ_{sz} in the reinforcing steel need to be no larger than the yield value f_y.

$$-f_y \leq \sigma_{sx} \leq f_y$$
$$-f_y \leq \sigma_{sy} \leq f_y \tag{3}$$
$$-f_y \leq \sigma_{sz} \leq f_y$$

The concrete principal stresses σ_{c1}, σ_{c2}, σ_{c3} need to be in compression or zero. This is because the concrete might not have tensile strength locally due to shrinkage cracks that can occur during hardening. In this paper the principal stresses are ordered from small to large

$$\sigma_{c3} \leq \sigma_{c2} \leq \sigma_{c1} \tag{4}$$

Therefore, it is sufficient to require that

$$\sigma_{c1} \leq 0. \tag{5}$$

In this paper the Mohr-Coulomb yield condition is used for preventing concrete compressive failure.

$$\frac{\sigma_{c1}}{f_t} - \frac{\sigma_{c3}}{f_c'} \leq 1 \tag{6}$$

where, f_c' is the uniaxial concrete compressive strength and f_t is the concrete mean tensile strength. Here, the tensile strength is larger than zero because it is an average value instead of a local value.

For load combinations related to the serviceability limit state, the crack width w needs to be limited.

$$w \leq w_{\max} \tag{7}$$

This condition is imposed for aesthetics and to prevent corrosion of the reinforcing steel.

In reinforced concrete beam design it is customary to include at least a minimum reinforcement. This is to ensure ductile failure and distributed cracking. However, in many situations the minimum reinforcement requirements result in much more reinforcement than reasonable. Therefore, in this paper it is not considered. Of course, a design engineer can decide to apply at least minimum reinforcement according to the governing code of practice.

3 CONCRETE STRESSES

The stresses in a structural part can be computed in a linear or non-linear finite element analysis. In this paper the stress tensor in a point is written as

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}. \tag{8}$$

From the material stress tensor the concrete stress tensor can be derived [9].

$$\begin{bmatrix} \sigma_{xx} - \rho_x\sigma_{sx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \rho_y\sigma_{sy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \rho_z\sigma_{sz} \end{bmatrix} \tag{9}$$

The concrete principal stresses are the eigenvalues of this tensor.

The invariants of the concrete stress tensor are

$$I_{c1} = \sigma_{cx} + \sigma_{cy} + \sigma_{cz} \tag{10}$$

$$I_{c2} = \sigma_{cx}\sigma_{cy} + \sigma_{cy}\sigma_{cz} + \sigma_{cz}\sigma_{cx} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2 \tag{11}$$

$$I_{c3} = \sigma_{cx}\sigma_{cy}\sigma_{cz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{cx}\sigma_{yz}^2 - \sigma_{cy}\sigma_{xz}^2 - \sigma_{cz}\sigma_{xy}^2 \tag{12}$$

where the concrete stresses are

$$\sigma_{cx} = \sigma_{xx} - \rho_x\sigma_{sx}$$
$$\sigma_{cy} = \sigma_{yy} - \rho_y\sigma_{sy} \tag{13}$$
$$\sigma_{cz} = \sigma_{zz} - \rho_z\sigma_{sz}.$$

It can be proved that the condition $\sigma_{c1} \leq 0$ is fulfilled if and only if $I_{c1} \leq 0$, $I_{c2} \geq 0$ and $I_{c3} \leq 0$ [9].

In this paper it is assumed that reinforced concrete is a ductile material. Prager's second law (lower bound theorem of plasticity theory) is applied to load combinations for the ultimate limit state [10, 11].

4 CRACK WIDTH COMPUTATION

The linear elastic strains computed by a finite element analysis could be used for determining the crack width. However, these strains would not be very accurate because they strongly depend on Young's modulus of cracked reinforced concrete which can only be estimated. On the other hand, the stresses do not depend on Young's modulus.[1] Therefore, the computation of

[1] Except for temperature loading and foundation settlements in statically indetermined structures. For these

crack widths starts from the stresses. In essence, the adopted equations are part of the Modified Compression Field Theory [12] simplified for the serviceability limit state and extended for three dimensional analysis.

Eqs (9) and (13) can be rewritten to

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = P \begin{bmatrix} \sigma_{c1} & 0 & 0 \\ 0 & \sigma_{c2} & 0 \\ 0 & 0 & \sigma_{c3} \end{bmatrix} P^T + \begin{bmatrix} \rho_x\sigma_{sx} & 0 & 0 \\ 0 & \rho_y\sigma_{sy} & 0 \\ 0 & 0 & \rho_z\sigma_{sz} \end{bmatrix} \tag{14}$$

where σ_{c1}, σ_{c2}, σ_{c3} are the concrete principal stresses and

$$P = \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{bmatrix}. \tag{15}$$

The columns in P are the vectors of the concrete principal directions. Note that in general these principal directions are not the same as the linear elastic principal directions.

Since yielding is supposed not to occur in the serviceability limit state, the constitutive relations for the reinforcing bars are linear elastic. The constitutive relation for compressed concrete is also approximated as linear elastic in the principal directions. Poisson's ratio is set to zero. The constitutive relation for tensioned concrete is

$$\sigma_{ci} = \frac{f_t}{1+\sqrt{500\varepsilon_i}} \quad i = 1,2,3 \tag{16}$$

where f_t is the concrete mean tensile strength [12]. For the crack width computation it is assumed that aggregate interlock can carry any shear stress in the crack. It is assumed that the concrete principal stresses and the principal strains have the same direction.

The principal strains ε_1, ε_2, ε_3 are the eigenvalues of the strain tensor.

$$\begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = P \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} P^T \tag{17}$$

From Eqs (14) to (17) the strain tensor can be solved numerically by the Newton-Raphson method.

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cases an accurate estimate of Young's modulus of cracked reinforced concrete needs be used in the linear elastic analysis. Alternatively, a physical nonlinear analysis can be used.

The CEB-fib Model Code 90 is applied for computing crack widths [13]. The mean crack spacings s for uniaxial tension in the reinforcement directions are

$$s_x = \frac{2}{3}\frac{d_x}{3.6\,\rho_x}, \quad s_y = \frac{2}{3}\frac{d_y}{3.6\,\rho_y}, \quad s_z = \frac{2}{3}\frac{d_z}{3.6\,\rho_z} \tag{18}$$

where d_x, d_y, d_z are the diameters of the reinforcing bars in the x, y, z direction. The crack spacing s in principal direction i is computed from

$$\frac{1}{s_i} = \frac{|\cos\alpha_i|}{s_x} + \frac{|\cos\beta_i|}{s_y} + \frac{|\cos\gamma_i|}{s_z} \quad i = 1,2,3. \tag{19}$$

The mean crack width in the principal direction i is

$$w_i = s_i(\varepsilon_i - \varepsilon_c - \varepsilon_s) \quad i = 1,2,3. \tag{20}$$

where ε_c is the concrete strain and ε_s is the concrete shrinkage. The value of ε_c is positive and the value of ε_s is negative. For simplicity, in this paper is assumed that they cancel each other out.

5 REINFORCMEMENT OPTIMISATION

The optimisation problem can be visualised in a graph (Fig. 2). The axis of this graph represent ρ_x, ρ_y and ρ_z. The condition $I_{c3} = 0$ is shown as a surface. The objective is to find the smallest possible value of $\rho_x+\rho_y+\rho_z$. The shape of the surface depends on the linear elastic stress tensor and on the steel stresses. Not only interior solutions but also corner solutions and boundary solutions are possible.

For each load combination related to the ultimate limit state four of such surfaces occur as a result of

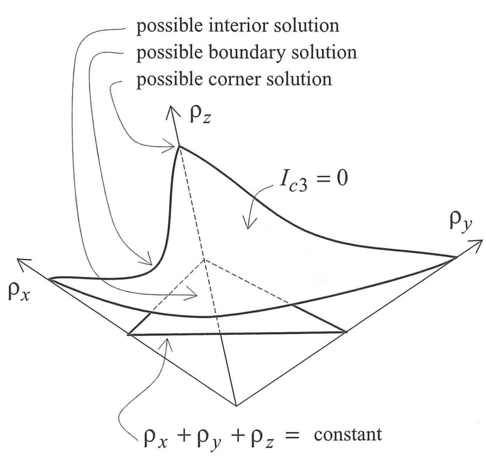

Figure 2. Conceptual presentation of the optimisation problem.

the equations $I_{c1} = 0$, $I_{c2} = 0$, $I_{c3} = 0$, $\sigma_{c1}/f_t - \sigma_{c3}/f_c' = 1$. For each load combination related to the serviceability limit state one surface occurs as a result of the equation $w = w_{\max}$. Each surface gives a lower bound to the amount of reinforcement. The minimum can be on a crossing line of two surfaces or on a crossing point of three surfaces.

At first sight, efficient use of the reinforcement requires that the steel yields in tension or in compression. However, sometimes it is necessary to consider less steel stress. This can be explained in a simple example. Suppose that one load combination requires a large amount of reinforcing steel. Suppose that another load combination results in hardly any stresses. If we insist on applying the yield stresses in both load combinations than in the second load combination the concrete is compressed considerably by the reinforcement and the Mohr-Coulomb constraint might not be fulfilled. In reality, the steel stresses and concrete stresses will be small too during the second load combination. Consequently, if the yield stress is used in the optimization, the Mohr-Coulomb condition can work as an artificial upper bound in the optimization problem instead of as a lower bound. This upper bound is removed when the steel stresses are properly reduced in the second load combination. Consequently, not only the reinforcement ratios but also the steel stresses σ_{sx}, σ_{sy}, σ_{sz} need to be varied in the optimisation problem.

6 REINFORCEMENT FORMULAS

The optimization problem is reduced considerably if only one load combination is present. It simplifies even further when the Mohr-Coulomb condition and crack control are ignored. In this case eleven sets of solutions can be derived for the optimal reinforcement [9]. Each of these sets fulfil the condition $I_{c3} = 0$ which means that one of the concrete principal stresses is zero.

$$\rho_x = 0, \quad \rho_y = 0, \quad \rho_z = \frac{I_3}{f_y(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)} \qquad (21\text{-}1)$$

$$\rho_x = 0, \quad \rho_y = \frac{I_3}{f_y(\sigma_{xx}\sigma_{zz} - \sigma_{xz}^2)}, \quad \rho_z = 0 \qquad (21\text{-}2)$$

$$\rho_x = \frac{I_3}{f_y(\sigma_{yy}\sigma_{zz} - \sigma_{yz}^2)}, \quad \rho_y = 0, \quad \rho_z = 0 \qquad (21\text{-}3)$$

$$\begin{aligned} \rho_x &= 0 \\ \rho_y &= \frac{\sigma_{yy}}{f_y} - \frac{\sigma_{xy}^2}{f_y\sigma_{xx}} \pm \left(\frac{\sigma_{xz}\sigma_{xy}}{f_y\sigma_{xx}} - \frac{\sigma_{yz}}{f_y}\right) \\ \rho_z &= \frac{\sigma_{zz}}{f_y} - \frac{\sigma_{xz}^2}{f_y\sigma_{xx}} \pm \left(\frac{\sigma_{xz}\sigma_{xy}}{f_y\sigma_{xx}} - \frac{\sigma_{yz}}{f_y}\right) \end{aligned} \qquad (21\text{-}4)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx}}{f_y} - \frac{\sigma_{xy}^2}{f_y\sigma_{yy}} \pm \left(\frac{\sigma_{yz}\sigma_{xy}}{f_y\sigma_{yy}} - \frac{\sigma_{xz}}{f_y}\right) \\ \rho_y &= 0 \\ \rho_z &= \frac{\sigma_{zz}}{f_y} - \frac{\sigma_{yz}^2}{f_y\sigma_{yy}} \pm \left(\frac{\sigma_{yz}\sigma_{xy}}{f_y\sigma_{yy}} - \frac{\sigma_{xz}}{f_y}\right) \end{aligned} \qquad (21\text{-}5)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx}}{f_y} - \frac{\sigma_{xz}^2}{f_y\sigma_{zz}} \pm \left(\frac{\sigma_{xz}\sigma_{yz}}{f_y\sigma_{zz}} - \frac{\sigma_{xy}}{f_y}\right) \\ \rho_y &= \frac{\sigma_{yy}}{f_y} - \frac{\sigma_{yz}^2}{f_y\sigma_{zz}} \pm \left(\frac{\sigma_{xz}\sigma_{yz}}{f_y\sigma_{zz}} - \frac{\sigma_{xy}}{f_y}\right) \\ \rho_z &= 0 \end{aligned} \qquad (21\text{-}6)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx} + \sigma_{xy} + \sigma_{xz}}{f_y} \\ \rho_y &= \frac{\sigma_{yy} + \sigma_{xy} + \sigma_{yz}}{f_y} \\ \rho_z &= \frac{\sigma_{zz} + \sigma_{xz} + \sigma_{yz}}{f_y} \end{aligned} \qquad (21\text{-}7)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx} + \sigma_{xy} - \sigma_{xz}}{f_y} \\ \rho_y &= \frac{\sigma_{yy} + \sigma_{xy} - \sigma_{yz}}{f_y} \\ \rho_z &= \frac{\sigma_{zz} - \sigma_{xz} - \sigma_{yz}}{f_y} \end{aligned} \qquad (21\text{-}8)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx} - \sigma_{xy} - \sigma_{xz}}{f_y} \\ \rho_y &= \frac{\sigma_{yy} - \sigma_{xy} + \sigma_{yz}}{f_y} \\ \rho_z &= \frac{\sigma_{zz} - \sigma_{xz} + \sigma_{yz}}{f_y} \end{aligned} \qquad (21\text{-}9)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx} - \sigma_{xy} + \sigma_{xz}}{f_y} \\ \rho_y &= \frac{\sigma_{yy} - \sigma_{xy} - \sigma_{yz}}{f_y} \\ \rho_z &= \frac{\sigma_{zz} + \sigma_{xz} - \sigma_{yz}}{f_y} \end{aligned} \qquad (21\text{-}10)$$

$$\begin{aligned} \rho_x &= \frac{\sigma_{xx}}{f_y} - \frac{\sigma_{xy}\sigma_{xz}}{f_y\sigma_{yz}} \\ \rho_y &= \frac{\sigma_{yy}}{f_y} - \frac{\sigma_{xy}\sigma_{yz}}{f_y\sigma_{xz}} \\ \rho_z &= \frac{\sigma_{zz}}{f_y} - \frac{\sigma_{xz}\sigma_{yz}}{f_y\sigma_{xy}} \end{aligned} \qquad (21\text{-}11)$$

where I_3 is the determinant of the linear elastic stress tensor.

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2 \quad (22)$$

Which of these eleven sets of formulas (21-xx) to use can be determined in four steps. First, calculate I_1, I_2 and I_3. If $I_1 \leq 0$, $I_2 \geq 0$ and $I_3 \leq 0$ than tension reinforcement is not needed. Second, ignore the sets of formulas that give negative reinforcement ratios. Third, calculate I_{c1} and I_{c2} by Eqs (13), (10), (11) and ignore the sets for which $I_{c1} > 0$ or $I_{c2} < 0$. Fourth, select the set of formulas for which $\rho_x + \rho_y + \rho_z$ is smallest.

With the results of step three the concrete compressive stress can be calculated and checked.

$$\sigma_{c3} = \frac{1}{2}I_{c1} - \sqrt{\left(\frac{1}{2}I_{c1}\right)^2 - I_{c2}} \geq -f_c' \quad (23)$$

The $\pm$ signs in Eq. (~~15~~21-4), (~~15~~21-5) and (~~15~~21-6) can be replaced by the absolute value. The proof for this is presented in Appendix 1.

7 BARRIER METHOD

The barrier method is a method for computational optimization [14]. In this method, a large cost is imposed on points that lie close to the boundary of the feasible region. This cost is called the barrier because it makes sure that a new point is not picked outside the feasible region.

$$\text{minimise } \rho_x + \rho_y + \rho_z + rB \quad (24)$$

where r is a factor that is reduced in subsequent steps and B is the barrier. A suitable barrier function for the problem of this paper is

$$B = \frac{0.01}{\rho_x} + \frac{0.01}{\rho_y} + \frac{0.01}{\rho_z} + \sum_{i=1}^{n_u}\left(-\frac{f_t}{\sigma_{c1,i}} + \frac{1}{1 - \frac{\sigma_{c1,i}}{f_t} + \frac{\sigma_{c3,i}}{f_c'}}\right) + \sum_{i=1}^{n_s}\frac{w_{\max}}{w_{\max} - w_i} \quad (25)$$

where n_u is the number of load combinations related to the ultimate limit state and n_s is the number of load combinations related to the serviceability limit state. The functions involved are

$$\sigma_{c1} = \sigma_{c1}(\rho_x\sigma_{sx}, \rho_y\sigma_{sy}, \rho_z\sigma_{sz})$$
$$\sigma_{c3} = \sigma_{c3}(\rho_x\sigma_{sx}, \rho_y\sigma_{sy}, \rho_z\sigma_{sz})$$
$$w = w(\rho_x, \rho_y, \rho_z)$$

Note that B is positive for feasible solutions. It goes to infinity if any of the constraints is almost violated.

An advantage of the barrier method compared to other methods of computational optimisation is that only interior points are evaluated. Interior points are "sufficient reinforcement" for which the computation of the crack width w converges quickly. The minimisation can be performed by any unconstrained optimisation algorithm such as the down-hill simplex method or Newton's method.

A good starting point for the Barrier method is the envelope of the requirements for the individual load combinations. The required reinforcement for a load combination related to the serviceability limit state can be quickly approximated by assuming that the reinforcement ratios are proportional to the steel stresses.

8 STRESS EXAMPLES

Table 1 shows 13 results of the proposed algorithm. The rows contain computation examples. Columns σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{xz}, σ_{yz} contain the input stresses in N/mm^2. The reinforcement yield stress is f_y = 500 N/mm^2 for each example. Column ρ_x, ρ_y, ρ_z contain the output reinforcement ratios in %. Column σ_{c1}, σ_{c2}, σ_{c3} contain the output principal concrete stresses. Column Eq. shows the formula number that gives the same result. Except for the last example all examples involve just one load combination. In the last computation example two load combinations are included.

Example 1 to 7 have also been used by Andreasen et al. [5]. Their results and the results in this paper are the same. Example 8 and 9 have also been studied by Foster et al. [7]. In example 8 the same results have been found. In example 9, Foster selected $\rho_x = 0.75\%$, $\rho_y = 0$, $\rho_z = 0.75\%$. Table 1 shows that the optimal reinforcement differs considerably. However, the total reinforcement is almost the same (Foster; $0.75 + 0.00 + 0.75 = 1.50\%$, Table 1; $0.89 + 0.00 + 0.57 = 1.46\%$). It is noted that Forster et al. selected this reinforcement without trying to find the optimum. In fact, the optimum is an edge solution which was not considered in their publications [7, 8].

Example 10 shows that in a plane stress state several formula sets provide the optimum reinforcement.

Example 11 and 12 are included for comparison with Example 13. Example 13 includes two load

Table 1. Stress computation examples.

Case	σ_{xx}	σ_{yy}	σ_{zz}	σ_{xy}	σ_{xz}	σ_{yz}	ρ_x	ρ_y	ρ_z	σ_{c3}	σ_{c2}	σ_{c1}	Eq.
1	1	2	3	−1	3	−4	1.00	1.40	2.00	−10.65	−5.35	.	21-10
2	−5	2	3	1	3	4	.	1.36	1.88	−10.31	−5.89	.	21-4−
3	−5	−6	3	1	3	4	.	.	1.69	−10.15	−6.30	.	21-1
4	−5	−6	−6	1	3	4	.	.	.	−10.44	−6.31	−0.24	
5	1	2	3	−1	−3	−4	0.60	1.00	2.00	−10.58	−1.42	.	21-8
6	1	−2	3	2	3	−4	0.50	0.13	1.80	−10.17	.	.	21-11
7	1	2	3	−1	2	4	0.40	1.00	1.80	−9.36	−0.64	.	21-7
8	2	−2	5	6	−4	2	2.40	0.40	1.40	−15.21	−0.79	.	21-8
9	−3	−7	.	6	−4	2	0.89	.	0.57	−14.76	−2.52	.	21-5+
10	3	.	10	.	5	.	1.60	.	3.00	−10.00	.	.	21-5−, 7, 10
11	15	.	.	.	.	.	3.00	.	.	.	.	.	21-3, 7, 9
12	.	.	.	5	.	.	1.00	1.00	.	−10.00	.	.	21-6−, 7, 8
13	15	.	.	.	.	.	3.00	0.33	.	−1.67	.	.	
	.	.	.	5	.	.	3.00	0.33	.	−16.67	.	.	

The dots (.) represent zeros (0) in order to improve readability of the table.

combinations. The volume reinforcement ratio is $\rho_x + \rho_y + \rho_z = 3.00 + 0.33 + 0.00 = 3.33\%$. Alternatively, we could have selected the envelope of the reinforcement requirements for the individual load combinations, which are 11 and 12. The volume reinforcement ratio applying the envelope method is $\max(3.00, 1.00) + \max(0.00, 1.00) = 4.00\%$. Consequently, the envelope method gives a safe approximation but it overestimates the required reinforcement substantially.

9 STRUCTURAL EXAMPLE

Figure 3 shows a square concrete block that is fixed at one face of the block. The block is loaded by a vertical force of 1000 kN over an area of 0.20×0.20 m (25 N/mm^2). Just one load is considered. Dead load (24 kN) is neglected. Young's modulus is 30000 N/mm^2 and Poisson's ratio is 0.15. The concrete compressive strength is 35 N/mm^2. Its tensile strength is 4 N/mm^2. The steel yield strength is 550 N/mm^2. One load case related to the ultimate limit state is considered. Load and resistance factors are not included.

The proposed algorithm has been implemented in a finite element program. An eight node brick element was used. The element dimensions are $0.10 \times 0.10 \times 0.10$ m. A linear elastic analysis is performed. The normal stress σ_{xx} is shown in Fig. 4. The required reinforcement ratios for the ultimate limit state are computed by the proposed algorithm (Figs. 5, 6 and 7).

The results provide sufficient information for a structural engineer to select bar diameters and bar spacing. Subsequently, the reinforcing cage can be designed by applying reinforcing principles (hoops, hooks, hairpins, development length). Note that not

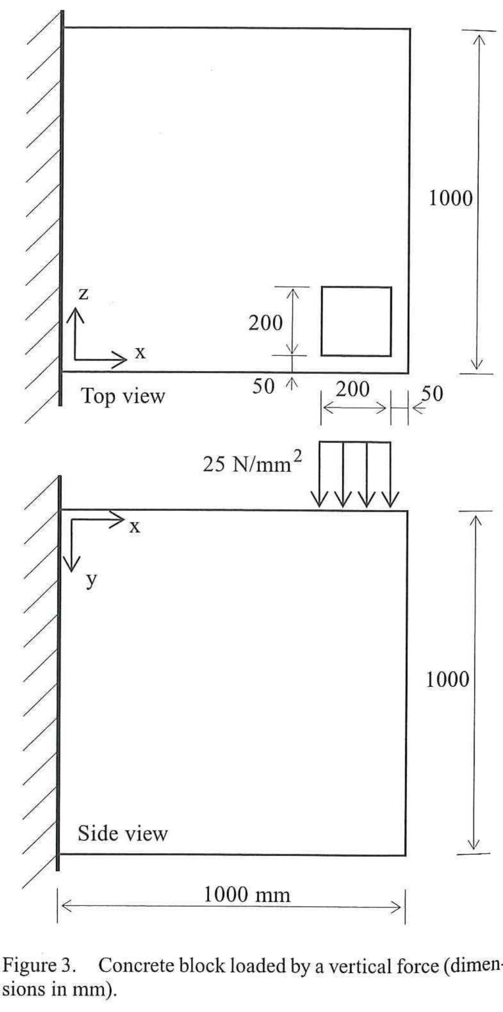

Figure 3. Concrete block loaded by a vertical force (dimensions in mm).

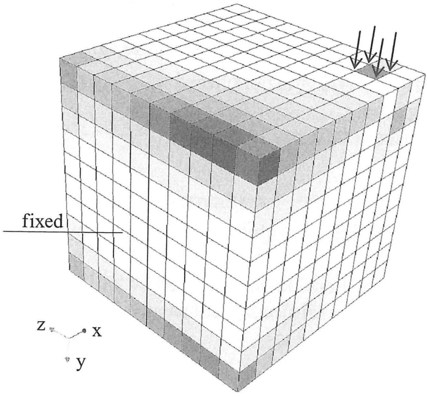

Figure 4. The normal stress σ_{xx} on the surface of the concrete block. The largest value is 6.17 N/mm^2. The smallest value is -8.52 N/mm^2.

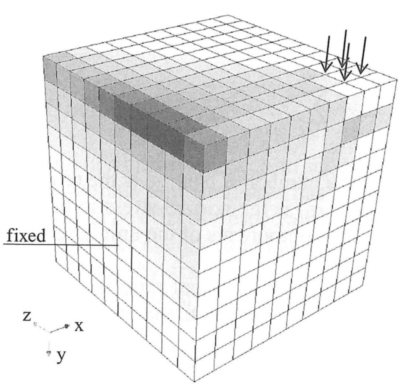

Figure 5. Optimal reinforcement ratio ρ_x The largest value is 1.48%.

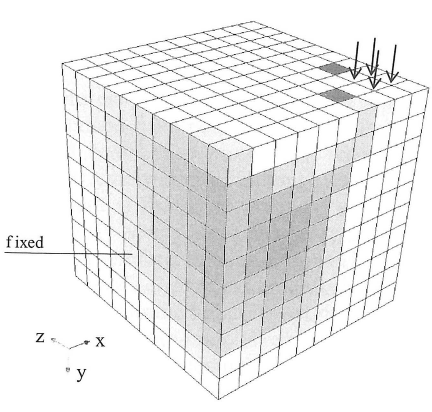

Figure 6. Optimal reinforcement ratio ρ_y The largest value is 0.56%.

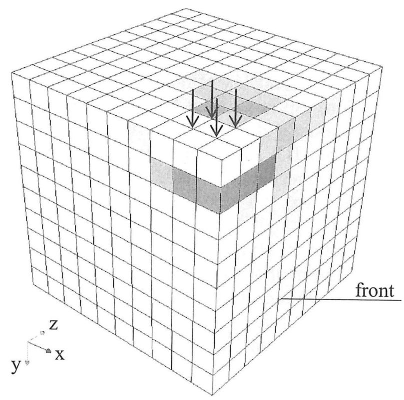

Figure 7. Optimal reinforcement ratio ρ_z The largest value is 1.02%.

only reinforcement for bending and shear are needed but also splitting reinforcement is needed for introducing the load into the concrete. The authors recommend that the reinforcement detailing is checked by mentally visualising the force flow with a strut-and-tie model. This does not mean that the reinforcement needs to be quantified with a strut-and-tie model. There is no need for this time consuming task because the required amounts are already determined by the proposed algorithm.

10 CONCLUSIONS

A numerical algorithm is proposed for computing the required reinforcement in solid concrete. It starts from the stresses in the integration points of a finite element model. In subsequent improvements the algorithm finds the reinforcement ratios ρ_x, ρ_y, ρ_z for which the sum is smallest. Constraints are imposed on the steel stresses and the concrete stresses for load combinations related to the ultimate limit state. A constraint is imposed on the crack widths for load combinations related to the serviceability limit state. The algorithm shows to be robust, fast and accurate.

LITERATURE

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APPENDIX 1

The $\pm$ sign in Eqs (21–4), (21–5) and (21–6) can be replaced by the absolute value. Here this is proven for Eqs (21, 5). Substitution of Eqs (21, 5) in Eqs (10), (11) and (12) gives

$$I_{c1} = \sigma_{yy} + \frac{\sigma_{xy}^2 + \sigma_{yz}^2}{\sigma_{yy}} \mp \left(\frac{\sigma_{yz}\sigma_{xy}}{\sigma_{yy}} - \sigma_{xz}\right) 2$$

$$I_{c2} = \frac{\mp\left(\frac{\sigma_{yz}\sigma_{xy}}{\sigma_{yy}} - \sigma_{xz}\right)(2\sigma_{yy}^2 + (\sigma_{yz} \mp \sigma_{xy})^2)}{\sigma_{yy}}$$

$$I_{c3} = 0$$

From $I_{c2} \geq 0$ it is concluded that the values of σ_{yy} and $\mp\left(\frac{\sigma_{yz}\sigma_{xy}}{\sigma_{yy}} - \sigma_{xz}\right)$ need to have the same sign.

Form $I_{c1} \leq 0$ it is concluded that this sign needs to be negative.

Consequently, $\sigma_{yy} < 0$ and $\pm\left(\frac{\sigma_{yz}\sigma_{xy}}{\sigma_{yy}} - \sigma_{xz}\right) \geq 0$.

Q.E.D.

APPENDIX 2

Reinforcing bars do not need to be in the x, y and z directions. For reinforcement in any direction the concrete stress tensor is

$$\begin{bmatrix} \sigma_{cx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{cy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{cz} \end{bmatrix}$$

$$\sigma_{cx} = \sigma_{xx} - \sum_{i=1}^{n} v_{xi}\,\rho_i\,\sigma_{si}$$

$$\sigma_{cy} = \sigma_{yy} - \sum_{i=1}^{n} v_{yi}\,\rho_i\,\sigma_{si}$$

$$\sigma_{cz} = \sigma_{zz} - \sum_{i=1}^{n} v_{zi}\,\rho_i\,\sigma_{si}$$

where n is the number of bars, ρ_i is the reinforcement ratio of bar i, σ_{si} is the normal stress in bar i and v_{xi}, v_{yi}, v_{zi} are the components of the unit length direction vector of bar i.

The volume reinforcement ratio is $\rho = \sum_{i=1}^{n} \rho_i$, which can be minimised with the proposed algorithm.